Review for Math 152H-1 Test #1

Material: You will be expected to know the following material. Where it says "you need to know the $\cdots$ theorem or definition" it means you need to know the statement as well as how to use it.

1. **Limits:** The test will only consider limits of functions, not sequences!

   (a) You will need to know how to compute $\lim_{x\to a} f(x)$, $\lim_{x\to a^{\pm}} f(x)$, and $\lim_{x\to\pm\infty} f(x)$ for functions similar to those in the homework.

   (b) You will need to know the definition of a limit and how it changes to include right and left hand limits, and limits at infinity. You may be asked to find δ so that ... or you may be asked to recognize that some statement is equivalent to the existence of a limit.

   (c) You should know what it means for $\lim_{x\to a} f(x) = \infty$ or $-\infty$ and how that is can be expressed by a definition.

   (d) You should be able to give examples of ways in which $f(x)$ can fail to have a limit at $x = a$. You should know that the value of $f(a)$ does not make a difference to $\lim_{x\to a} f(x)$ and that $f(x)$ need not be defined at a for the limit to exist. You should know that you can use sequences or left and right hand limits to show that a limit does not exist.

   (e) You should know what three conditions are required for a function to be continuous at a point and how to find the sets upon which a given function is continuous. You should know when you can extend a function to be continuous at points where it is not defined or re-define a function to make it continuous (these are called removable singularities). You should know when this process fails. You should know that this is related to the limit definition of a derivative.

   (f) You should know the intermediate value theorem and how to use it to show that an equation has at least one solution.

   (g) You should know the following limits:

   $$\lim_{x\to 0} \frac{sin\,x}{x^n} \qquad\qquad \lim_{x\to 0} \frac{1-cos\,x}{x^n}$$

   (this last limit can be computed using $cos\,x = 1 - 2\,sin^2\frac{x}{2}$).

   (h) You should know the squeeze theorem.

   (i) You should be able to use limits to find vertical and horizontal asymptotes (section 2.4, 2.5) and describe the behavior of $f(x)$ near a vertical asymptote in terms of limits.

2. **Derivatives:**

   (a) You will need to know that a function $f(x)$ is **defined** to be differentiable at a point a in its domain when

   $$\lim_{x\to a} \frac{f(x)-f(a)}{x-a} \qquad\qquad \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

   exist. You should understand why these two limits are equivalent. You should know what it means to be right and left differentiable (section 3.1) and that a function is differentiable if and only if it is *both* right and left differentiable and the right and left derivatives are equal. Since this is a limit, all that you know about limits applies here.

   (b) You should know that the expression inside the second limit above is the average rate of change for $f(x)$ over $[a, a+h]$ (section 2.7), and that the limit gives the instantaneous rate of change for the function

   (c) Graphically, $f'(a)$ is the slope of the tangent line. The average rate of change is the slope of the line through $(a, f(a))$ and $(a+h, f(a+h))$ You should be able to find the *equation* of the tangent line to $f(x)$ at a.

   (d) You should be able to give examples of how a function fails to be differentiable at a point (section 3.1). You should know what it means for the left and right hand derivatives to exist but differ in value.

(e) *You should be able to calculate $f'(x)$, the derivative function, using the definition above or by using the rules given in class (sections 3.2, 3.4, and the first part of 3.5).*

(f) You should know that $f'(x)$ existing at a implies that $f(x)$ is continuous at a, but that the reverse is not true (think $y = |x|$ at 0). Furthermore, $f'(x)$ can exist everywhere, but that does not imply that $f'(x)$ is continuous.

This is a summary, and may not include everything in the homework. Except where explicitly stated otherwise, you will also need to know what we have covered in the homework.

Review Problems: Please note: these are meant to be challenging!

1. Calculate the following limits:

(1) $\lim\limits_{x\to 2} \dfrac{x^2 - 2x}{x(\sqrt{x+2} - 2)}$ (2) $\lim\limits_{x\to 1^-} \dfrac{x^2 - 3x + 2}{x^2 - 2x + 1}$

(3) $\lim\limits_{x\to\infty} \dfrac{2\sqrt{x} + \frac{1}{x}}{3 + \sqrt{x}}$ (4) $\lim\limits_{x\to 2} \dfrac{sin(\pi\, x^2 - 2\pi\, x)}{x^2 - 4}$

(5) $\lim\limits_{x\to 0^+} \dfrac{1 - cos\, x}{x^{\frac{3}{2}}}$ (6) $\lim\limits_{h\to 0} \dfrac{(16+h)^{\frac{1}{4}} - 2}{h}$

2. Compute the following derivatives in any manner you prefer:

(1) $\dfrac{d}{dx}\left(10\, x^{-2} + \dfrac{2}{\sqrt{x}} - 4\right)$ (2) $\dfrac{d^2}{dx^2}\, tan\, 2x$

(3) $\dfrac{d}{dx}\left(\sqrt{1+2x} - 1\right)^5$ (4) $\dfrac{d}{dx}\sqrt{(1+x^2)(sin\, 2x\, cos\, x)}$

3. *Use the definition of the derivative* to compute the derivative of $y = \frac{1}{x^2 + 2\,x} + 3$ (for this it's best not to use the version with an h). Find the equation of the tangent line for this function at $x = 1$.

4. Where is
$$y = 3 + \frac{x^2 + 2\,x - 3}{(x+1)(x^2 - 1)}$$
continuous? Can we extend this function to be continuous at any point where it is currently not continous. What does its graph look near $x = -3$? what happens near $x = +1$? near $x = 1$? Does the function have any horizontal asymptotes? If so what are their equations?

5. You are given that $\lim\limits_{x\to 0^+} f(x) = 3$ and $\lim\limits_{x\to 0^-} f(x) = 1$, what is

$\lim\limits_{x\to 0^+} f(x^3 - x)$ $\lim\limits_{x\to 0^-} f(x^3 - x)$

$\lim\limits_{x\to 0} f(x^3 - x)$ $\lim\limits_{x\to 0^-} f(x^2 - x^4)$

6. If $\lim\limits_{x\to 2} \frac{3x+1}{\sqrt{g(x)}} = 3$ what can you say about $\lim\limits_{x\to 2} g(x)$? Suppose $\lim\limits_{x\to 2} \frac{x^2-4}{g(x)-3} = 5$, what can you say about $\lim\limits_{x\to 2} g(x)$?

7. Given $\epsilon > 0$ find δ so that $\left|\sqrt{x+3} - 2\right| < \epsilon$ when $0 < \left|x - 1\right| < \delta$. For each $\epsilon > 0$ is there always a $\delta > 0$ so that $\left|\sqrt{x+3} - 3\right| < \epsilon$ when $0 < x < \delta$?

8. Is $y = \sqrt{(x-1)^2}$ differentiable at 1? What happens to the graph at $x = 1$? What is the equation of the tangent line to this function at $x = 2$? Why is
$$y = y = \begin{cases} x^2 & x \leq 0 \\ x+3 & x > 0 \end{cases}$$
not differentiable at 0? What does its graph look like? Why is $y = x^{\frac{1}{5}}$ not differentiable at 0, what does its graph look like?

9. Suppose
$$f(x) = \begin{cases} g(x) & x \leq 1 \\ x^2 + a\,x + b & x \geq 1 \end{cases}$$
where $g(x)$ is differentiable on $\mathbb{R}$ with $g(1) = 3$ and $g'(1) = 5$. What are the values of a and b for which $f(x)$ is both continuous and differentiable for $x \in \mathbb{R}$? If $g''(x)$ exists everywhere, and $g''(1) = 3$, is $f''(x)$ defined at 1?

10. Show that
$$y = \begin{cases} x^3 cos(\frac{1}{x^2}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$
is differentiable at 0. Is it continuously differentiable at 0? Same questions for
$$y = \begin{cases} x^3 cos(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

11. **A challenge problem:** For each $\epsilon > 0$ is it possible to find a $\delta > 0$ so that $\left|\left((1+x)^\pi - 4^\pi\right) - 5(x-3)\right| < \epsilon\left|x-3\right|$ when $0 < \left|x-3\right| < \delta$? (**Hint:** Can you relate this to derivatives?)