Homework for Math 152H-1 September 13

Reading: Read pgs 127- 130 in section 2.6 in the book. Some of the topics there will help on the homework, although you have enough to do them already. Here's a reminder:

Definition: A function $f(x)$ is continuous at a if

1. a is in the domain of $f(x)$
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$

A function is continuous if it is *continuous at each point in its domain.* If this is an interval $[a, b]$, it is continuous at a if the limit above can be replaced by $\lim_{x \to a^+} f(x) = f(a)$ and at b if $\lim_{x \to b^-} f(x) = f(b)$ (we use right and left limits because that's all that we have to work with).

Continuous functions have the following properties:

1. If $f(x)$ is continuous on $[a, b]$ then $f(x)$ attains a maximum and a minimum on this set. That is there there are numbers m and M such that $n \leq f(x) \leq M$ for all x in $[a, b]$ *and* there are points c, d in $[a, b]$ where $f(c) = m$ and $f(d) = M$.
2. If $f(x)$ is continuous on $[a, b]$ and C is a constant with $f(a) < C < f(b)$ then there is at least one point c where $f(c) = C$ (the intermediate value theorem).

Homework:

(1) On Monday I will choose some $\epsilon > 0$. Will you be able to find a δ so that $\left|\sqrt{x-3} - 2\right| < \epsilon$ when $\left|x - 6\right| < \delta$, regardless of which ϵ I choose? Why or why not? Will you be able to find $N > 0$ so that $\left|\frac{4x^2-2x}{x^2} - 4\right| < \epsilon$ when $x > N$?

(2) Explain why $f(x) = \left|x\right|$ is continuous on $\mathbb{R}$, but that $g(x) = \dfrac{\left|x\right|}{x}$ is not.

(3) It can be shown that $-\left|x\right| < sin\, x < \left|x\right|$ for all x and that $0 \leq 1 - cos\, x \leq \left|x\right|$ (these can be shown using right triangles and the definition of $sin\, x$ and $cos\, x$). What does this say about $\lim_{x \to 0} sin\, x$ and $\lim_{x \to 0} cos\, x$ (we're proving what you already know). Now consider the following:

$$\lim_{h \to 0} sin(x + h) = \lim_{h \to 0} \left(sin\, x\, cos\, h + cos\, x\, sin\, h\right) = sin\, x \cdot 1 + cos\, x \cdot 0 = sin\, x$$

where the first equality comes from an identity for $sin\, x$ (again, verified from the definitions of sine and cosine), the second comes from the limits you just verified and the limit laws. What does this tell you about the function $sin\, x$? (Think section 2.6)

(4) Find where the following functions are continuous (you may assume that polynomials, sine and cosine, and square roots, cube roots, etc are continuous. For more help look at example 8, pg 128-129):

$$y = \tfrac{1}{x+1} + 2x^2 \qquad\qquad y = \sqrt{3x - 1}$$

$$y = \sqrt{\tfrac{x-1}{x+2}} \qquad\qquad y = \tfrac{1}{1-\left|x\right|} + \sqrt{x}$$

$$y = csc\, x \qquad\qquad y = cos\left(\tfrac{\pi}{\sqrt{2-3x}}\right)$$

(5) If $f(x)$ and $g(x)$ are continuous at $x = 0$ show that $f(x)/g(x)$ need not be continuous at 0 (find a counter-example).

(6) If $h(x) = f(x) \cdot g(x)$ is continuous at $x = 0$ must *both* $f(x)$ and $g(x)$ be continuous at 0?

(7) There is a function which is continuous on $[-1, 1]$ but on $[-1, 0) \cup (0, 1]$ equals

$$\frac{\sqrt{4+x}-2}{x(x+2)}$$

What is the value of this function at 0? Can you find an expression for this function?

(8) Find the domain of

$$g(x) = \frac{sin(x^2 - x - 2)}{x^2 - 5x + 3}$$

Where is this function continuous? Can we extend the function, by defining it at points currently not in the domain, so that it is continuous at these points?

(9) What value of c will make the following function continuous on $\mathbb{R}$?

$$f(x) = \begin{cases} sin\, x + c & x \leq \frac{\pi}{2} \\ x^2 + 2\, x & x > \frac{\pi}{2} \end{cases}$$

(10) Explain why $f(x) = sin\, \frac{1}{x}$ cannot be defined at 0 so that the resulting function is continuous on $\mathbb{R}$. How should we define $g(x) = x^2 sin\, \frac{1}{x}$ at 0 so that it is continuous?.