MATH 461: Homework #16

No additional problems, only an argument to read. Make sure you understand why everything in the following is true.

Let $(X_i,\, d_i)$, $i \in \mathbb{N}$ be metric spaces with $diam(X_i, d_i) = M$. We have seen that

$$\prod_{i \in \mathbb{N}} X_i \qquad\qquad d(\{x_i\}, \{y_i\}) = \sum_{i=1}^{\infty} \frac{d_i(x_i, y_i)}{2^i}$$

is a metric space. Here we show that the metric topology is the product topology. This uses the following lemma:

Lemma: Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on X, with bases $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively. Suppose for each $p \in X$ and $\mathcal{O} \in \mathcal{B}_1$ with $p \in \mathcal{O}$, there is a $\mathcal{U} \in \mathcal{B}_2$ such that $p \in \mathcal{U} \subset \mathcal{O}$. Then $\mathcal{T}_1 \subset \mathcal{T}_2$.

A base for the product topology on $\prod_{i \in \mathbb{N}} X_i$ is

$$\{\prod_{i=1}^{n} B_{r_i}(x_i) \times \prod_{i=n+1}^{\infty} X_i\}$$

where we include every such set with $n > 0$, $\{x_i\} \in \prod X_i$ and $r_i > 0$. Choose such a basic open set, B, around $\{x_i\}$ and let $\epsilon < min\{r_1/2, \ldots, r_n/2^n\}$. Then $B_\epsilon(\{x_i\})$ contains $\{x_i\}$ and is contained in B. This follows since $d(\{x_i\}, \{y_i\}) < \epsilon$ implies that $d_i(x_i, y_i)/2^i < \epsilon$ for each i. Thus, if $i < n+1$ we will have $d_i(x_i, y_i) < r_i$. Note that since B includes the whole factor for $i > n$, we don't need to check any requirement for $i > n+1$. Using the lemma, we have that $\mathcal{T}_{prod} \subset \mathcal{T}_{metric}$.

For the other direction, we need to find such a basic open set for the product topology inside $B_\epsilon(\{x_i\})$. We do this by choosing N such that $\frac{\epsilon}{2} > \frac{M}{2^N}$. Then

$$B = \{\prod_{i=1}^{N} B_{\frac{\epsilon}{2N}}(x_i) \times \prod_{i=N+1}^{\infty} X_i\}$$

is contained in $B_\epsilon(\{x_i\})$. Let $x = \{x_i\}$ and $y = \{y_i\}$ be any point other in B. Let $z = \{x_i\}_{i=1}^{N} \times \{y_i\}_{N+1}^{\infty}$, i.e. the point whose first N coordinates are those of $\{x_i\}$, and whose remaining coordinates are those of $\{y_i\}$. Then $d(x, z) = \sum_{i>N} d(x_i, y_i)/2^i \le M \sum_{i>N} 2^{-i} < \epsilon/2$ and $d(z, y) = \sum_{i=1}^{N} d_i(x_i, y_i)/2^i < N\frac{\epsilon}{2N}$. Therefore, B is contained in $B_\epsilon(x)$ and $\mathcal{T}_{metric} \subset \mathcal{T}_{prod}$.