Solutions to homework for September 1

Homework: Prove by mathematical induction:

1.
$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

Solution: For $n = 1$ the sum on the left is 1 and the fraction on the right is $\frac{1(2)}{2} = 1$. So the formula is true for $n = 1$. Now assume that the formula is true for n. Then we have

$$1 + 2 + 3 + \ldots + n + (n+1) = \frac{n(n+1)}{2} + (n+1) = (n+1)\left(\frac{n}{2} + 1\right) = \frac{(n+1)(n+2)}{2}$$

If the formula is true for n then it is true for $n+1$. Since it is true for $n = 1$, the formula is true for all n.

2.
$$1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Solution: This is very similar to the previous case. For $n = 1$ the sum on the left is 1 and the fraction on the right is $\frac{1(2)(3)}{6} = 1$. So the formula is true for $n = 1$. Now assume that the formula is true for n. Then we have

$$\begin{aligned} 1^2 + 2^2 + 3^2 + \ldots + n^2 + (n+1)^2 &= \tfrac{n(n+1)(2n+1)}{6} + (n+1)^2 = \tfrac{n(n+1)(2n+1)+6(n+1)^2}{6} \\ &= \tfrac{(n+1)(2n^2+n+6n+6)}{6} = \tfrac{(n+1)(2n^2+7n+6)}{6} = \tfrac{(n+1)(n+2)(2n+3)}{6} \end{aligned}$$

Since $2n+3 = 2(n+1)+1$, this is just the formula for $n+1$. If the formula is true for n then it is true for $n+1$. Since it is true for $n = 1$, the formula is true for all n.

3. Since $\left|\, a + b \,\right| \le \left|\, a \,\right| + \left|\, b \,\right|$, show that

$$\left|c_1 + c_2 + \ldots + c_n\right| \le \left|c_1\right| + \left|c_2\right| + \ldots + \left|c_n\right|$$

Solution: I have told you that the case for $n = 2$ is true. Now we use that to prove the more general case. Assume that for any numbers:

$$\left|c_1 + c_2 + \ldots + c_n\right| \le \left|c_1\right| + \left|c_2\right| + \ldots + \left|c_n\right|$$

Now consider

$$\left|c_1 + c_2 + \ldots + c_n + c_{n+1}\right| = \left|(c_1 + c_2 + \ldots + c_n) + c_{n+1}\right|$$

We think of $c_1 + \ldots + c_n$ as a and c_{n+1} as b, and then use $\left|\, a + b \,\right| \le \left|\, a \,\right| + \left|\, b \,\right|$. This gives

$$\left|c_1 + c_2 + \ldots + c_n + c_{n+1}\right| \le \left|c_1 + c_2 + \ldots + c_n\right| + \left|c_{n+1}\right|$$

By assumption, we know that $\left|c_1 + c_2 + \ldots + c_n\right| \le \left|c_1\right| + \left|c_2\right| + \ldots + \left|c_n\right|$, so we have

$$\left|c_1 + c_2 + \ldots + c_n + c_{n+1}\right| \le \left|c_1\right| + \left|c_2\right| + \ldots + \left|c_n\right| + \left|c_{n+1}\right|$$

We know that the inequality is true for all numbers when $n = 2$. If the inequality is true for n, we have seen that it must be true for $n+1$. Therefore, no matter how many terms, c_i, there are, we may use the inequality above.

Problem: Let A and B be positive, real numbers. Prove the AM-GM inequality:

$$\frac{A+B}{2} \ge \sqrt{AB}$$

Solution: Since $x^2 \geq 0$ for any numbere x, we must have $(\sqrt{A} - \sqrt{B})^2 \geq 0$. It is equal to 0 only when $A = B$. Now expanding the square gives $A - 2\sqrt{AB} + B \geq 0$ or, upon re-writing, $\frac{A+B}{2} \geq \sqrt{AB}$. There is equality only if $A = B$.

Challenge Problem: Consider the sequence from the last homework

$$b_n = \frac{1}{2}\big(b_{n-1} + \frac{1}{b_{n-1}}\big) \qquad\qquad b_1 = 2$$

Show that this is decreasing sequence that is bounded below, justifying our assumption that it has a limit. Here are the steps:

1. Show that if $b_{n-1} > 1$ then $b_n < b_{n-1}$.
   Solution: We replace b_n with the formula above and ask is $\frac{1}{2}\big(b_{n-1} + \frac{1}{b_{n-1}}\big) < b_{n-1}$. To find when this happens we solve the inequality: $\big(b_{n-1} + \frac{1}{b_{n-1}} < 2b_{n-1} \Leftrightarrow \frac{1}{b_{n-1}} < b_{n-1}$ or $b_{n-1}^2 > 1$. The solutions to this last inequality are when b_{n-1} is in the set $(-\infty, -1) \cup (1, \infty)$. So when $b_{n-1} > 1$ the inequality is true and we have $b_n < b_{n-1}$.

2. Show that if $b_{n-1} > 1$ then $b_n > 1$. To do this use the AM-GM inequality from the previous problem.
   Solution: Since $b_{n-1} > 0$, we have $\frac{1}{b_{n-1}} > 0$. If we apply the AM-GM inequality from the previous exercise we obtain

   $$b_n = \frac{1}{2}\big(b_{n-1} + \frac{1}{b_{n-1}}\big) \geq \sqrt{b_{n-1} \cdot \frac{1}{b_{n-1}}} = 1$$

   There is equality only when $b_{n-1} = 1$, but we have assumed that this is not the case.

Now we use induction. Since $b_1 = 2 > 1$, we have $b_1 > b_2 > 1$ by the properties above. Assume that $b_{n-1} > b_n > 1$. Then by the properties above we have $b_n > b_{n+1} > 1$. Hence this must be true for all n. But then $b_1 > b_2 > b_3 > \cdots > b_n \cdots > 1$ and the sequence $\{b_n\}$ is decreasing and bounded below. Such a sequence has a limit.

Another problem involving bounded, decreasing or increasing sequences: Suppose we have closed intervals in the number line, $I_j = [a_j, b_j]$ (the set of x with $a_j \leq x \leq b_j$), and we have one for each $j = 1, 2, 3, \ldots$. Further assume that $I_j \subset I_{j-1}$ (i.e. $a_{j-1} \leq a_j$ and $b_j \leq b_{j-1}$), and that $b_j - a_j \to 0$ as $j \to \infty$. Explain why there is precisely one and only one real number that is in *all* the intervals.

Solution: The sequence $\{a_j\}$ is increasing and bounded above by any of the b_j. Likewise $\{b_j\}$ is decreasing and bounded below. Thus both sequences have limits, L_a and L_b. However, as $j \to \infty$, the terms in the sequence must get close to the limits, and by assumption must get close to each other. This cannot happen if $L_a \neq L_b$ as they would be separated by some distance on the number line. Thus $L_a = L_b$ and we call this number L. But $a_j \leq L \leq b_j$ for each j and thus L is in all the intervals. Since the length of the intervals goes to zero only L can be in all the intervals. Thus there is one and only one number in all the intervals. The situation described in the problem is called haveing a nested sequence of closed intervals.