RESEARCH SUMMARY FOR LAWRENCE ROBERTS

I am interested in the following subjects:

(1) **Heegaard-Floer Homology:** Especially the knot Floer homology of fibered knots in three manifolds, the Heegaard-Floer homology of fibered three manifolds, and calculations for open book decompositions of three manifolds.
(2) **Lefschetz Fibrations:** In particular, calculating Heegaard-Floer invariants of four dimensional cobordisms equipped with a Lefschetz fibration or broken Lefschetz fibration.
(3) **Khovanov Homology:** In particular, as a means to approaching the first two items on this list.
(4) **Smooth Four Manifolds:** More or less generally.

Background: Synopsis of the work of P. Ozsváth and Z. Szabó

In defining Heegaard-Floer homology, P. Ozsváth and Z. Szabó construct chain complexes for a closed, oriented three manifold, Y, and chain maps for each oriented four dimensional cobordism with two ends. When taking homology this yields a map $F_W : HF_*(Y_1) \to HF_*(Y_2)$ for each W^4 with $\partial W^4 = -Y_1 \cup Y_2$. The homology comes in four different but related versions, with $\widehat{HF}$ and HF^+ being the most studied. In fact, the chain complexes and maps decompose using the additional data of $Spin^c$ structures on the various manifolds. The cobordism maps yield invariants of closed, oriented, smooth four manifolds equipped with $Spin^c$ structures by considering certain decompositions of the cobordism between three-spheres found by removing two four balls from the closed four manifold. This construction was inspired by Seiberg-Witten theory, which also provides invariants of smooth, oriented four-manifolds, and is conjectured to yield to same invariants. For more information see [6], [7], [9].

There is a further refinement of Heegaard-Floer homology. Let $K \subset Y$ be a null-homologous knot, bounding an embedded surface Σ. K then induces a $\mathbb{Z}$-filtration on the chain complex $\widehat{CF}(Y)$. The homology of the graded module for this filtration, which is an invariant of the isotopy class of the knot, is called the knot Floer homology of (Y, K), and is a direct sum $\oplus_{i\in\mathbb{Z}}\widehat{HFK}(Y, K; i)$, where i is the filtration index, as determined by Σ. The spectral sequence induced from this filtration converges to $gr\,\widehat{HF}(Y)$ in finitely many steps. Furthermore, when $Y = S^3$, taking the Euler characteristic of $\widehat{HFK}(S^3, K; i)$ yields the coefficient of t^i in the Alexander polynomial of K, a classical invariant of the knot. There is a generalization of this construction to a link in S^3 and its multi-variable Alexander polynomial. For more information see [5].

Past research: summaries of recent papers

Heegaard-Floer homology and string links. arKiv:math.GT/0607244, 57 pgs. submitted for publication

Definition. *Choose k points $p_1, \ldots, p_k$ in D^2. A k-stranded "string link" in $D^2 \times I$ is a proper embedding, $\coprod_{i=1}^k f_i$ of $\coprod_{i=1}^k I_i$ into $D^2 \times I$, where $I_i = [0,1]$ and $f_i : I_i \to D^2 \times I$, such that $f_i(0) = p_j \times 0$ and $f_i(1) = p_s \times 1$. We call the string link "pure" if $j = s$ for each interval. These are considered up to free isotopy, i.e. those isotopies allowing the points on $D^2 \times \{0,1\}$ to move on each end of $D^2 \times I$*

In this paper, I prove:

Theorem. *For each string link S with k strands and each $\overline{v} \in \mathbb{Z}^k$, there is a group $\widehat{HF}(S; \overline{v}, \mathbb{Q})$ invariant under free isotopies of S. These groups have the property that*

$$\sum_{\overline{v} \in \mathbb{Z}^k} \chi(\widehat{HF}(S; \overline{v}, \mathbb{Q})) t_1^{v_1} \cdots t_k^{v_k} = \tau(S)$$

where $\tau(S)$ is the analog of a link's Alexander polynomial, found from the universal abelian cover of $(D^2 \times I) - S$. In addition I describe the behavior of this invariant under compositions of string links, which generalizes the composition of braids, and calculate it for alternating string links. Furthermore, I extend the theory to d-based links in arbitrary three manifolds.

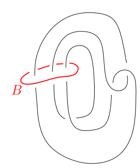
Rational blow downs in Heegaard-Floer homology arKiv:math.GT/0607675, 35 pgs., *Communications in Contemporary Mathematics*. Vol 10, No. 4, August 2008. pgs. 491-522

P. Ozsváth and Z. Szabó have define an diffeomorphism invariant $\Phi(X, \mathfrak{s})$ of a closed, oriented four manifold, X, equipped with a $Spin^c$ structure, $\mathfrak{s}$, which is supposed to recover the Seiberg-Witten invariants, [9]. It was known from Seiberg-Witten theory that if X contains an embedded copy of certain negative definite four manifolds, described by plumbing disc bundles over spheres, then this piece can be removed and replaced with a rational homology ball, without altering some of the Seiberg-Witten invariants. R. Fintushel and R. Stern found the first non-obvious examples of this behavior, which has since been generalized by Jongil Park, Zoltan Szabo, Andras Stipsicz, and Jonathan Wahl. I analyzed the Ozsváth-Szabó invariants for this situation, in a manner similar to R. Fintushel and R. Stern, and found a general argument which proved the same relationship between the Φ-invariants for the original and surgered manifolds, for all of the examples then known. Knowing this information allows one to verify that the Ozsváth-Szabó invariants will be the same as the Seiberg-Witten invariants for four-manifolds obtained by log transforms on fibers of elliptic surfaces (if one ignores signs). Furthermore, the proof indicated a more general construction of such examples.

On knot Floer homology in double branched covers. arKiv:math.GT/0706.0741, 32 pgs., submitted

Little is known about the Heegaard-Floer homology of fibered three manifolds, or its cousin for a fibered pair (Y, K), called the knot Floer homology, see [3]. The paper reports results on a class of fibered three manifolds and knots with hyperelliptic monodromy, building on an intriguing relationship establish by P. Ozsváth and Z. Szabó between branched double covers and another homology theory for links in S^3 called Khovanov homology, [8], [4].

Let $\mathbb{L}$ be a link in $A \times I$ where A. We assume that $A \times I$ is embedded in S^3 and that a component B, called the axis, linking $A \times I$ is added as in the diagram below. We always assume throughout that $\mathbb{L}$ intersects the spanning disc of B in an odd number of points.



Let $\Sigma(\mathbb{L})$ be the branched double cover of S^3 over $\mathbb{L}$, and let $\widetilde{B}$ be the pre-image of B in $\Sigma(\mathbb{L})$. Then $\widetilde{B}$ is a null-homologous knot in $\Sigma(\mathbb{L})$. When $\mathbb{L}$ is a braid, the pre-image, under the covering map, of the open book of discs with binding B is an open book with binding $\widetilde{B}$. I examined the knot Floer homology of the binding, ignoring the $Spin^c(Y)$ structures:

$$\widehat{HFK}(\Sigma(\mathbb{L}), \widetilde{B}, i) = \bigoplus_{\{\bar{\mathfrak{s}}\,|\,\langle c_1(\underline{\mathfrak{s}}), [F]\rangle = 2i\}} \widehat{HFK}(\Sigma(\mathbb{L}), \widetilde{B}, \underline{\mathfrak{s}})$$

where $\underline{\mathfrak{s}}$ is a relative $Spin^c$ structure for $\widetilde{B}$ on $Y - K$ and $[F]$ is the homology class of a pre-image of a spanning disc for B. The Euler characteristic of the left side for $\mathbb{L}$ a braid is the coefficient of t^i in the symmetrized version of $\det(I - tM)$ where M is the monodromy action on $H_1(F; \mathbb{Z})$.

In this setting, I proved

Proposition. *Let $\mathbb{L}$ be a link in $A \times I \subset \mathbb{R}^2 \times \mathbb{R}$ as above. Let $\mathbb{L}'$ be $\mathbb{L}$ with two copies of the center of the annulus, unlinked from $\mathbb{L}$. There is a spectral sequence whose E^2 term is isomorphic to the reduced Khovanov skein homology of the mirror, $\overline{\mathbb{L}}'$, in $A \times I$ with coefficients in $\mathbb{F}_2$ and which converges to $\oplus_{i \in \mathbb{Z}} \widehat{HFK}(\Sigma(\mathbb{L}) \#^2 (S^1 \times S^2), \widetilde{B} \# B(0,0), i, \mathbb{F}_2)$.*

P. Ozsváth and Z. Szabó first constructed such a spectral with E^2-page given by the reduced Khovanov homology of the mirror of the link, L, and which converged to $\widehat{HF}(\Sigma(L)))$, [8]. This proposition generalizes that result, where the Khovanov skein homology is an invariant for links in $A \times I$ defined by M. Asaeda, J. Przytycki, and A. Sikora, [1].

When $\mathbb{L}$ is a braid, this gives a starting point for computing the knot Floer homology of the binding of an open book with a sufficiently nice, hyperelliptic mondoromy. In particular, I gave an algorithm for calculating the reduced Khovanov skein homology for alternating braids which results in

Theorem. *Let $\mathbb{L}$ be a non-split, alternating braid in $A \times I$, with $\det(\mathbb{L}) \neq 0$, and which intersects the spanning disc for B in an odd number of points. Then the $\mathbb{Z}/2\mathbb{Z}$-grading of every element of $\widehat{HFK}(\Sigma(\mathbb{L}), \widetilde{B}, F, i, \mathbb{F}_2)$ is determined by the parity of i. Thus the dimension of the homology for each i is determined by the coefficient of t^i in $\det(I - tM)$.*

The Heegaard-Floer homology for the result of 0-surgery on $\widetilde{B}$ can then be computed from the knot Floer homology, providing the first examples of computations for fibered three manifolds with pseudo-Anosov monodromy foor a fiber of genus $g \geq 2$.

This approach also gives information about the relationship between the Heegaard-Floer contact invariant for the contact structure associated to the open book and O. Plamanevskaya's invariant in Khovanov homology. This approach has since been used by O. Plamanevskaya and J. Baldwin to give examples of contact structures which can be proven to be tight using Khovanov homology.

On knot Floer homology for some fibered knots. arKiv:math.GT/0706.0743, 22 pgs., submitted

I extend the technology of the preceding paper to recover the last theorem for knot Floer homology with $\mathbb{Z}$-coefficients. This requires a slightly different approach as the signs in the spectral sequence described above have yet to be made explicit. In the final section, I analyze the Heegaard-Floer groups for certain fibered three manifolds. Let $\gamma_i, 1 \leq i \leq 2g$ be a collection of simple, closed, non-separating curves on a genus g surface, such that γ_i intersects γ_j only if $j = i \pm 1$, and then the intersection occurs at a single point. Call this situation a linear chain of simple loops. I proved:

Proposition. *Let M_ϕ be the fibered three manifold determined by a fiber F^g and monodromy $D_1^{n_1} \cdots D_{2g}^{n_{2g}}$ where $n_i \geq 0$ and the Dehn twist D_i occurs along the curve γ_i in a linear chain of simple loops on F^g. Let $\mathcal{S}$ be a collection of loops in F consisting of n_i parallel copies of γ_i for each loop γ_i in the linear chain. Then*

$$HF^+_{\mathbb{Z}/2\mathbb{Z}}(M_\phi, \mathfrak{s}_{g-2}) \cong H^*(F \backslash \mathcal{S})$$

as $H^*(F, \mathbb{Z}/2\mathbb{Z})$-modules, where the action is by cup product on the right side of the isomorphism and by the H_1-action on the left. The Heegaard-Floer group is the direct sum of the homologies over all Spin^c structures pairing with the fiber to give $2g-4$.

This proposition is a calculation in Heegaard-Floer homology similar to those of P. Seidel, I. Smith, and E. Eftekhary, for the Floer cohomology of symplectomorphisms of surfaces. In particular, under the same assumptions, the Floer cohomology of a symplectomorphism compatible with the monodromy will also be isomorphic to $H^*(F \backslash \mathcal{S})$.

Note on the Heegaard-Floer link surgery spectral sequence, math.GT/0808.2817

This paper consists of two parts: First, a review of the constructions of a spectral sequence computing the hat theory of Heegaard-Floer homology from a link surgery picture for a three manifold. This repeats the argument of P. Ozsváth and Z. Szabó in a slightly more precise form. In the second part, we use this increased precision to show that the spectral sequence is independent of many of the underlying choices made in its construction, and develop a theory of maps on these spectral sequences in induced by cobordisms between three manifold. This paper is supposed to be a foundation to answering the questions described in the section on future work below.

Finding bounds for the τ-invariant of a satellite knot, in preparation

Using knot Floer homology, P. Ozsváth and Z. Szabó defined an invariant, $\tau(K) \in \mathbb{Z}$, for any knot, K in S^3. The absolute value of this number is a lower bound for the smooth four-ball genus of the knot. M. Hedden provided bounds for τ on a cable of a knot. In this paper, I provide a slightly weaker set of bounds for all satellites:

Proposition. *Let $S_r(C, P)$ be the r-twisted satellite knot formed from a companion, C, in S^3 and a pattern, P, in $S^1 \times D^2$. Let l be the intersection number of P with D^2, with P oriented so that $l \geq 0$, and n_+ be the number of positive interesections. Let*

$$D(S_r(C,P)) = \tau(S_r) - \big(\tau(P) + l\,\tau(C) + \frac{l(l-1)}{2}\,r\big)$$

then when $r \neq 0$, we have

$$-(1+l) \leq D(S_r) \leq n_+(P) + l \qquad \text{when } r < 2\tau(C) - 1$$

$$-n_+(P) - l \leq D(S_r) \leq 1 + l \qquad \text{when } r > 2\tau(C) + 1$$

Furthermore, for every $r \in \mathbb{Z}$ we have

$$-n_+(P) - l \leq D(S_r) \leq n_+(P) + l$$

Heegaard-Floer homology and string links, II, in preparation

This paper is a sequel to the first paper above. Consider the situation where we close the string link in $S^1 \times D^2$ by gluing another copy of $D^2 \times I$ to the first to obtain $S^1 \times D^2$ and joining the points $p_s \times \{0\}$ to the points $p_s \times \{1\}$ using the trivial braid. The complement of the resulting link in $S^1 \times D^2$ is homeomorphic to the complement of a link in S^3 found taking the link already formed union an unknot which is the meridian of $S^1 \times D^2$. We let L be this link and consider the link Floer homology $\widehat{HFL}(L)$. This invariant decomposes into summands, $\widehat{HFL}(L, \vec{h})$, along relative $Spin^c$ structures on $S^3 - L$, and the support of $\widehat{HFL}(L)$ in that decomposition determines the minimal genus of surfaces in $S^3 - L$ with boundary on L. Then,

Proposition.
a) For S, a pure string link, $\widehat{HF}(S) \otimes V_k \cong \bigoplus_{\vec{h} \in H} \widehat{HFL}(L; \vec{h})$, where V_k is a rank 2^k-free abelian group depending only on k, and H is the set of relative $Spin^c$ structures pairing with D to give $1 - k$.
b) The string link L is freely isotopic to a braid if and only if $\widehat{HF}(L) \cong \mathbb{Z}_{(0)}$ where $\widehat{HF}(L) = \oplus_{\overline{v}} \widehat{HF}(L; \overline{v})$

The "if" direction can be shown using the work of A. Juhasz on the Heegaard-Floer homology of balanced, sutured manifolds.

Future Research

I intend to study the Heegaard-Floer invariants of four-manifolds equipped with symplectic structures, i.e. a closed two form ω such that $\omega \wedge \omega > 0$. Due to a theorem of S. Donaldson, [2], the study of symplectic four manifolds can be approached by understanding certain fibrations on the four manifold whose generic fiber is a smooth surface, but allowing a finite number of carefully controlled singular fibers. These fibrations can always be found once we blow-up the symplectic four manifold sufficiently many times.

To do this we must start with the Heegaard-Floer homology of fibered three manifolds: those equipped with a map $f : Y \to S^1$ without critical points. I can construct a chain complex for a fibered three manifold by taking a certain page in the a spectral sequence arising from the monodromy. Achiral Lefschetz fibrations between fibered three manifolds would give rise to chain maps on this complex. The resulting maps on the homology would be a first approximation to the cobordism maps on Heegaard-Floer homology. As every symplectic closed four manifold gives rise to such cobordisms between $S^1 \times \Sigma^g$ for some g, this could potentially provide a handle for computing the Ozsváth-Szabó four manifold invariants for symplectic manifolds.

I hope this construction will help resolve the following questions, inspired by the author's paper "On knot Floer homology in double branched covers," which already achieves some of this program for hyperelliptic allowable Lefschetz fibrations between fibered knots. In fact there is an analog of the Khovanov skein homology which is the E^2-page of a spectral sequence converging to $\widehat{HFK}(Y, K)$ for any fibered knot, a fact independently discovered by J. Baldwin. Furthermore, there are analogs for any self-indexed S^1-valued Morse function on a closed three manifold. The content of the paper "Notes on the Heegaard-Floer Spectral Sequence" is that the spectral sequence is independent of the underlying choices in Heegaard diagrams, under suitable equivalences, and that the Heegaard-Floer cobordism maps can be approximated by maps between the spectral sequences. The next step is to study the following situation:

The Khovanov skein homology is a functor from the category whose objects are braids embedded in S^3 and whose morphisms are surface braids (i.e. embedded surfaces in $A \times I^2$ which are branched coverings of an annulus with prescribed singularities). The resulting maps on groups depend only on the isotopy class of the surface braid. Do these maps form a first approximation for the maps induced by cobordisms on the Heegaard-Floer, and knot Floer, homologies of the branched double covers? A similar question was asked by P. Ozsváth and Z. Szabó in [8].

Such a result would provide a guide to understanding cobordism maps for achiral Lefschetz fibrations, as the branched double covers decribed here have fibrations with Lefschetz type singularities.

The author is also researching the following questions:

(1) The knot Floer homology of a fibered knot is isomorphic to $\mathbb{Z}$ in the indices $\pm g(K)$, where $g(K)$ is the minimal genus of a spanning surface. Indeed, Yi Ni proved that this property characterizes fibered knots. However, as a consequence these groups do not reflect the monodromy in any respect. The monodromy first appears in the indices $\pm(g(K)-1)$. What, if anything, can be said in general about this group?
(2) A contact structure on Y^3 is supported by many open books, related by positive stabilizations and destabilizations. J. Etnyre and B. Ozbagci defined an invariant of contact structures by asking for the minimal genus of the page of such open books, under the restriction that the binding be connected. I believe this invariant can be studied using Heegaard-Floer homology, which would likely provide a lower bound for this invariant.
(3) Does a cobordism in $S^3 \times I$ between oriented knots in $S^3 \times \{0,1\}$ induce a map on the knot Floer homologies?
(4) Let Y be a balanced, sutured three manifold. Is there a cobordism invariant for $S^1 \times Y$ which can be calculated from the Euler characteristic of the sutured Floer homology in different $Spin^c$ structures. It seems that such an invariant would have use in studying the knot surgery construction of R. Fintushel and R. Stern.

References

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