(1)

(a) For any $\epsilon > 0$ can you find an $\delta > 0$ such that $1 < x < 1 + \delta$ implies that $\left|\dfrac{sin(x^2-1)}{x-1} - 1\right| < \epsilon$.
Explain your answer using limits.

The question is asking whether $\lim\limits_{x\to 1^+} \dfrac{sin(x^2-1)}{x-1} = 1$. It is 1^+ since x must be greater than 1. In fact, the limit is 2, as can be seen using L'ôpital's rule or by multiplying by $\frac{x+1}{x+1}$ and using $\lim\limits_{h\to 0} \frac{sin\, h}{h} = 1$. The answer is **no**.

(b) For any $\epsilon > 0$ is there an $\delta > 0$ so that $|x| < \delta$ implies that $\left|x^2 sin(\dfrac{1}{x^3})\right| < \epsilon$?

The question is asking whether $\lim\limits_{x\to 0} x^2 sin(\dfrac{1}{x^3}) = 0$. Since $sin\, u$ is always between -1 and 1, while $x^2 \to 0$ as $x \to 0$, the squeeze theorem tells us that the limit is indeed 0. The answer is **yes**.

(2) Compute the derivative of $f(x) = \sqrt{x} - \frac{1}{x^2}$ using the definition of the derivative.

(3) Without evaluating the integral, calculate the derivative of

$$F(x) = \int_{2x}^{3x} (1 - t^2)\, dt$$

where is $F(x)$ increasing? decreasing? Does it have any local maxima? Where is it concave up/down? Does it have any inflection points? What is $F(0)$? Graph this function!

$F'(x) = (1-(3x)^2)\frac{d}{dx} 3x - (1-(2x)^2)\frac{d}{dx} 2x = 3 - 27x^2 - 2 + 8x^2 = 1 - 19x^2$. $F(x)$ is increasing when $|x| < \frac{1}{\sqrt{19}}$ and decreasing when $|x| > \frac{1}{\sqrt{19}}$. It has a local minimium at $-\frac{1}{\sqrt{19}}$ and a local maximum at $\frac{1}{\sqrt{19}}$. $F''(x) = -38x$ so it is concave up for $x < 0$ and concave down for $x > 0$. The graph looks like a cubic polynomial with negative leading coefficient (on x^3). In fact, $F(x)$ is a cubic polynomial as actually integrating will demonstrate.

(4) Compute the following integrals:

$$\int \frac{sec^2\, x}{(1+tan\, x)^2}\, dx \qquad\qquad \int_0^{\frac{\pi^2}{4}} \frac{sin\sqrt{x}}{3\sqrt{x}}\, dx$$

$$\int_0^8 \frac{1}{\sqrt{1+\sqrt{1+x}}}\, dx$$

For the first substitute $u = 1 + tan\, x$ to get $-(1+tan\, x)^{-1} + C$. For the second, substitute $u = \sqrt{x}$ to get $\int_0^{\frac{\pi}{2}} \frac{2}{3} sin\, u\, du$ and then integrate. For the last substitute $u = 1 + \sqrt{1+x}$ and write $du = \frac{1}{2\sqrt{1+x}}\, dx$ or $2(u-1)\, du = dx$.

(5) Find the tangent line to $\sqrt{x}y + x\, sin\, y = \frac{\pi}{2} + 1$ at $(1, \frac{\pi}{2})$. By how much should we change y to approximate a solution when $x = 1.01$?

Implicit differentiation yields $\frac{y}{2\sqrt{x}} + \sqrt{x}y' + sin\, y + xy'\, cos\, y = 0$. Substituting in the point gives $\frac{\pi}{4} + y' + 1 + y'\, cos \frac{\pi}{2} = 0$ or $y' = -1 - \frac{\pi}{4}$ since $cos\, \frac{\pi}{2} = 0$. The equation of the tangent line is $y = \frac{\pi}{2} - \left(1 + \frac{\pi}{4}\right)(x-1)$. To approximate a solution corresponding to $x = 1.01$ we plug into the tangent line and see what we get: $y = \frac{\pi}{2} - \frac{1}{100} \cdot \left(1 + \frac{\pi}{4}\right)$, so y changes by approximately $-\frac{1}{100} \cdot \left(1 + \frac{\pi}{4}\right)$

(6) Find the area of the bounded region between $y = x^3 - x$ and $y = x^2 - 1$. Now compute the same area using right handed Riemann sums with equal width rectangles and a limit as $n \to \infty$.

The two curves intersect where $x^3 - x = x^2 - 1$, i.e. $x^3 - x^2 - x + 1 = 0$ which factors as $(x-1)^2(x+1) = 0$. Thus the curves intersect at ± 1. In fact they are tangent at $+1$ since they have tangent lines of the same slope at the intersection point. In any case. Draw a graph to see that $x^3 - x$ is the upper boundary on the interval $[-1, 1]$. Thus the area is given by $\int_{-1}^{1} x^3 - x - x^2 + 1\, dx$ which is straightforward to evaluate.

The Riemann sum will have rectangles of width $\frac{1-(-1)}{n}$. Thus the sum is

$$\frac{2}{n}\sum_{k=1}^{n}\left((-1+\frac{2k}{n})^3 - (-1+\frac{2k}{n}) - (-1+\frac{2k}{n})^2 + 1\right)$$

One then expands each of the terms and collects according to the power of k to get

$$\frac{2}{n}\sum_{k=1}^{n}((-1+1-1+1) + (3-1+2)\frac{2k}{n} + (-3-1)(\frac{(2k)^2}{n^2}) + \frac{(2k)^3}{n^3})$$

$$= \frac{2}{n}\sum_{k=1}^{n}(8\frac{k}{n} - 16\frac{k^2}{n^2} + 8\frac{k^3}{n^3})$$

Now use the formulas for summing the first n squares and cubes. Don't worry! Nothing this algebraic will appear on an exam.

(7) Use an integral to compute the following limit:

$$\lim_{n\to\infty} \sqrt{\frac{3}{n}} \sum_{k=1}^{n} \frac{1}{\sqrt{k}}$$

Hint: Use a function $f(x) = x^s$ with $s < 0$.

Let's re-write to get an n instead of a $\sqrt{n}$ in the denominator of fraction in front of the sum:

$$\lim_{n\to\infty} \frac{\sqrt{3}}{n} \sum_{k=1}^{n} \frac{\sqrt{n}}{\sqrt{k}}$$

Following the hint, we should be looking for $\frac{k}{n}$'s in the denominator of some fraction. This can be achieved as

$$\lim_{n\to\infty} \frac{\sqrt{3}}{n} \sum_{k=1}^{n} \frac{1}{\sqrt{\frac{k}{n}}}$$

This is now a Riemann sum converging to $\sqrt{3}\int_0^1 \frac{1}{\sqrt{x}}\, dx$. Notice that you don't need to do anything with the $\sqrt{3}$ (although you can, you just get a more complicated integral).

(8) Water flows into a conical filter with radius $3'$ and height $4'$. At every moment the water makes it through the filter at a rate of $2\ ft^3/hr$. The rate at which the water flows into the filter changes according to the height of the water already in the filter (denoted by h) according to $\frac{dV_{in}}{dt} = \frac{12}{(1+h)^2}$. How fast is the height changing when the water is $2'$ high in the filter? If we let the water flow in indefinitely, starting with an empty filter, will the filter ever overflow? What is the highest the water will get in the filter?

If the water is at height h, then the radius of the surface of the water, r, satisfies $\frac{r}{h} = \frac{3}{4}$ from similar triangles. Thus the volume, $V(r,h) = \frac{1}{3}\pi r^2 h$ can be written as $V(h) = \frac{3\pi}{16}h^3$. Therefore, $\frac{dV}{dt} = \frac{9\pi}{16}h^2\frac{dh}{dt}$. On the other hand, $\frac{dV}{dt} = \frac{12\pi}{(1+h)^2} - 2$, the rate at which water flows in minus the rate at which it flows out. Therefore,

$$\frac{9\pi}{16}h^2\frac{dh}{dt} = \frac{12}{(1+h)^2} - 2$$

When $h = 2$, this equation gives $\frac{dh}{dt} = -\frac{8}{27\pi}$. The negative indicates that the flow of water into the tank is much less at this height than the amount flowing out. To answer the second two questions, note that if $\frac{12}{(1+h)^2} - 2 > 0$ the height is rising. This is true when $12 > 2(1+h)^2$ or $\sqrt{6} - 1 > h > 0$. When $h = \sqrt{6} - 1$ the level stays constant. So the water will rise until it reaches this level and then stay at that level.

(9) Find the largest volume of a cylindrical can with radius r and height h which can be made using less than or equal to $96\pi\ ft^2$ of sheet aluminum (including both top and bottom).

The volume of the can is $\pi r^2 h$. The constraint amounts to requiring that $2\pi rh + 2\pi r^2 \leq 96$. The first term is the are of the curved surface (circumference of the base times the height) and the second term is the are of the top and bottom and bottom of the can. Using as much aluminum as we can will surely give greater volume, so we solve $2\pi rh + 2\pi r^2 = 96$ for h to get $h = \frac{\frac{48}{-}\pi r^2}{\pi r}$. Plugging into the volume formula gives $V(r) = r(48 - r^2) = 48r - r^3$. Note that to have positive volume we need $0 \leq r \leq \sqrt{48}$. Taking the derivative gives $V' = 48 - 3r^2$ and $V'' = -6r$. The critical point is $x = 4$ and $V''(4) < 0$ so it is a local maximum. Furthermore, as the endpoints of the interval give 0 volume, it must also be the absolute maximum. For $r = 4$, $V(r) = 4(48 - 16) = 4 \cdot 32 = 128\,ft^3$.