MATH 461: Homework #27

One way to look at components in $(X, \mathcal{T})$ is to say $x \sim y$ if there is a connected subspace E_{xy} such that $x, y \in E_{xy}$. This is an equivalence relation whose equivalence classes are the components of X. (You should check this!).

To get at the question of when components are open, another equivalence relation has been defined. Let $x \approx y$ if there is no decomposition of $X = H \cup K$, with H, K disjoint, non-empty open sets, and $x \in H$, $y \in K$. Call an equivalence class under this relationship a "quasi-component". Prove

(1) $\approx$ is an equivalence relation on X.
(2) Let C_x be the component of x and Q_x be the quasi-component. Show that $C_x \subset Q_x$.
(3) Suppose for $x \in X$, C_x is open, then $C_x = Q_x$.
(4) Consider the subspace of $\mathbb{R}^2$ defined as the union of the lines $y = 0$, $y = 1$ and the rectangles with vertices at $(\pm n, \frac{1}{n})$ and $(\pm n, 1 - \frac{1}{n})$ for $n \geq 1$. Consider the point $(0, 1)$. Show that $C_{(0,1)}$ is the line $y = 1$, but that $Q_{(0,1)}$ also contains $\{y = 0\}$.