Section 5.3

Exercise 5.3.1: Let $f : [a,b] \to \mathbb{R}$ be differentiable with $f'(x)$ continuous on $[a,b]$. Since $f'(x)$ is continuous on $[a,b]$ we know that $|f'(x)|$ is continuous as well, since it is the composition of continuous functions. Since $[a,b]$ is compact and $f'(x)$ is continuous, there is a point $c \in [a,b]$ where $|f'(c)| = M$ is a maximum. Now, the conditions on $f(x)$ allow an application of the mean value theorem to $f(x)$ on any interval $[d,e] \subset [a,b]$. This tells us there is another point $c' \in [d,e]$ with

$$\frac{f(e)-f(d)}{e-d} = f'(c')$$

In absolute value, $|f'(c')| \leq M$ since M is the maximum. But this implies that for any $d < e$ in $[a,b]$ we have

$$\left|\frac{f(e)-f(d)}{e-d}\right| \leq M$$

Changing the order of e and d will not change the term on the left, so this implies that for any $x, y \in [a,b]$ we will have the desired conclusion (let $d = x$ and $e = y$). Hence $f(x)$ is Lipschitz.

Exercise 5.3.3: (a) Since $h(x)$ is differentiable on $[0,3]$ we know that it is also continuous. Thus $f(x) = h(x) - x$ is continuous as well on that interval. But $f(0) = -1$, $f(1) = -1$ and $f(3) = 1$ using the values in the problem. By the intermediate value theorem there is a point $d \in [1,3]$ where $f(d) = 0$ and thus $h(d) = d$.
(b) The conditions on $h(x)$ allow us to apply the mean value theorem on the interval $[0,3]$ since the function is continuous there and differentiable on $(0,3)$ (in fact on a larger set). But $h(3) - h(0) = 2 - 1 = 1$ and $3 - 0 = 3$. The mean value theorem on $[0,3]$ thus ensures us that there is a point $c \in (0,3)$ with $h'(c) = \frac{1}{3}$
(c) First apply the mean value theorem on $[1,3]$ to find a point $e \in (1,3)$ with $h'(e) = 0$. We know that $e \neq c$, where c is the point found in part b, since the derivative takes different values at these points. Then Darboux's theorem tells us that there is a point between e and c where h' must equal $\frac{1}{4}$ since this value is between $h'(e)$ and $h'(c)$, and derivatives have the intermediate value property on closed intervals where they are defined.

Exercise 5.3.5: Suppose that $f(x)$ has two fixed points, d and e with $d < e$, in an interval where $f'(x)$ is defined and $f'(x) \neq 1$. Then $f(d) = d$ and $f(e) = e$. In addition, $f(x)$ is continuous on $[d,e]$ since it is differentiable on $[d,e]$. Furthermore, $f(x)$ is differentiable on (d,e) as well. The mean value theorem applies on $[d,e]$ and guarantees the existence of a point $c \in (d,e)$ with

$$f'(c) = \frac{f(e)-f(d)}{e-d} = \frac{e-d}{e-d} = 1$$

which contradicts that $f'(x) \neq 1$ on the interval containing $[d,e]$.

Exercise 5.3.7:(a) Let $f : (a,b) \to \mathbb{R}$ be an increasing function that is also differentiable on (a,b). If $c \in (a,b)$ then

$$f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c}$$

Let $x_n \to c$ with $x_n > c$, then we have that $f(x_n) - f(c) \geq 0$ since the function is increasing and $x_n - c > 0$. Since the limit above exists, we must have $f'(c) = \lim_{n\to\infty} \frac{f(x_n)-f(c)}{x_n - c} \geq 0$ since the fraction in this limit is ≥ 0. Thus, for each point $c \in (a,b)$ we must have $f'(c) \geq 0$. Now assume that $f'(c) \geq 0$ for each $c \in (a,b)$. To show that the function is increasing, we need to know that $x < y$ with $x, y \in (a,b)$ implies $f(x) \leq f(y)$. But f on $[x,y]$ is differentiable and thus continuous, so we may apply the mean value theorem to conclude that there is a $c \in (x,y)$ with

$$f'(c) = \frac{f(y)-f(x)}{x-y}$$

Since $c \in (x,y) \subset (a,b)$ we have, by assumption, that $f'(c) \geq 0$. Furthermore, $y > x$ makes the denominator positive. Thus the numerator must be ≥ 0, or $f(y) \geq f(x)$.

(b) For $g(x)$ in the statement of the problem, we calculate

$$g'(0) = \lim_{x \to 0} \frac{\frac{x}{2} + x^2 \, sin\left(\frac{1}{x}\right) - 0}{x - 0} = \lim_{x \to 0} \frac{1}{2} + x \, sin\left(\frac{1}{x}\right) = \frac{1}{2}$$

Thus $g'(0) > 0$. However, the function is not increasing on any interval $(-\delta, \delta)$ containing 0. We see this by computing the derivative of $g(x)$ on $(-\infty, 0) \cup (0, \infty)$ to get

$$g'(x) = \frac{1}{2} + 2x \, sin\left(\frac{1}{x}\right) - cos\left(\frac{1}{x}\right)$$

If this is negative at any point in $(-\delta, \delta)$ then the function is not increasing, as shown in the previous part of the problem. So we choose points where $cos\left(\frac{1}{x}\right)$ is as big as possible, namely equal to 1. This occurs when $\frac{1}{x} = 2k\pi$ for $k \in \mathbb{Z}$, i.e. for $x_k = \frac{1}{2k\pi}$. If we let $k \in \mathbb{N}$, then we obtain a sequence $x_k \to 0$ which must then enter $(-\delta, \delta)$. On the other hand, $sin\left(\frac{1}{x_k}\right) = sin(2k\pi) = 0$ for all $k \in \mathbb{N}$, so the middle term will be 0 for each of these points. Thus $g'(x_k) = \frac{1}{2} + 0 - 1 = -\frac{1}{2} < 0$. For k large enough this point will be in $(-\delta, \delta)$.

Exercise 5.3.8: Let $g : (a, b) \to \mathbb{R}$ be differentiable at a point $c \in (a, b)$. We assume that $g'(c) > 0$ (the case where $g'(c) < 0$ will be done below). We *cannot* use the mean value theorem since we only know that the function is differentiable at a single point. Instead we use the definition of the derivative. Since

$$0 < g'(c) = \lim_{x \to c} \frac{g(x) - g(c)}{x - c}$$

there is an $\epsilon > 0$ such that $g'(c) - \epsilon > 0$ and a $\delta > 0$ such that $0 < |x - c| < \delta$ implies

$$\left| \frac{g(x) - g(c)}{x - c} - g'(c) \right| < \epsilon$$

This further implies that

$$\frac{g(x) - g(c)}{x - c} > g'(c) - \epsilon > 0$$

When $0 < x - c < \delta$ we have that $g(x) > g(c)$ in order for the fraction to be bigger than 0. When $0 > x - c > -\delta$ we have that $g(x) < g(c)$ so that both numerator and denominator will be negative. In either case, there is a δ-neighborhood of $x = c$ in which $g(x) \neq g(c)$ for $x \neq c$. To address the case where $g'(c) < 0$, let $h(x) = -g(x)$. Then $h'(c) = -g'(c) > 0$. Thus there is a neighborhood of c where $-g(x) \neq -g(c)$ unless $x = c$. Multiplying by -1 yields $g(x) \neq g(c)$ on the same neighborhood.

To relate to the previous problem: that $g'(0) = \frac{1}{2}$ in 5.3.7 tells us that there is no other point in a sufficiently small neighborhood $(-\delta, \delta)$ with $g(x) = g(0) = 0$. In fact, all points with $x \in (0, \delta)$ will have $g(x) > 0$ and all points $x \in (-\delta, 0)$ will have $g(x) < 0$. However, there is still room for the function to increase and decrease within these conditions, as seen from 5.3.7 (b).