MATH 461: Homework #6

Let $(X,\, d)$ be a metric space. We say that $(X,\, d)$ is *bounded* if the "largest" distance you can find between any two points is finite. More technically, we require the following to be finite:

$$\sup\big\{d(x,y)\big|x,\, y\in X\big\}$$

Recall that the supremum of a set in $\mathbb{R}$ is the least upper bound of the set. This number is called the diameter of $(X,\, d)$ and is denoted $\operatorname{diam}(X,\, d)$. Being bounded says that for any two points $x,y\in X$ we have $0\le d(x,y)\le \operatorname{diam}(X,\, d)$.

1) Let $(X,\, d)$ be any metric space. Prove that $D(x,y)=\min\{2,d(x,y)\}$ is also a metric for X. Is (X,D) a bounded metric space? If it is, what is its diameter?

2) Let $(X_i,\, d_i)$ be a metric space for each $i\in\mathbb{N}$. Assume further that $\operatorname{diam}(X_i,\, d_i)=M$ for all i. Show that

$$X=\prod_{i\in\mathbb{N}}X_i \qquad\qquad d(\{x_i\},\{y_i\})=\sum_{i=1}^{\infty}\frac{d_i(x_i,y_i)}{2^i}$$

is a bounded metric space. What is its diameter?

3) A *pseudo-metric* space (X,p) is a set X and a function $p: X\times X\to\mathbb{R}$ such that

(1) $p(x,y)\ge 0$
(2) $p(x,y)=p(y,x)$
(3) $p(x,z)\le p(x,y)+p(y,z)$

In other words, like a metric space but without the condition that points must be a non-zero distance apart. You've already seen one of these: $\mathbb{R}^2$ with $p\big((x_1,x_2),(y_1,y_2)\big)=\big|x_1-x_2\big|$.

Let (X,p) be a pseudo-metric space.

a) Let $x,y\in X$ have $x\sim y$ if and only if $d(x,y)=0$. Show that $\sim$ is an equivalence relation on X.
b) Let Y be the set of equivalence classes under this equivalence relation, and let $[x]\in Y$ be the x's equivalence class. Show that (Y,d) is a metric space, where $d([x],[y])=p(x,y)$.

Note: If $x_i\to x$ in a pseudo-metric space, then $x_i\to y$ for every $y\in[x]$. That sequences can have more than one limit is a severe demerit and why pseudo-metric spaces aren't all that useful in themselves. However, they are used, as in this exercise, as a stepping stone to obtaining a true metric space.