Homework and Pre-Class reading for Math 152H-1 August 30

Homework:

1. Do the following limits exist? If so, what do they equal? If not, is the sequence diverging to $\pm\infty$?

$$\lim_{n\to\infty} 2+\frac{(-1)^n}{n} \qquad \lim_{n\to\infty}\frac{cos(n)}{\sqrt{n}}$$

hint : how large is cos(n)?

$$\lim_{n\to\infty}\sqrt{\frac{n}{n+1}} \qquad \lim_{n\to\infty}\frac{3n^2+2\,n}{5n^2+4}$$

$$\lim_{n\to\infty}\frac{2^n}{n^2} \qquad \lim_{n\to\infty}\frac{2^n}{n!}$$

   Reminder: $n! = n(n-1)(n-2)\cdots 2\cdot 1$, so $3! = 3\cdot 2\cdot 1 = 6$.

2. Let $a_n = 1+\frac{1}{2}+\cdots+\frac{1}{2^n}$. What is $\lim_{n\to\infty} a_n$?

3. Finish the calculation of the area of a regular polygon begun in class: take a circle of radius 1, divide the circumference into n equal arcs, and join the endpoints of these arcs by straight lines. Inside the circle you will have a regular n-gon. In class, we divided the polygon into n congruent triangles, each with two sides of length 1 and angle $\frac{2\pi}{n}$ between them. Use this to find the area of the polygon (you will need to use the fact that the area of a triangle is $\frac{1}{2}ab\sin(\theta)$ where θ is the angle between two sides of length a and b. Verify this using trigonometry). Now what happens to the polygon and its area as $n\to\infty$?

Reading: Look at table 1.1 and make sure you understand the $[1,2)$ notation versus $1\le x<2$ notation.

First, we will use the following notation:

$$\mathbb{Z} = \{0,\pm1,\pm2,\pm3,\ldots\} \qquad \text{the integers}$$

$$\mathbb{Q} = \{\tfrac{p}{q}\big|p\in\mathbb{Z}, q\in\mathbb{Z}, q\neq 0\} \qquad \text{the rational numbers}$$

$$\mathbb{R} = \text{the real numbers}$$

The real numbers are all numbers on the number line: e.g. 1.1, $\sqrt{2}$, $-\frac{3}{4}$, $\pi = 3.1415\ldots$. These can be represented by decimal expansions like $121.34567\ldots$ where the part after the decimal can be finite, repeating infinite, or infinte but not repeating.

Second, we will say that a sequence $\{a_n\}_{n=1}^{\infty}$ is **non-decreasing** if $a_{n+1}\ge a_n$ for each n, i.e. the terms generally get larger, and **non-increasing** if $a_{n+1}\le a_n$. On the other hand, they are called increasing if the terms are strictly larger ($a_{n+1}>a_n$ for every n), and decreasing if the terms are strictly smaller. Finally, a sequence is **bounded above** if there is a number M such that $a_n\le M$ for all M, i.e. all the terms are less than M; it is **bounded below** if there is an N such that $a_n\ge N$ for all n, i.e. the terms are all greater than N. It is bounded if there is an N and an M such that $N\le a_n\le M$ for all n.

Here are the questions to ponder for Wednesday:

1. Which of the following sequences is non-decreasing? non-increasing? bounded?
   (a) $a_n = 2+\dfrac{(-1)^n}{n}$
   (b) $a_n = a_{n-1}+\sqrt{n}$ and $a_1 = 1$ (calculate a few terms first, each term is found from its predecessor)
   (c) $b_n = \frac{1}{2}\big(b_{n-1}+\frac{1}{b_{n-1}}\big)$ and $b_1 = 2$ (we'll do this one in class, so don't worry)
2. Is a bounded sequence of real numbers necessarily convergent? What if it is non-decreasing?

3. Is every real number the limit of a sequence of distinct rational numbers? (distinct means no term in the sequence is equal to any other. To do this think about what you know about decimal expansions of real numbers, but be careful about terminating decimals!)

"God created the integers, all else is the work of men" - Leopold Kronecker, a mathematician of the 19^{th} century.

Start with the natural numbers: $\{1, 2, 3, \ldots\}$. These we can add or multiply to get another natural number, but we can't always subtract to get a natural number ($2-3 = -1$, ughh), or divide ($\frac{2}{3}$). By including the negatives of the natural numbers, and 0, we get $\mathbb{Z}$, the integers. Now we can add, multiply, and subtract, but we still can't divide any two integers to get an integer. To resolve the last difficulty, we use fractions of integers, and get the rational numbers, $\mathbb{Q}$. Of course, there are many ways to represent the same rational number by quotients of integers $\frac{2}{3} = \frac{4}{6} = \frac{-6}{-9}$, but we know how to deal with this. These we can add by finding the least common denominator, multiply, divide, and subtract. Additionally, there is a number 0 such that $0 + \frac{p}{q} = \frac{p}{q}$, and an number 1 which when multiplied by any number returns the number. There are inverses and negatives. The arithmetic is the same as you are used to.

How then to get to the real numbers, $\mathbb{R}$. These are the numbers on the number line. Are they all rational? And if not, what additional properties do they have that distiguish them. $\sqrt{2}$ is not rational, as the following argument (due essentially to Euclid) shows:

> Suppose $\sqrt{2} = \frac{p}{q}$, where $p, q \in \mathbb{Z}$ and we assume that the fraction has been reduced, i.e. that p and q have no common factors, then $2q^2 = p^2$. Each side is an integer, so we know that p^2 must be even. Hence p must be even since an odd number squared is odd. But then p^2 is divisible by 4. So $2q^2 = 4(\textit{some integer})$ or $q^2 = 2(\textit{some integer})$, hence q is also even. But then both p and q are divisible by 2 contradicting the fact that the fraction can be assumed to be reduced. Hence there can be no such fraction.

What do we know about the reals? We know that they have decimal expansions. So for instance $\sqrt{2} = 1.41421356\ldots$ and the decimal goes on forever. We can construct the reals then as limits of sequences of rational numbers in the following way: $1, 1.4, 1.41, 1.414, 1.4142, \ldots$ (Why are these rational?). Clearly we can do this for any decimal. So, the reals can be thought of as limits of rational numbers. In fact, this is somewhat unsatisfying, because it already preseumes we know about the real numbers. In a more advanced course, what would be done is to specify a certain type of sequence of rational numbers (of which the one above is an example), describe when two such sequences should be equal (i.e. if we had the real numbers already, their limits would be equal), and how to add, multiply, divide, subtract, etc such sequences. This would then be a "construction" of the reals using only the rationals. We will forgo all of this.

The upshot: at the heart of the real numbers, the numbers we use to measure the world, and thus the basis for science and engineering, is the concept of a limit.