MATH 461: Homework #4

1) Use the Cantor-Schroeder-Bernstein theorem to show that the following sets are all equivalent to $\mathbb{R}$

a) $[0,1]$
b) (a,∞)
c) $\{\,(x,y)\in\mathbb{R}^2 \mid x^2+y^2=1\,\}$

Note: All *intervals* in $\mathbb{R}$ are equivalent to $\mathbb{R}$. An interval is any set of the form (a,b), $[a,b]$, $(a,b]$, $[a,b)$, and (a,∞), $[a,\infty)$, $(-\infty,a)$, $(-\infty,a]$.

2) Show that $A\sim B$ and $C\sim D$ implies $A\times C\sim B\times D$.

Note: We use this to prove that $\mathbb{R}^2\sim\mathbb{R}$. First we prove $(0,1)^2\sim(0,1)$ using the CSB theorem. Let $(x,y)\in(0,1)^2$ and write x and y as infinite decimals, neither ending in repeating 9's. Now define a new decimal by alternating between the entries in the expansions of x and y. This defines a map $f:(0,1)^2\to(0,1)$. (i.e. the $2i-1$ entry in $f(x,y)$ is the i^{th} entry in x, the $2i^{th}$ entry in $f(x,y)$ is the i^{th} entry in y). This map is $1-1$ and thus can be used in the CSB theorem. The map in the other direction is easier to obtain, and I'll leave it up to you.

3) Here's one of the great applications of set theory to mathematics. It requires a slight preamble to explain its importance. You know about rational numbers, and their complement in $\mathbb{R}$, the irrational numbers. There are also different types of irrationals. For example, $\sqrt{2}$ satisfies an equation $x^2-2=0$. Does π satisfy such an equation? This gave rise to the following definition. The set of *algebraic numbers* is

$$\mathbb{A}=\left\{\,r\in\mathbb{R} \mid r \text{ satisfies } a_nx^n+a_{n-1}x^{n-1}+\ldots a_1x+a_0=0,\ \text{for } n\in\mathbb{N}, a_i\in\mathbb{Z}\,\right\}$$

In other words, those numbers which are roots to polynomials with *integer* coefficients. Any rational number is algebraic, since it is the root of a linear equation: $qx-p=0$. We've just seen that $\sqrt{2}$ is algebraic. Are all numbers in $\mathbb{R}$ algebraic?

The answer is NO! and you are now going to prove it. Show that $\mathbb{A}$ is countable. You will need to use the facts about the products and unions of countable sets. First show that the roots to n^{th} degree polynomials constitute a countable set and then take a union.

Since $\mathbb{A}$ is countable and $\mathbb{A}\subset\mathbb{R}$, the non-algebraic numbers, $\mathbb{R}-\mathbb{A}$ is not countable (why?). In other words, there are far more non-algebraic numbers than algebraic. However, it is very, very hard to prove that a given number, say π, is non-algebraic. In fact, e and π are both non-algebraic. Non-algebraic numbers are called *transcendental* numbers. Their existence was known before set theory was invented, but only through difficult arguments. With set theory one could establish their existence much more easily.

On Monday, we will begin metric spaces. It might be useful to look at chapter 1 of Sutherland.