Homework and Pre-Class reading for Math 152H-1 August 6

Homework Solutions:

1. Given $\epsilon > 0$. Find N so that for $n > N$ the terms of $a_n = \frac{n^2+2}{4n^2+1}$ satisfy $\left|a_n - \frac{1}{4}\right| < \epsilon$. This is the crucial step in showing that $\lim_{n\to\infty} a_n = \frac{1}{4}$.

   Compute $\left|a_n - \frac{1}{4}\right| = \frac{7}{4(4n^2+1)}$. We now ask when is this less than ϵ. So we solve $\frac{7}{16n^2+4} < \epsilon$ for n. This gives $16n^2 + 4 > \frac{7}{\epsilon}$ or $n > \frac{1}{4}\sqrt{\frac{7}{\epsilon} - 4}$. So we can choose $N_\epsilon = \frac{1}{4}\sqrt{\frac{7}{\epsilon} - 4}$. Remember that we only care about ϵ when it is very small, so $\frac{7}{\epsilon} - 4$ will be a positive number.

2. Show that $b_n = 3n - 1$ diverges to ∞ by using the definition.

   We need to show that for each $M > 0$ (thought of as very large), $b_n > M$ for all n bigger than some N (which depends on M). So we solve to find this N, as in (1). $3n - 1 > M$ when $n > \frac{M+1}{3}$. Setting $N = \frac{M+1}{3}$ will do the job. Since we can find such an N for any M that you might be given, the sequence diverges to infinity.

3. Use the definition to explain why $a_n \to L$ implies that $C \cdot a_n \to C \cdot L$ where C is a constant (i.e. does not change with n, you're just multiplying all the terms in the sequence by C). Try not to get anxious about the absence of any specific numbers.

   Consider $\left|C \cdot a_n - C \cdot L\right| = \left|C\right|\left|a_n - L\right|$. By the definition of $a_n \to L$ there is an N such that for $n > N$, $\left|a_n - L\right| < \frac{\epsilon}{\left|C\right|}$. IF we choose $n > N$ then upon putting these together we get:

   $$\left|C \cdot a_n - C \cdot L\right| = \left|C\right|\left|a_n - L\right| < \left|C\right| \cdot \frac{\epsilon}{\left|C\right|} = \epsilon$$

   Hence for each $\epsilon > 0$, if we go far enough along the sequence we can be sure that $\left|C \cdot a_n - C \cdot L\right| < \epsilon$. Notice that the effect of C only changes how long we need to wait.

4. Prove the squeeze theorem: If $a_n \to L$ and $b_n \to L$ (the same limit) and $a_n \leq c_n \leq b_n$ then $c_n \to L$. You might want to write $\left|a_n - L\right| < \epsilon$ as $L - \epsilon < a_n < L + \epsilon$, and likewise for b_n and c_n.

   Choose $\epsilon > 0$. Suppose for $n > N_a$ we have $L - \epsilon < a_n < L + \epsilon$, and for $n > N_b$ we have $L - \epsilon < b_n < L + \epsilon$. Since there are two sequences we need both N_a and N_b, one for each sequence. However, if $n > \max\{N_a, N_b\}$ then both sets of inequalities will be true. For those n we will have

   $$L - \epsilon < a_n \leq c_n \leq b_n < L + \epsilon$$

   And thus for $n > \max\{N_a, N_b\}$ we will have $\left|c_n - L\right| < \epsilon$. Try drawing a picture to see that this is a lot simpler than the proof makes it appear.

5. Use the definition to explain why $1, -1, 1, -1, \ldots$ has no limit. Hint: Suppose the limit is L. Calculate the distance from L to 1 and -1 separately. Show that there is no N_ϵ for $\epsilon = \frac{1}{2}$ so that for *all* $n > N$, etc.

   Suppose it did reach a limit L. Then for each $n > N_{\frac{1}{2}}$ we would have $\left|L - a_n\right| < \frac{1}{2}$. When n is even $a_n = 1$, thus $-\frac{1}{2} < L - 1 < \frac{1}{2}$ or $\frac{1}{2} < L < \frac{3}{2}$. Put more transparently, in order for the terms with even n to be within $\frac{1}{2}$ of a limit means the limit must be within $\frac{1}{2}$ of 1. But then L is at least $\frac{3}{2}$ from -1. Thus the terms with n odd cannot get close enough to L. Simply put, that 1 and -1 are some distance from each other means there can be no L close to both.