MATH 461: Homework #10

1) Let $A \subset (X, d)$. Let $d|_A$ be the metric on A found by restricting that of (X, d), i.e. $d|_A(a_1, a_2) = d(a_1, a_2)$. Prove the following:

(1) If $B^A_r(a)$ is the open ball of radius r around a in A, then $B^A_r(a) = B^X_r(a) \cap A$.
(2) A subset $O \subset A$ is open in A if and only if there is an open set $\mathcal{O} \subset X$ such that $O = \mathcal{O} \cap A$.

You should use the result about open balls to prove the more general case.

2) Let (X_i, d_i) be a metric space for each $i \in \mathbb{N}$. Assume further that $\text{diam}(X_i, d_i) = M$ for all i. Let

$$X = \prod_{i \in \mathbb{N}} X_i \qquad\qquad d(\{x_i\}, \{y_i\}) = \sum_{i=1}^{\infty} \frac{d_i(x_i, y_i)}{2^i}$$

In an earlier homework, you showed that this is a metric space. Choose $r_i < \frac{M}{2}$ for $i \in \mathbb{N}$. Show that the subset

$$\prod_{i \in \mathbb{N}} B_{r_i}(x_i) \subset X$$

is *not* a neighborhood of $\{x_i\} \in X$. Note that $B_{r_i}(x_i)$ is the ball of radius r_i around x_i in (X_i, d_i).

Ok, so here's the situation for first countable topological spaces. First, the definition

Definition A topological space $(X, \mathcal{T})$ is first countable if for every $x \in X$ there exists a countable collection of open sets $\mathcal{N}_x = \{V_i \in \mathcal{T} | x \in V_i, i \in \mathbb{N}\}$ such that for every neighborhood N of x, there is at least one $V_i \subset N$.

First, we can assume these are nested. If they are not, let $V'_i = \cap_{j \leq i} V_i$. Then $V'_1 = V_1$, $V'_2 = V_1 \cap V_2 \subset V_i$, and $V'_i \subset V_i$ for all i. Since $V'_i = V_i \cap V'_{i-1}$, these are nested: $V'_{i-1} \supset V'_i$. Furthermore, since any neighborhood of $x \in X$ contains one of the V_i, it will also contain V'_i.

So, let's say we have a nested collection $V_1 \supset V_2 \supset V_3 \supset \ldots$ of open sets about x, such that any other neighborhood contains at least one of these sets. It will also contain all the sets with larger index. Furthermore, we may assume that $V_i \neq V_{i+1}$, or else throw out one of these two sets (and relabel the indices for everything following). If we pick points $x_i \in V_i - V_{i+1}$, then $x_i \to x$. For if N is a neighborhood of x, then $V_i \subset N$ for some i. Indeed, $V_j \subset N$ for all $j \geq i$, and thus $x_j \in N$ for all $j \geq i$ as well. This is all we need to see that $x_i \to x$.

Here's the point that caused me difficulty in class. I gave you a way to find a subcollection of the V_i that would be nested, even if the V_i weren't; however, what I didn't do correctly was to ensure that this subcollection would still be a neighborhood base. The construction above resolves that. Since my collection was not necessarily a neighborhood base, I had difficulty seeing that $\cap V_i$ could not contain an open set (if it did, then the sequence I was trying to construct might not wind up in that set, and then we wouldn't have convergence). This is not a problem if the V_i form a neighborhood base, since any such open set, being a neighborhood of x, must contain a V_i and thus could not be contained in $\cap V_i$ unless the neighborhood base were finite and the open set equal to the smallest open set in the base.

The case of a finite base is actually taken care of in the argument above, but if there is only finitely many V_i – say V_n is the last – then pick $x_j = x_n$ for all $n \geq j$. Since every neighborhood contains a V_i, it must contain V_n. Since the sequence eventually stays in V_n, it must stay in any other neighborhood, and thus converges to x.

If the list V_i is infinite, the set $H = \cap_{i \in \mathbb{N}} V_i$ is interesting. This set must contain x, and must also be contained in any neighborhood of x since $H \subset V_i$ for all i. This also shows that it is not a neighborhood of x, since it would then contain a V_i and thus be equal to V_i making the neighborhood base finite. What are the possibilities for such a set? When we get to topological spaces we will discuss some axioms. In one of these we will assume that given any two distinct points in the space, there are at least two open sets, each a neighborhood of one of the points, but not the other. With this assumption H can only be $\{x\}$ since if there were another point in H we could find an open set containing x and not the other point, and hence a V_i containing x and not the other point.

As an example of more complicated behavior, consider the pseudo-metric $|x_1 - y_1|$ on $\mathbb{R}^2$ (which you've seen several times). Any neighborhood of $(0,0)$ contains the set of points within $\frac{1}{n}$ of $(0,0)$ for some n. But this is a vertical strip, as you've shown, and the intersection of all such vertical strips is the y-axis.

In a first countable space, we can now show "if every neighborhood of x intersects A, then there is a sequence $a_i \in A$ such that $a_i \to x$". This was the problem from class. Pick a nested countable neighborhood base, $\{V_i\}$, for x. Since $A \cap V_i \neq \emptyset$ choose $a_i \in A \cap V_i$. First $a_i \in A$, but since $a_i \in V_i$ for all i, the argument above shows that $a_i \to x$ as well.