Homework for Math 152H-1 October 9

Reading: Start reading section 4.1, make sure you know the definitions.

Homework: Quadratic Approximation

(1) You have a function $f(x)$ which has both first and second derivatives at $x = a$. The quadratic approximation to $f(x)$ at $x = a$ is the parabola:

$$Q(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

Show that $Q(x)$ has the following properties:

1. $Q(a) = f(a)$
2. $Q'(a) = f'(a)$
3. $Q''(a) = f''(a)$

(2) Find the linearization and quadratic approximations to the following functions at $x = 0$

1. $y = x^2 + 2x$
2. $y = cos\, x$
3. $y = (1+x)^{-1}$
4. $y = x(x^2-1)^3$

(3) Using $f(x) = \sqrt{x}$ near $x = 4$ calculate three approximations of $\sqrt{4.01}$: 1) $f(4)$, 2) $L_4(4.01)$ and $Q_4(4.01)$ where L_4 and Q_4 are the linearization and quadratic approximations to $f(x)$ at $x = 4$. Which is closest to the actual value $\sqrt{4.01} = 2.00249843...$?

(4) Find the linearization and quadratic approximation of the curve defined by $x^2y + y^3x = 6$ near $(2, 1)$. Describe the shape of the curve near this point. Find an approximate value of y such that $(2.01)^2y + 2.01y^3 = 6$. (**Hint:** use your linearization)

(5) You are given that there is a function $y = f(x)$ such that $f'(x) = 1 - [f(x)]^2$ and $f(0) = 2$. Plug 0 into the equation for the derivative to find $f'(0)$. Can you find $f''(0)$? What are the linearization and quadratic approximation to $f(x)$ at $x = 0$?

(6) You have a function $f(x)$ which has first, second, and third derivatives at $x = a$. Find the values of a_0, a_1, a_2, and a_3 so that $C(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3$ has the following properties:

1. $C(a) = f(a)$
2. $C'(a) = f'(a)$
3. $C''(a) = f''(a)$
4. $C'''(a) = f'''(a)$

Find $C(x)$ for $y = x^2 + 2x$ and $y = sin\, x$ at $x = 0$.

(7) One way of saying that a function $f(x)$ is twice differentiable at $x = a$ is to say that we can find numbers b_0, b_1, and b_2 so that

$$E(h) = f(a+h) - \left(b_0 + b_1 \cdot h + b_2 \cdot h^2\right)$$

satisfies $\lim_{h\to 0} \frac{E(h)}{h^2} = 0$. If we can do this, $b_0 + b_1 \cdot h + b_2 \cdot h^2$ is just $Q(a+h)$ where $Q(x)$ is the quadratic approximation to $f(x)$ at $x = a$. Suppose we have $f(x)$ and $g(x)$ twice differentiable with

$$f(a+h) = f(a) + f'(a) \cdot h + \frac{f''(a)}{2} \cdot h^2 + E_1(h)$$

$$g(a+h) = g(a) + g'(a) \cdot h + \frac{g''(a)}{2} \cdot h^2 + E_2(h)$$

Take the product and write:

$$f(a+h)g(a+h) = c_0 + c_1 \cdot h + c_2 \cdot h^2 + E(h)$$

where c_0, c_1, and c_2 are *numbers*. Verify that $\lim_{h\to 0} \frac{E(h)}{h^2} = 0$ ($E(x)$ will depend upon many different terms including E_1 and E_2). Explain how you can find the first and second derivatives of $f(x) \cdot g(x)$ at $x = a$ from this information.