# Review List Solutions (not done in class)

3) If $\sum_{i=1}^{\infty} a_i$ converges and $\sum_{i=1}^{\infty} b_i$ diverges, does $\sum_{i=1}^{\infty} \big(2\,a_i - b_i\big)$ converge or diverge?

**Solution:** Intuition suggests that this should diverge. At least, if $\sum b_i$ diverges because $b_i \to 2$ (as an example) then there's now way the sum we're interested can converge. How to prove this? Note the appearance of $b_i$ in each term of the new series. We will use theorem 2.7.1 to obtain $\sum b_i$ from the new series. Note that theorem 2.7.1 can only be applied to convergent series. So we assume that $S = \sum(2a_i - b_i)$ converges, and we know $A = \sum a_i$ converges. By the theorem $\sum -2a_i$ also converges, so $\sum(2a_i - b_i) - \sum 2a_i = \sum -b_i$ must converge to $S - 2A$. But then $\sum b_i$ would converge to $2A - S$, which contradicts that $\sum b_i$ diverges. Therefore, $\sum(2a_i - b_i)$ must diverge.

8) Let $a_n$ be a monotone increasing sequence. When is $\{a_n\}$ a closed subset of $\mathbb{R}$?

**Solution:** There are two possibilities for a montone increasing sequence: 1) it is bounded above and therefore has a limit $L$, or 2) it is unbounded. Let's consider these two seperately. If $a_i \to L$ then for $\{a_n\}$ to be closed we must have $L = a_N$ for some $L \in \mathbb{R}$ and some $N \in \mathbb{N}$. Otherwise, $\{a_n\}$ would have a limit point (namely $L$) not contained in the set. Since the sequence is increasing we would need $a_m = L$ for all $m \geq N$ as well. The set is thus $\{a_1, \ldots, a_{n-1}, L\}$, with each point isolated. The sequence is $a_1, a_2, \ldots, a_{N-1}, L, L, \ldots, L, \ldots$. If the sequence is unbounded, then the sequence has no limit (convergent sequences are bounded, see section 2.3). Let $x \in \mathbb{R}\backslash\{a_i\}$. Since $(a_i)$ is unbounded, either $x < a_0$ or there is an $N$ such that $a_{N-1} < x < a_N$. Thus letting $\epsilon$ be half the distance to the nearest element in $\{a_i\}$ results in a non-zero number (it is here that the argument breaks down for convergent monotone increasing seqeunces). But then $V_\epsilon(x) \subset \mathbb{R}\backslash\{a_i\}$ so $\mathbb{R}\backslash\{a_i\}$ is open, and thus $\{a_i\}$ is closed. Thus all unbounded monotone increasing sequences result in a closed subset.

9) Let $A \subset B \subset \mathbb{R}$. Show that $\overline{A} \subset \overline{B}$.

**Solution:** $\overline{A} = A \cup L_A$, and since $A \subset B$ if $L_A \subset L_B$ then $\overline{A} \subset \overline{B}$. Let $x \in L_A$. Then there is a sequence $a_i \to x$ with $a_i \in A$ and $a_i \neq x$. But $A \subset B$ implies $a_i \in B$ for all $i \in \mathbb{N}$. Thus there is a sequence $b_i = a_i$ with $b_i \neq x$ and $b_i \to x$. Hence $x \in L_B$.

10) Call $S \subset \mathbb{R}$ complete if every $(s_i) \subset S$ that is a Cauchy sequence in $\mathbb{R}$, has its limit in $S$. What subsets of $\mathbb{R}$ are complete?

**Solution:** Convergent sequences and Cauchy sequences are the same in $\mathbb{R}$. So we are asking for those subsets of $\mathbb{R}$ for which any sequence drawn from $S$ that we know converges in $\mathbb{R}$ has its limit in $S$. This is one of the characterizations of closed subsets of $\mathbb{R}$ given in class.