Reading for Math 152H-1 September 13

There are a couple of topics which I have not given much time to, but which we should go through at least once. First, recall that our definition of $\lim_{x \to a} f(x)$ is the following:

Definition: $\lim_{x \to a} f(x) = L$ if an only if there is an open set $(c_1, a) \cup (a, c_2)$ such that for any sequence in this set, $a_n \to a$, we have $b_n = f(a_n) \to L$.

Here I wanted to record how this can be used. First, we immediately obtain that if $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$ then

1. $\lim_{x \to a} f(x) + g(x) = L + M$
2. $\lim_{x \to a} f(x)g(x) = L \cdot M$
3. $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M}$, when $M \neq 0$.
4. $\lim_{x \to a} C \cdot f(x) = C \cdot L$ for C a constant.
5. (squeeze theorem) If $M = L$ and $f(x) \leq h(x) \leq g(x)$ for x in $(c_1, a) \cup (a, c_2)$, then $\lim_{x \to a} h(x) = L$

These follow from the corresponding theorem for sequences in the following way. For any sequence $a_n \to a$, we consider the sequences $b_n = f(a_n)$ and $c_n = g(a_n)$. Then $b_n \to L$ and $c_n \to M$, so $b_n + c_n \to L + M$ by the properties of limits of sequences. But then for any sequence $a_n \to a$, we have $f(a_n) + g(a_n) \to L + M$. This is what we need to conclude the first property mentioned above. These rules are the rules we have been implicitly using to compute limits. Furthermore, they apply to right and left hand limits and limits as $x \to \infty$. In each case we need only consider a restricted set of sequences (those with a_n from (a, c_2) for $\lim_{x \to a^+}$, for example). Clearly the properties do not change.

These properties allow us to conclude that $\lim_{x \to a} P(x) = P(a)$ for any polynomial $P(x) = C_n x^n + C_{n-1} x^{n-1} + \cdots + C_0$. Why? We know that $\lim_{x \to a} x = a$, hence $\lim_{x \to a} x^n = \lim_{x \to a} x \cdot x \cdots \cdot x = a \cdot a \cdots \cdot a = a^n$. Thus $\lim_{x \to a} C_n x^n = C_n a^n$. Now we use the limit of a sum is the sum of the limits to conclude that $\lim_{x \to a} P(x) = P(a)$. We didn't need to use sequences or ϵ's to do this, once we know the rules!

Likewise we have: if $\lim_{x \to a} f(x) = L$ and $L > 0$ then $\lim_{x \to a} \sqrt{f(x)} = \sqrt{L}$. Using the above idea, let a_n be any sequence convergin to a and consider $b_n = f(a_n)$, a sequence converging to L. We have seen in the notes that $\sqrt{b_n} \to \sqrt{L}$ if $b_n \to L > 0$. Hence for any $a_n \to a$ $\sqrt{f(a_n)} \to \sqrt{L}$. This is what we needed to see that $\lim_{x \to a} \sqrt{f(x)} = \sqrt{L}$.

Second, let me comment upon the following problem from the last homework: suppose $\lim_{x \to 2} \dfrac{f(x) - 5}{x - 2} = C$; what is $\lim_{x \to 2} f(x)$? What is this statement actually telling you? Consider the limit

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

What this is really saying is that very, very close to 0, $\sin x$ acts like x. In other words we can approximate $\sin x$ by x for x very, very close to zero. This clearly cannot be true away from 0 since $\sin x$ is always between -1 and 1 and x can get extremely big. Nevertheless, according to my calculator $\sin(0.00234) = 0.00233999...$ (you must use radians for this!) which is very close to 0.00234. Sometimes we write $\sin x \sim x$ as $x \to 0$. As another example,

$$\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2} \Rightarrow \cos x \sim 1 - \frac{1}{2} x^2$$

for $x \to 0$. If you graph $\cos x$ and $1 - \frac{1}{2} x^2$ you will see that near 0, they are very close indeed. However, we need to answer

> How close must x be to a for my approximation to be a good one, i.e. not to far away from the actual values of the function?