MATH 461: Homework #5

You should look at the examples of metric spaces in section 2.2 of Sutherland.

1) Explain why the following are *not* metric spaces. Note all of the properties that fail.

a) $(\mathbb{R}^2, d)$ with $d\big((x_1, x_2), (y_1, y_2)\big) = \big|x_1 - y_1\big|$.

b) $(\mathbb{R}^2, d)$ with $d\big((x_1, x_2), (y_1, y_2)\big) = min\{\big|x_1 - y_1\big|, \big|x_2 - y_2\big|\}$

2) For the non-metric in part 1.a) above, we can still formulate a notion of limit: $x_i \to x$ if for each $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that $d(x_i, x) < \epsilon$ when $n > N$. To what points (there's more than one!) does the sequence $(\frac{1}{n}, 0)$ converge? What do the balls $B_r(x)$ look like for this space? (This is an example of a *pseudo-metric* space; more later).

3) Show that the following are metric spaces:

a) Let (X, d) be given by X a finite set of points ("cities") in $\mathbb{R}^2$, joined by a finite set of non-overlapping curves ("highways"). The highways should only intersect at cities. Assign to each highway a strictly positive length. We define $d(x, y)$ to be the shortest length of any route between two cities, x and y, found by adding the lengths of all the highways in the route. This works even when two cities are not joinable by highways("you can't get there from here"), if you say such cities are a very large distance apart (how large?).

b) On the rational numbers, $\mathbb{Q}$, define $|\,|_5 : \mathbb{Q} \to \mathbb{R}$ by writing $\frac{p}{q} = 5^s \frac{m}{n}$, where 5 does not divide m or n, and defining $|\frac{p}{q}|_5 = 5^{-s}$. For example,

$$|\tfrac{3}{50}|_5 = |5^{-2} \cdot \tfrac{3}{2}|_5 = 5^2$$
$$|\tfrac{375}{7}|_5 = 5^{-3}$$

Define $d(q_1, q_2)$ as $|q_1 - q_2|_5$. Show that this is a metric. (Rational numbers are close if they differ by a number divisible a large number of times by 5. This can be done for any prime in a similar way. These metrics play an important role in the number theory.)