Homework and Pre-Class reading for Math 152H-1 August 30

This section begins with part of the reading for Friday:
We have seen how to show that an increasing sequence that is bounded above converges. What do we do about sequences which are not increasing or decreasing? For example, how do we *prove* that $2 + \frac{(-1)^n}{n}$ converges to 2 or that $n - \frac{1}{n}$ diverges to infinity instead of just bouncing around. In general, we first have to guess what happens and guess the limit if we think there is one. We then check our guess using the **definition** of convergence. Ok, we've talked about convergence already and we have an intuitive idea of what it means: as $n \to \infty$, the values of a_n should "get closer, and closer" to some specific value L. Why do we need a definition? Because "closer and closer" is too vague; when we have complicated sequences we need something better. In particular, when we want to understand the properties of limits, we need the definition. So here it is:

Definition: We say that a sequence $\{a_n\}$ converges to a limit L when for each $\epsilon > 0$, there is a number N_ϵ such that when $n \geq N_\epsilon$ we have $\big|a_n - L\big| < \epsilon$.

You might say that this doesn't look any better, but it is. In particular, it gives specific mathematical properties to be checked. Furthermore, we have ways to manipulate inequalities, so checking convergence gets turned into algebraic calculation rather than guessing. Note, however, that you need to have an L in mind to be able to use this. What the definition really does is say, how close do you want to be to L? within ϵ? then you need only wait until $n > N_\epsilon$ and *all* the subsequent terms will be within ϵ of L, i.e. if we choose ϵ very small, then they will be very close to L. Now by choosing ϵ even smaller, and waiting until the new N_ϵ, we can force the terms to be even closer.

Here's an example: Lets shown that $\frac{1}{n}$ converges to 0. Given ϵ, we choose $N_\epsilon = \frac{1}{\epsilon}$. Then when $n > N$, $\big|\frac{1}{n} - 0\big| = \big|\frac{1}{n}\big| < \frac{1}{N} = \epsilon$. So for each ϵ we have found an N_ϵ and verified the inequality. Therefore 0 is the limit.

I mentioned on the first day that we could talk about an infinite list by giving a rule for calculating each term (rather than wasting away our lives actually writing down the list). This definition builds upon that. Instead of having to know the entire sequence to determine whether we are getting "closer, and closer", we need only verify that after some point all the terms are squeezed near the limit by the inequality. Here's another example:

Let $b_n = \frac{3n+1}{2n-1}$. We will show that $b_n \to \frac{3}{2}$. To do this, we need to know how to choose N_ϵ for a given ϵ. For that we first compute how far away the terms are from the limit:

$$\Big|\frac{3n+1}{2n-1} - \frac{3}{2}\Big| = \frac{5}{2n-1}$$

Now when $\frac{5}{2n-1} < \epsilon$ we will be okay. This occurs when $2n - 1 > \frac{5}{\epsilon}$ or $n > \frac{1}{2} + \frac{5}{2\epsilon}$. We thus take N_ϵ to be $\frac{1}{2} + \frac{5}{2\epsilon}$.

Here's the new stuff:
The laws underlying the use of limits in computations: suppose $a_n \to L$ and $b_n \to M$ then

1. $a_n \pm b_n \to L \pm M$
2. $a_n \cdot b_n \to L \cdot M$
3. $C \cdot a_n \to C \cdot L$ for any number C
4. If $M \neq 0$ then $\dfrac{a_n}{b_n} \to \dfrac{L}{M}$

To prove these, it is useful to have the triangle inequality. For any finite set of real numbers $C_1, \ldots, C_n$ the following is true:

$$\big|C_1 + C_2 + \ldots + C_n\big| < \big|C_1\big| + \big|C_2\big| + \ldots + \big|C_n\big|$$

This is very useful.

For example: suppose we wish to show that $a_n + b_n \to L + M$. We need to show that $\big|(a_n + b_n) - (L + M)\big|$ can be forced to remain smaller than any real number, ϵ, by letting n get larger and larger. We know that $\left|a_n - L\right|$ gets smaller than any real number as $n \to \infty$, as does $\left|b_n - M\right|$. Using the triangle inequality:

$$\left|(a_n + b_n) - (L + M)\right| = \left|(a_n - L) + (b_n - M)\right| \leq \left|a_n - L\right| + \left|b_n - M\right|$$

If we wait long enough both terms on the right get smaller than $\frac{\epsilon}{2}$. At that point the expression on the left gets smaller than ϵ and will remain so.
We also argued that if $a_n > 0$ and $a_n \to L$ and $L > 0$, then $\sqrt{a_n} \to \sqrt{L}$. Here's the proof we gave written out correctly. Choose $\epsilon > 0$. Since $a_n \to L$, we can choose N large enough for $\left|a_n - L\right| < \sqrt{L}\epsilon$. Then

$$\left|\sqrt{a_n} - \sqrt{L}\right| = \frac{\left|a_n - L\right|}{\left|\sqrt{a_n} + \sqrt{L}\right|} < \frac{1}{\sqrt{L}}\left|a_n - L\right| < \epsilon$$

since $a_n > 0$ implies $\sqrt{a_n} + \sqrt{L} > \sqrt{L}$.

We used two facts $\left|C \cdot D\right| = \left|C\right| \cdot \left|D\right|$ and $(E^2 - F^2) = (E - F)(E + F)$. You should learn both of these. In the argument above, we use $E = \sqrt{a_n}$ and $F = \sqrt{L}$. Then $C = \sqrt{a_n} - \sqrt{L}$ and $D = \sqrt{a_n} + \sqrt{L}$.

Definition: A sequence $\{a_n\}$ diverges to $+\infty$ if for any poitive real number, M, no matter how large, there is an N_M such that for every term with $n > N_M$, we have $a_n > M$.

The point is to force all the terms after a certain point to be larger than M. Since we can do this for any M – in particular, we can do it even if we keep making M larger and larger – the sequence is forced to go off to $+\infty$. How do we modify this definition to get diverging to $-\infty$?