Review for Math 152H-1 Test #1

Review Problem Answers and Solutions:

1. Calculate the following limits:

(1) $\lim_{x\to 2} \dfrac{x^2-2x}{x(\sqrt{x+2}-2)}$ (2) $\lim_{x\to 1^-} \dfrac{x^2-3x+2}{x^2-2x+1}$

(3) $\lim_{x\to\infty} \dfrac{2\sqrt{x}+\frac{1}{x}}{3+\sqrt{x}}$ (4) $\lim_{x\to 2} \dfrac{sin(\pi\, x^2-2\pi\, x)}{x^2-4}$

(5) $\lim_{x\to 0^+} \dfrac{1-cos\, x}{x^{\frac{3}{2}}}$ (6) $\lim_{h\to 0} \dfrac{(16+h)^{\frac{1}{4}}-2}{h}$

Answers: (1) 4, (2) ∞, (3) 2, (4) $\frac{\pi}{2}$, (5) 0, replace $cos\, x$ using $cos\, x = 1-2\,sin^2(\frac{x}{2})$, (6) $\frac{1}{4}(16)^{-\frac{3}{4}} = \frac{1}{32}$, write down the limit definition of the derivative for $f(x)=x^{\frac{1}{4}}$ at $x=16$ (using the version with “h”) and compare to the above.

2. Compute the following derivatives in any manner you prefer:

(1) $\dfrac{d}{dx}\left(10\,x^{-2}+\dfrac{2}{\sqrt{x}}-4\right)$ (2) $\dfrac{d^2}{dx^2}\, tan\, 2x$

(3) $\dfrac{d}{dx}\left(\sqrt{1+2x}-1\right)^5$ (4) $\dfrac{d}{dx}\sqrt{(1+x^2)(sin\, 2x\, cos\, x)}$

(1) $-20x^{-3}-x^{-\frac{3}{2}}$, (2) $8\,sec^2(2x)tan(2x)$, (3) $5(\sqrt{1+2x}-1)^4\frac{1}{\sqrt{1+2x}}$, (4) $\frac{1}{2\sqrt{(1+x^2)(sin\,2x\,cos\,x)}}\big(2x\,sin\,2x\,cos\,x+(1+x^2)(2cos\,2x\,cos\,x-sin\,2x\,sin\,x)\big)$

3. *Use the definition of the derivative* to compute the derivative of $y=\frac{1}{x^2+2\,x}+3$ (for this it's best not to use the version with an h). Find the equation of the tangent line for this function at $x=1$.

We calculate

$$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=\lim_{x\to a}\frac{\frac{1}{x^2+2\,x}+3-\frac{1}{a^2+2\,a}-3}{x-a}=\lim_{x\to a}\frac{\frac{a^2-x^2+2a-2x}{(x^2+2\,x)(a^2+2\,a)}}{x-a}$$
$$=\lim_{x\to a}\frac{(a-x)(a+x+2)}{(x^2+2\,x)(a^2+2\,a)(x-a)}=\lim_{x\to a}-\frac{(a+x+2)}{(x^2+2\,x)(a^2+2\,a)}=-\frac{2a+2}{(a^2+2a)^2}$$

At $a=1$ we have $f'(1)=-\frac{4}{3^2}=-\frac{4}{9}$. Furthermore $f(1)=\frac{10}{3}$ so the equation of the tangent line is $y=f'(a)(x-a)+f(a)$ or $y=-\frac{4}{9}(x-1)+\frac{10}{3}$.

4. Where is

$$y=3+\frac{x^2+2\,x-3}{(x+1)(x^2-1)}$$

continuous? Can we extend this function to be continuous at any point where it is currently not continous. What does its graph look near $x=-3$? what happens near $x=+1$? near $x=1$? Does the function have any horizontal asymptotes? If so what are their equations?

The numerator factors as $(x+3)(x-1)$ and the denominator factors as $(x+1)^2(x-1)$. So our function is

$$y = 3 + \frac{(x+3)(x-1)}{(x+1)^2(x-1)} = 3 + \frac{x+3}{(x+1)^2}$$

when $x \neq 1$. So the domain of the function is $(-\infty,-1) \cup (-1,1) \cup (1,\infty)$. It is also continuous there since polynomials are continuous and the quotient of continuous functions is continuous away from where the denominator is zero. That the $x-1$ terms cancel tell us that we can remove the singularity at 1, to the function on the right, which is continuous on $(-\infty,-1) \cup (-1,\infty)$. To find how to redefince the function to be continuous at 1 we compute:

$$\lim_{x\to 1} 3 + \frac{x^2+2x-3}{(x+1)(x^2-1)} = \lim_{x\to 1} 3 + \frac{x+3}{(x+1)^2} = 3 + \frac{4}{2^2} = 4$$

Now the limit exists at 1, we define y for $x=1$ to be 4, so the limit and the function value are equal. Hence the function is continuous. At $x=-3$ we see that $y=3$. For $x=+1$, we have seen that there is a removable singularity, so the graph of the original function has a point missing. For $x=-1$ there is a vertical asymptote. Indeed, since $\lim\limits_{x\to -1^+} = \lim\limits_{x\to -1^-} = \infty$, both the graph on either side of -1 tends to positive infinity. Finally, we can find horizontal asymptotes by calculating $\lim\limits_{x\to\pm\infty} y$. If we divide top and bottom of the fraction by x^3 we'll see that these limits of the fraction are both 0. Thus $\lim\limits_{x\to\pm\infty} y = 3$, so $y=3$ will be a horizontal asymptote as either $x\to\infty$ or $x\to-\infty$.

5. You are given that $\lim\limits_{x\to 0^+} f(x) = 3$ and $\lim\limits_{x\to 0^-} f(x) = 1$, what is

$$\lim_{x\to 0^+} f(x^3-x) \qquad \lim_{x\to 0^-} f(x^3-x)$$

$$\lim_{x\to 0} f(x^3-x) \qquad \lim_{x\to 0^-} f(x^2-x^4)$$

As $x \to 0^+$ $x^3 - x = x(x-1)(x+1)$ tends to 0 from the *left* since $x(x-1)(x+1)$ is negative for $x \in (0,1)$. So the answer to the first is 1. The answer to the secon dis 3. The third does not exist. For the fourth $x^2 - x^4 = x^2(1-x)(1+x)$ which is positive both as $x \to 0^+$ or as $x \to 0^-$, thus $\lim\limits_{x\to 0} f(x^2-x^4) = 3$ (so the limit from either side will as well).

6. If $\lim\limits_{x\to 2} \frac{3x+1}{\sqrt{g(x)}} = 3$ what can you say about $\lim\limits_{x\to 2} g(x)$? Suppose $\lim\limits_{x\to 2} \frac{x^2-4}{g(x)-3} = 5$, what can you say about $\lim\limits_{x\to 2} g(x)$?

   For the first $\lim\limits_{x\to 2} g(x) = (\frac{7}{3})^2$ when it exists. For the second $\lim\limits_{x\to 2} g(x) = 3$. In either case can the limit not exist?

7. Given $\epsilon > 0$ find δ so that $\left|\sqrt{x+3} - 2\right| < \epsilon$ when $0 < |x-1| < \delta$. For each $\epsilon > 0$ is there always a $\delta > 0$ so that $\left|\sqrt{x+3} - 3\right| < \epsilon$ when $0 < x < \delta$?

   We solve $-\epsilon < \sqrt{x+3} - 2 < \epsilon \Leftrightarrow -\epsilon + 2 < \sqrt{x+3} < \epsilon + 2$. Squaring both sides, and noting that for $\epsilon < 1$ $-\epsilon + 2 > 0$, we have $\epsilon^2 - 4\epsilon + 4 < x + 3 < \epsilon^2 + 4\epsilon + 4$. Subtracting 4 gives $\epsilon^2 - 4\epsilon < x - 1 < \epsilon^2 + 4\epsilon$. Now, since $\epsilon > 0$ we have $4\epsilon - \epsilon^2 < \epsilon^2 + 4\epsilon$, so when $\epsilon^2 - 4\epsilon < x - 1 < -\epsilon^2 + 4\epsilon$ the previous inequality is also true, and by working backwards we will get what we want. So we could choose $\delta = 4\epsilon - \epsilon^2$. Note that this last step is forced on us to find a single δ which works for $0 < |x-1| < \delta$. The idea of a limit doesn't require the δ to be the same on the left and right, but that's the definition we're using so we have to conform to it.

8. Is $y = \sqrt{(x-1)^2}$ differentiable at 1? What happens to the graph at $x=1$? What is the equation of the tangent line to this function at $x=2$? Why is

$$y = y = \begin{cases} x^2 & x \leq 0 \\ x+3 & x > 0 \end{cases}$$

not differentiable at 0? What does its graph look like? Why is $y = x^{\frac{1}{5}}$ not differentiable at 0, what does its graph look like?

First, $y = \sqrt{(x-1)^2} = |x-1|$. It is not differentiable at 1 because when $x > 1$ it has slope $+1$ and when $x < 1$ it has slope -1. There is a "corner" at 1. Near 2 however, the function agrees with $y = x - 1$. As a line is its own tangent, this is also the tangent line. The second function is not differentiable at 0 because it is not continuous there. The last function has derivative $\frac{1}{5}x^{-\frac{4}{5}}$. It thus has an infinite slope tangent line at 0.

9. Suppose

$$f(x) = \begin{cases} g(x) & x \leq 1 \\ x^2 + a\,x + b & x \geq 1 \end{cases}$$

where $g(x)$ is differentiable on $\mathbb{R}$ with $g(1) = 3$ and $g'(1) = 5$. What are the values of a and b for which $f(x)$ is both continuous and differentiable for $x \in \mathbb{R}$? If $g''(x)$ exists everywhere, and $g''(1) = 3$, is $f''(x)$ defined at 1?

For the function to be continuous we need $1 + a + b = 3$. This follows from $g(x)$ and $x^2 + ax + b$ both being continuous on $\mathbb{R}$, so all we need to do is match the right and left hand limits, which will be $g(1)$ and $1 + a + b$ (since the functions are continuous!). For the function to be differentiable, we must have a unique slope at 1, thus $5 = 2 + a$. Note that the first slope is given to you, while the second follows from the fact that $x^2 + a\,x + b$ is continuously differentiable. Thus $a = 3$ and $1 + 3 + b = 3$ implies $b = -1$. Howevere, not matter how we try we cannot make the second derivatives match up, so $f''(x)$ is not defined at 1 (only the x^2 term contributes to the second derivative). If we had the freedom to consider $c\,x^2 + a\,x + b$, then it can be done.

10. Show that

$$y = \begin{cases} x^3 cos(\frac{1}{x^2}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is differentiable at 0. Is it continuously differentiable at 0? Same questions for

$$y = \begin{cases} x^3 cos(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

For the first, we compute

$$\lim_{h \to 0} \frac{h^3 cos(\frac{1}{h^2}) - 0}{h} = \lim_{h \to 0} h^2 cos(\frac{1}{h^2}) = 0$$

where the last comes from the squeeze theorem. TO see if it is continuously differentiable, we use the rules for derivatives to computs $f'(x) = 3x^2 cos(\frac{1}{x^2}) - 2\,sin(\frac{1}{x^2})$ when $x \neq 0$. For $x \neq 0$ all the terms in this expression are continuous, hence it is continuously differentiable away from 0. At 0 $\lim_{x \to 0} f'(x)$ does not exist because of the $sin(\frac{1}{x^2})$ term. Hence it is not continuously differentiable. For the second function, we proceed as before, and discover that again the derivative is 0 at $x = 0$. However, when $x \neq 0$ we have $f'(x) = 3x^2 cos(\frac{1}{x}) - x\,sin(\frac{1}{x})$. And now we can use the squeeze theorem to show that each of these terms has limit equal to 0 at $x = 0$. Hence it is continuously differentiable. (For $x\,sin(\frac{1}{x})$ we use $-|x| \leq x\,sin(\frac{1}{x}) \leq |x|$ for the squeeze. We need the absolute values to ensure that for x negative, the inequalities are still true.)

11. **A challenge problem:** For each $\epsilon > 0$ is it possible to find a $\delta > 0$ so that $\big|\big((1+x)^\pi - 4^\pi\big) - 5(x-3)\big| < \epsilon\big|x - 3\big|$ when $0 < \big|x - 3\big| < \delta$? (**Hint:** Can you relate this to derivatives?)

Re-write as

$$\Big|\frac{\big((1+x)^\pi - (1+3)^\pi\big)}{x-3} - 5\Big| < \epsilon$$

when $0 < |x - 3| < \delta$. This is asking whether $\lim_{x \to 3} \frac{\big((1+x)^\pi - (1+3)^\pi\big)}{x-3} = 5$. Now we recognize the limit as the calculation of the derivative of $f(x) = (1+x)^\pi$ at $x = 3$. That derivative is $\pi(1+x)^{\pi-1}$ at 3 or $\pi \cdot 4^{\pi-1}$. This turns out to be bigger than 60.