MATH 461: Some general constructions of metric spaces

Here are two nice ways to construct a metric space. The first is similar to the facts used in class. The second is a very general approach, employed frequently in analysis.

I. Let (X, d) be a metric space and let $f : [0, \infty) \to \mathbb{R}$ be a function. Assume that $f(0) = 0$ and $f(x) > 0$ for $x > 0$. Assume further that $x \leq y$ implies $f(x) \leq f(y)$ (i.e. f is increasing). Since $d : X \times X \to \mathbb{R}$ we can compose with f to form a new map $D_f : X \times X \to \mathbb{R}$ given by $D_f(x, y) = f(d(x, y))$. The conditions on f ensure that $D_f(x, y) \geq 0$ and $D_f(x, y) = 0$ implies that $x = y$. If we add one further condition we can make D_f into a metric. That condition is concavity: if a, b are real numbers with $a + b = 1$ assume that $f(ax + by) \geq af(x) + bf(y)$. The graph of such a function lies above any line segment between two points on the graph. This property allows us to obtain the triangle inequality. With a little effort, you can prove all this; it's just a bit tricky at the end.

When f is twice differentiable, $f'' \leq 0$ for all x implies that f is concave (so it's what we call concave down in calculus). This allows us to prove that $\sqrt{d(x, y)}$ and $\frac{d(x,y)}{1+d(x,y)}$ are also metrics. The last is a bounded metric with diameter 1.

II. In analysis one often encounters the following scenario. Let V be a vector space over $\mathbb{R}$. As a reminder, this is a set with two operations: addition of vectors and multiplication by scalars ($\mathbb{R}$ in this case) that mimics the properties of vectors in $\mathbb{R}^3$. If you need a more detailed definition, come and ask me. In analysis, these often come with a map $|| \cdot || : V \to \mathbb{R}$ with the properties that

(1) $||v|| \geq 0$ for all $v \in V$
(2) $||v|| = 0$ if and only if $v = 0$
(3) $||\lambda \cdot v|| = |\lambda| \, ||v||$ for any $\lambda \in \mathbb{R}$.
(4) $||v_1 + v_2|| \leq ||v_1|| + ||v_2||$ for all $v_1, v_2 \in v$

A map satisfying these properties is called a norm and is a generalization of the absolute value function on $\mathbb{R}$. A pair $(V, || \cdot ||)$ is called a normed vector space.

If these properties seem similar to those of a metric, that's because $d(v_1, v_2) = ||v_1 - v_2||$ will be a metric when $|| \cdot ||$ is a norm. The third property of the norm can be interpretted geometrically as saying that multiplication by λ stretches distances by a factor of $|\lambda|$.

An example of a normed vector space is $\mathbb{R}^n$ with $||x||_1 = |x_1| + |x_2| + \ldots + |x_n|$. The metric we obtain from this norm is one of the three we discussed in class. Two other norms on $\mathbb{R}^n$ are $\sqrt{\sum_{i=1}^{n} x_i^2}$ and $max_i\{|x_i|\}$. These give the other two metrics.

An easy way to get a norm is to use an inner product. There are norms which do not get constructed this way, but it's so common its worth mentioning. An inner product is a bilinear map $V \times V \to \mathbb{R}$, denoted $\langle v, w \rangle$, such that 1) $\langle v, v \rangle \geq 0$ and equals 0 only when $v = 0$ and 2) $\langle v, w \rangle = \langle w, v \rangle$. These properties are chosen to mimic the dot product from multi-variable calculus. These properties ensure that $||v|| = \sqrt{\langle v, v \rangle}$

is a norm. The fourth property defining a norm is derived from the Cauchy-Schwarz inequality:

$$\langle v, w\rangle \leq ||v||\,||w||$$

by considering $\langle v+w, v+w\rangle$ and using the bilinearity. Of the three norms on $\mathbb{R}^n$ given above, only $\sqrt{\sum_{i=1}^{n} x_i^2}$ is constructed from an inner product.

Here's an example of the full procedure: Let V be the continuous functions $f : [-1,1] \to \mathbb{R}$. Addition is defined by $(f+g)(x) = f(x)+g(x)$, and multiplication by scalars is defined by $(\lambda \cdot f)(x) = \lambda(f(x))$. V is then a vector space. Let's say you care more about the size of the function near zero than at the endpoints. We can encode this through the following inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\big(1-x^2\big)\,dx$$

It requires a little work to see that this is an inner product (but it is). So we would then have a norm

$$||f|| = \sqrt{\int_{-1}^{1} f(x)^2\big(1-x^2\big)\,dx}$$

and a metric

$$d(f,g) = \sqrt{\int_{-1}^{1} \big(f(x)-g(x)\big)^2\big(1-x^2\big)\,dx}$$

Note that the distance between the functions is heavily influenced by their difference near 0 and not at all influenced by their difference at the endpoints since $1-x^2$ vanishes at the endpoints regardless of the values of f and g. Thus we have a metric with the desired property. What function to choose as $1-x^2$ depends upon the exact nature of how you wish to weight the influence of each point in $[-1,1]$.

All the examples from lecture so far can be constructed by choosing the right V and $||\cdot||$. Not all come from an inner product, however.