Solutions to Mid-term #1

Problem 1: (10 pts) You are given the function

$$g(x) = \frac{2x^2 + 4\,x - 16}{x^2 - 5\,x + 6}$$

Calculate $\lim_{x \to -4} f(x)$, $\lim_{x \to 2} f(x)$, $\lim_{x \to 3^+} g(x)$ and $\lim_{x \to 3^-} g(x)$. Graph $g(x)$ near 3. What is/are the horizontal asymptote(s) for $g(x)$?

First we factor numerator and denominator to get

$$g(x) = \frac{2(x+4)(x-2)}{(x-2)(x-3)}$$

The function isn't defined at 2, but when $x \neq 2$, it is equal to $\frac{2(x+4)}{x-3}$. Thus

a) $\lim_{x \to -4} g(x) = 0$

b) $\lim_{x \to 2} g(x) = \frac{2(6)}{-1} = -12$

c) $\lim_{x \to 3^+} g(x) = +\infty$ since $x + 4 > 0$ and $x - 3 > 0$ when $x > 3$. On the other hand, when $-4 < x < 3$ we have $x + 4 > 0$ and $x - 3 < 0$, so $\lim_{x \to 3^-} g(x) = -\infty$. For the graph come Monday.

d) To find the horizontal asymptotes, calculate $\lim_{x \to -\infty} g(x)$ and $\lim_{x \to \infty} g(x)$. Both are done thus:

$$\lim_{x \to \infty} g(x) = \lim_{x \to \infty} \frac{2 + \frac{4}{x} - \frac{16}{x^2}}{1 - \frac{5}{x} + \frac{6}{x^2}} = \frac{2}{1} = 2$$

Problem 2: (10 pts)

(a) Using any method your prefer, compute the derivative of

$$f(x) = sin(\frac{\pi}{2} + x^2)$$

$$\frac{d}{dx} f(x) = \frac{d}{dx} sin(\frac{\pi}{2} + x^2) = cos(\frac{\pi}{2} + x^2) \frac{d}{dx} x^2 = 2x\, cos(\frac{\pi}{2} + x^2)$$

(b) Use your calculation from part (a) to evaluate

$$\lim_{h \to 0} \frac{sin(\frac{\pi}{2} + h^2) - 1}{h}$$

Justify your answer.

How to use the previous part? Try the limit definition of the derivative:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{sin(\frac{\pi}{2} + (a+h)^2) - sin(\frac{\pi}{2} + a^2)}{h}$$

To match this with the derivative we need $sin(\frac{\pi}{2} + a^2) = 1$ and $sin(\frac{\pi}{2} + (a+h)^2) = sin(\frac{\pi}{2} + h^2)$. The easiest thing to try is $a = 0$. Both equalities are then satisfied, so the value of the limit is simply $f'(0)$. From part (a) $f'(x) = 2x\, cos(\frac{\pi}{2} + x^2)$ and $f'(0) = 0$.

(c) What is the equation of the tangent line to $f(x)$ at $x = 0$?

From part (b), the slope of this tangent line is 0 (or use part (a) directly). It must go through the point $(0, sin(\frac{\pi}{2} + 0^2) = (0, 1)$. Using the point-slope form the equation of the tangent line is $y - 1 = 0(x - 0)$ or $y = 1$.

Problem 3: (10 pts) Is there a solution to

$$x^2 - cos\, x = 0$$

for some $x \geq 0$?

x^2 is continuous and $cos\, x$ is continuous. Therfore $f(x) = x^2 - cos\, x$ is continuous. However, $f(0) = 0^2 - cos\, 0 = -1$, whereas $f(\pi) = \pi^2 - cos\, \pi = \pi^2 + 1$. Since $f(0) < 0$ and $f(\pi) > 0$, by the intermediate value theorem there is a point $c \in [0, \pi]$ such that $f(c) = 0$. This is equivalent to $c^2 - cos(c) = 0$.

Problem 4: (10 pts) Suppose $h(x)$ is the function

$$h(x) = \begin{cases} \dfrac{sin(\sqrt{x} - 1)}{x - 1} & \text{when } 0 \leq x < 1 \\ \frac{1}{4}(x + 1) & \text{when } x \geq 1 \end{cases}$$

Where is $h(x)$ continuous?

As $\frac{1}{4}(x + 1)$ is a polynomial it is continuous on $(1, \infty)$. $x - 1$, $\sqrt{x} - 1$ and $sin\, x$ are continuous for $x \geq 0$. Since the composition and quotient of continuous functions is continuous, $h(x)$ is also continuous on $[0, 1)$. That leaves the point $x = 1$ to be checked. We know $h(1) = \frac{1}{4}(1 + 1)$, so we need to check 1) whether $\lim_{x \to 1} h(x)$ exists, and 2) does it equal $\frac{1}{2}$. We calculate the left and right hand limits:

$$\lim_{x \to 1^+} h(x) = \lim_{x \to 1^+} \frac{1}{4}(x + 1) = \frac{1}{2}$$

$$\lim_{x \to 1^-} h(x) = \lim_{x \to 1^-} \frac{sin(\sqrt{x} - 1)}{x - 1} = \lim_{x \to 1^-} \frac{sin(\sqrt{x} - 1)}{\sqrt{x} - 1} \cdot \frac{\sqrt{x} - 1}{x - 1} = \lim_{x \to 1^-} \frac{\sqrt{x} - 1}{x - 1} = \lim_{x \to 1^-} \frac{1}{\sqrt{x} + 1} = \frac{1}{2}$$

Since these both exist and are equal, we have $\lim_{x \to 1} h(x) = \frac{1}{2} = h(1)$. The function is also continuous at 1. Therefore, the function is continuous on $[0, \infty)$.

Problem 5:(10 pts) Let

$$f(x) = \begin{cases} 2x + 1 & x \leq 1 \\ 3x - 1 & x > 1 \end{cases}$$

If I choose any $\epsilon > 0$ is it *always* possible to find $\delta > 0$ so that $|f(x) - 3| < \epsilon$ for all $0 < x - 1 < \delta$? What does this say in terms of limits? (Read the inequalities carefully!!)

Note that we are only asking for $0 < x - 1 < 1$ or $1 < x < 1 + \delta$. The rest is like the definition of the limit. So, whether I can find such a δ for any ϵ depends upon whether $\lim_{x \to 1^+} f(x) = 3$. The limit is from the right since we are only concerned with $x > 1$. However, $\lim_{x \to 1^+} f(x) = 3(1) - 1 = 2$, so the answer is **NO**.

Problem 6: (10 pts) Suppose that $f(x)$ has domain equal to $\mathbb{R}$ and $-x^2 \leq f(x) \leq x^2$ near $x = 0$. Use the definition of the derivative to show that $f'(0) = 0$.

We use the definition of the derivative on $f(x)$ at 0, so

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{f(x)}{x}$$

All we know about $f(x)$ is that $-x^2 \le f(x) \le x^2$. Plugging in $x = 0$ tells us that $0 \le f(0) \le 0$, so $f(0) = 0$. Using what we know, we also deduce $-|x| \le \frac{f(x)}{x} \le |x|$. The absolute values appear because if $x < 0$ and we divide, we change the direction of the $\le$ signs. To handle both $x > 0$ and $x < 0$, we need the absolute value signs (I let this slide while grading, but you should be aware of this problem). But now we can use the squeeze theorem to see that $\lim_{x\to 0} \frac{f(x)}{x} = 0$. Therefore $f'(0) = 0$.

Use this fact to show that

$$y = \begin{cases} x^2 & x \in \mathbb{Q} \\ -x^2 & x \notin \mathbb{Q} \end{cases}$$

is differentiable at 0. Explain why it is not differentiable anywhere else. (This is an example of a function that is differentiable at only one point!)

The function is differentiable at 0 because it clearly satisfies $-x^2 \le y \le x^2$ (in fact is always equal to one or the other of these). By the preceding, $y'|_{x=0} = 0$. On the other hand, between any two rationals there is an irrational, and between any two irrationals there is a rational. So when $x \neq 0$, the function is constantly jumping back and forth across the x-axis. In short it is not continuous at any other point, and thus cannot be differentiable there.