Homework for Math 152H-1 September 1

I mentioned in class the prinicple of mathematical induction. To read more about it, see Appendix A1 in the book. The idea is the following: Suppose we have an infinite number of things we wish to prove: P_1, P_2, P_3, $\ldots$. For example, suppose we wanted to prove that

$$2^n \geq n^2 \text{ for } n \geq 4$$

Our P_i is then the inequality for $n = i$: $2^i \geq i^2$. Rather than verifying each inequality we do the following.

1. For $n = 4$ the inequality is true $16 = 2^4 \geq 4^2 = 16$.
2. If the inequality is true for n, then it is true for $n+1$: $(n+1)^2 = n^2 + 2n + 1 < n^2 + 3n$ since $1 < n$. Since $3 < n$ we also have $n^2 + 3n < n^2 + n \cdot n < 2n^2$. Hence $(n+1)^2 < 2n^2$. But $2(n^2) < 2 \cdot 2^n = 2^{n+1}$ because the inequality is true for n. Hence, $(n+1)^2 < 2^{n+1}$.

This proves it for all n. Once we know it's true for $n = 4$, we have seen that it must be true for $n = 5$. Then it must be true for $n = 6$, etc. The second step guarantees that this never fails.

Here's another example: Prove that $n^3 - n$ is always divisible by 3 for $n = 1, 2, 3, \ldots$. For $n = 1$, $1^3 - 1 = 0$, and 0 is divisible by every number. Now assume that the result is true for $n^3 - n$ and consider $(n+1)^3 - (n+1) = n^3 + 3n^2 + 3n + 1 - n - 1 = n^3 - n + 3n^2 + 3n$. Since 3 divides $n^3 - n$ and $3(n^2 + n)$, it must divide $(n+1)^3 - (n+1)$. We have shown that is is true for $n = 1$, and that if it is true for n then it is true for $n + 1$, so it must be true for all n.

Homework: Prove by mathematical induction:

1. $$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$
2. $$1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$
3. Since $\left| a + b \right| \leq \left| a \right| + \left| b \right|$, show that
$$\left| c_1 + c_2 + \ldots + c_n \right| \leq \left| c_1 \right| + \left| c_2 \right| + \ldots + \left| c_n \right|$$

A comment: I wasn't actually going to give any homework on the increasing, bounded above sequence property. However, I have included here another example which you can try to do, with some hints as to how to do it.

Problem: Let A and B be positive, real numbers. Prove the AM-GM inequality:

$$\frac{A + B}{2} \geq \sqrt{AB}$$

Hint: $(\sqrt{A} - \sqrt{B})^2 \geq 0$. When does it equal 0?

Challenge Problem: Consider the sequence from the last homework

$$b_n = \frac{1}{2}\left(b_{n-1} + \frac{1}{b_{n-1}}\right) \qquad b_1 = 2$$

Show that this is decreasing sequence that is bounded below, justifying our assumption that it has a limit. Here are the steps:

1. Show that if $b_{n-1} > 1$ then $b_n < b_{n-1}$.
2. Show that if $b_{n-1} > 1$ then $b_n > 1$. To do this use the AM-GM inequality from the previous problem.

Now what conclusion can you draw using induction and these two properties?

Another problem involving bounded, decreasing or increasing sequences: Suppose we have closed intervals in the number line, $I_j = [a_j, b_j]$ (the set of x with $a_j \leq x \leq b_j$), and we have one for each $j = 1, 2, 3, \ldots$. Further assume that $I_j \subset I_{j-1}$ (i.e. $a_{j-1} \leq a_j$ and $b_j \leq b_{j-1}$), and that $b_j - a_j \to 0$ as $j \to \infty$. Explain why there is precisely one and only one real number that is in *all* the intervals.