MATH 461: Homework #8

1) Let (X, d) be a *metric* space. Suppose $x_i \to x$ and $x_i \to y$ (note: it's the same sequence converging to both). Show that $x = y$. In other words, show that in a metric space the limits of sequences are unique. Note that we have already seen that this fails for a pseudo-metric space. You may assume the result from 1.b) of the previous homework.

2) Consider $X = \mathbb{R}^2$, and let

$$d_1(\vec{x}, \vec{y}) = |x_1 - y_1| + |x_2 - y_2| \tag{1}$$

$$d_2(\vec{x}, \vec{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \tag{2}$$

Show that: if a sequence $x_i \to x$ for one of these metrics, then $x_i \to x$ for *both* of these metrics.

3) Let X be the set of continuous, real-valued functions on $[-1, 1]$. Equip X with the metric $d(f, g) = \sup\{f(x) - g(x) | x \in [-1, 1]\}$ (so f and g are within r of each other if an only if $-r \leq f(x) - g(x) \leq r$ for every x in $[-1, 1]$). Let $K > 0$ and let M_K be the subset of X such that

$$|f(x) - f(y)| \leq K |x - y|$$

for every x and y in $[0, 1]$. Show that

1) M_K is closed.

2) $\bigcup_{K \geq 0} M_K$ is not closed. **Hint**: Try to find a sequence which limits to $f(x) = x^{\frac{1}{3}}$. This works because the tangent line at $x = 0$ is vertical for this function, and thus has infinte slope.