**Homework and Pre-Class reading for Math 152H-1 September 11**

Here are some comments on how to calculate limits of functions that will be helpful for the homework above. First I review the example from Friday's reading:

$$\lim_{x\to 1} \frac{x^2-1}{x-1}$$

At 1 both the numerator and the denominator are 0. If we had to use what happened at 1, we'd have no idea what to do. Is it 0 because the numerator is 0, or undefined since the denominator is 0. In fact "$\frac{0}{0}$" is meaningless; when we see that it tells us that we are doing something wrong. So plugging in will not always work. So how do things differently? We use algebra: combine terms that are not yet combined, factor, rationalize denominators, etc. Manipulate the expression until you can get something better. In this case, we can factor using the expression $a^2-b^2=(a-b)(a+b)$:

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{(x-1)(x+1)}{x-1} = \lim_{x\to 1} x+1 = 2$$

The point is that since we don't care about what happens at 1, the cancellation of the $x-1$ terms is fine. Away from $x=1$, these terms are non-zero numbers. Surely though, we have done something by such a drastic change in the function. In fact, we have changed the domain of the function. $\frac{x^2-1}{x-1}$ has domain $(-\infty,1)\cup(1,\infty)$, since the expression makes no sense when we try to plug in $x=1$. When we cancel, we find a function $x+1$ which gives the same values on $(-\infty,1)\cup(1,\infty)$, but has domain $\mathbb{R}$, and thus tells us what to do at $x=1$. How do we know when we need to do this sort of algebra? Here are some hints, but anytime you're stuck it's a good thing to try.

> Indicators that you are doing something wrong, and need to do more work: "$\frac{0}{0}$", "$\frac{\infty}{\infty}$", "$\infty-\infty$", $0^0$. Basically expressions which don't make sense.

Now consider the following example:

$$\lim_{x\to ?} \frac{(x-1)(x-2)}{x^2(x-2)(x-1)}$$

where we will put in different values for ?. First, the domain of the expression is $(-\infty,0)\cup(0,1)\cup(1,2)\cup(2,\infty)$ since we cannot have a 0 in the denominator. If we replace ? by 2, then we can use the trick of extending the domain by cancelling the $(x-2)$ terms. The limit is then:

$$\lim_{x\to 2} \frac{(x+1)(x-2)}{x^2(x-2)(x-1)} = \lim_{x\to 2} \frac{(x+1)}{x^2(x-1)} = \frac{3}{4}$$

What happens as $x\to 1$? The numerator approaches a finite value $(-2)$, but the denominator goes to 0. So as we get closer to 1, we get a finite number over something really close to 0, and this is really large or really negative. In fact, if we get closer to 1 using numbers larger than 1 then $x-1$ is positive, and the limit goes to $+\infty$ (since the rest of the expression will be positive as well). But if we approach using numbers less than 1, $x-1$ is negative and the limit goes to $-\infty$. Since we get different answers, there is no limit. A limit where we restrict $x$ to be smaller than $a$ is denoted by $\lim_{x\to a^-}$ whereas a limit where we restrict $x$ to be larger than $a$ is denoted by $\lim_{x\to a^+}$. We have found that

$$\lim_{x\to 1^-} \frac{(x+1)(x-2)}{x^2(x-2)(x-1)} = -\infty \qquad \lim_{x\to 1^+} \frac{(x+1)(x-2)}{x^2(x-2)(x-1)} = +\infty$$

What happens as $x\to 0$? Again, the expression is going to $\pm\infty$, but now when we come from the right (numbers greater than 0) and when we come from the left (numbers less than 0), $x^2$ will be positive, and we can say that

$$\lim_{x\to 0} \frac{(x+1)(x-2)}{x^2(x-2)(x-1)} = -\infty$$

It is $-\infty$ because although $x^2$ is always positive near 0, the rest of the expression is negative.

What happens as $x\to -1$? This is straightforward. The numerator gets closer to zero, and the denominator gets closer to $(-1)^2\cdot -3\cdot -2 = 6$, so the fraction is going to 0. This is very nice, and it seems to be just plugging $-1$ into the expression. So one question that we will want to answer is

When can we evaluate a limit simply by plugging $a$ into $f(x)$?

$a$ must be in the domain of $f(x)$, but this is **not** enough. To see this consider

$$f(x) = \begin{cases} \frac{x^2-1}{x-1} & x \neq 1 \\ \sqrt{2} & x = 1 \end{cases}$$

The domain of this function is $\mathbb{R}$ since we now know how to evaluate $f(1) = \sqrt{2}$, but away from 1, the function is $\frac{x^2-1}{x-1}$. We have already seen that as $x \to 1$, this expression tends to 2. Even with the additional value at 1, the limit as $x \to 1$ is still 2.

When trying to calculate

$$\lim_{x\to 2} \frac{x-2}{\sqrt{x+2}-2}$$

we see that both numerator and denominator goes to 0. So we are in one of those bad cases. To calculate this we can rationalize the denominator:

$$\lim_{x\to 2} \frac{x-2}{\sqrt{x+2}-2} = \lim_{x\to 2} \frac{(x-2)(\sqrt{x+2}+2)}{(\sqrt{x+2}-2)(\sqrt{x+2}+2)} = \lim_{x\to 2}(\sqrt{x+2}+2) = 4$$

When the square root Is in the numerator, we can still try this. It will put a square root in the denominator, but if that helps with evaluating the limit, we won't care. It's often helpful to simplify your expressions. Don't be afraid to do algebra! Your efforts will often be wasted without it.

There are many limits where it is not obvious how to proceed:

$$\lim_{x\to 0} \frac{sin\ x}{x} \qquad \lim_{x\to 0} \frac{1-cos\ x}{x^2}$$

Both of these give $\frac{0}{0}$, but how should we go about manipulating them?

Finally, since the value at $a$ does not matter – $a$ doesn't have to be in the domain – we can consider what happens as $x \to \infty$ (or as $x \to -\infty$). Again we would like the expression to go to a single, finite value. Now we require that all the points in some interval $(c, \infty)$ or $(-\infty, c)$ are in the domain of $f$. To evaluate these limits is a lot like evaluating the limit of a sequence:

$$\lim_{x\to\infty} \frac{1}{x} = 0$$

$$\lim_{x\to\infty} \frac{3x^2+2}{2x^2+x} = \lim_{x\to\infty} \frac{3+\frac{2}{x^2}}{2+\frac{1}{x}} = \frac{3}{2}$$

$$\lim_{x\to\infty} \frac{3\sqrt{x}+2}{1+2x} = \lim_{x\to\infty} \frac{3+\frac{2}{\sqrt{x}}}{\frac{1}{\sqrt{x}}+2\sqrt{x}} = \frac{3}{1+\lim\limits_{x\to\infty} 2\sqrt{x}} = 0$$

$$\lim_{x\to\infty} cos\ x = d.n.e.$$

where "d.n.e" means does not exist. This last limit does not exist because $cos\ x$ just keeps oscillating back and forth between $-1$ and 1 as $x$ gets bigger and bigger.

See sections 2.2, 2.4, 2.5 for more on computing limits.