This assignment is slightly different. I am going to give you an argument and intersperse it with a couple of conceptual problems. Your job is to read the argument and try to answer any questions that will occur to you, noting those points which are confusing so we can discuss them. This likely will be very challenging, as it's the hardest topic we'll cover; we will discuss the argument next Monday, so if you get stuck don't worry.

Consider the vector field

$$\vec{F}(x,y,z) = P(x,y,z)\,\vec{\imath}+Q(x,y,z)\,\vec{\jmath}+R(x,y,z)\,\vec{k}$$

defined on all of $\mathbb{R}^3$, and such that the partial derivatives of P, Q, R are continuous. Let $\mathcal{R}$ be a simply connected region in $\mathbb{R}^2$ with boundary γ, to which Green's theorem applies, and let $\mathcal{S}$ be $\{(x,y,f(x,y))|(x,y)\in\mathcal{R}\}$ with boundary $C = \{(x,y,f(x,y)|(x,y)\in\gamma\}$ where $f(x,y)$ is a function with at least continuous second partial derivatives.

1) Draw a picture!! for example draw the case $\mathcal{R} = \{x^2+y^2 \leq 9\}$ and $f(x,y) = x^2+y^2$ and label all the pieces.

Let's compute

$$\oint_C \vec{F} \bullet d\vec{r}$$

To fully define this line integral we need to choose a direction to travel around the boundary. We will do this by choosing a parameterization. Let $(x(t), y(t)), 0 \leq t \leq 1$ be a parameterization of γ which travels counter-clockwise around $\mathcal{R}$ (So that we can use it in Green's theorem). Then $(x(t), y(t), f(x(t), y(t)))$ $0 \leq t \leq 1$ is a parameterization of C.

2) Now, show that

$$\oint_C \vec{F} \bullet d\vec{r} = \int_0^1 P\,x'(t) + Q\,y'(t) + R\big(\frac{\partial f}{\partial x}\,x'(t) + \frac{\partial f}{\partial y}\,y'(t)\big)\,dt$$

3) Explain why this is just

$$\oint_\gamma \big(P' + R'\frac{\partial f}{\partial x}\,\big)\,dx + \big(Q' + R'\frac{\partial f}{\partial y}\,\big)\,dy$$

where $P'(x,y) = P(x,y,f(x,y))$, $Q'(x,y) = Q(x,y,f(x,y))$, $R'(x,y) = R(x,y,f(x,y))$.

If we apply Green's theorem we get

$$\oint_C \vec{F} \bullet d\vec{r} = \iint_{\mathcal{R}} \frac{\partial}{\partial\,x}\left(Q' + R'\frac{\partial f}{\partial y}\right) - \frac{\partial}{\partial\,y}\left(P' + R'\frac{\partial f}{\partial x}\right)\,dx\,dy$$

4) Show that

$$\frac{\partial}{\partial\,x}\left(Q' + R'\frac{\partial f}{\partial y}\right) = \frac{\partial Q}{\partial x} + \frac{\partial R}{\partial x}\frac{\partial f}{\partial y} + \frac{\partial Q}{\partial z}\frac{\partial f}{\partial x} + R\frac{\partial^2 f}{\partial x \partial y}$$

and

$$\frac{\partial}{\partial\,y}\left(P' + R'\frac{\partial f}{\partial x}\right) = \frac{\partial P}{\partial y} + \frac{\partial R}{\partial y}\frac{\partial f}{\partial x} + \frac{\partial P}{\partial z}\frac{\partial f}{\partial y} + R\frac{\partial^2 f}{\partial y \partial x}$$

(keep in mind the definition of $P'(x,y), Q'(x,y), R'(x,y)$ and the chain rule!!) Use this to simplify the double integral one gets from Green's theorem. Put a star besides the result, and put to one side.

The unit normal to $\mathcal{S}$ at $(x, y, f(x,y))$ is

$$\vec{n} = \frac{\vec{r}_x \times \vec{r}_y}{||\vec{r}_x \times \vec{r}_y||}$$

where we let $\vec{N} = \vec{r}_x \times \vec{r}_y = \vec{k} - \frac{\partial f}{\partial x}\,\vec{\imath} - \frac{\partial f}{\partial y}\,\vec{\jmath}$ (check this).

5) Compute $\nabla \times \vec{F}$ and $\left(\nabla \times \vec{F}\right) \bullet \vec{N}$. Compare this last expression to what's inside the double integral arising in Green's theorem (the one you placed a star beside). (Hint: these should be equal!)

6) Now rewrite the surface integral:

$$\int\!\!\int_S \left(\nabla \times \vec{F}\right) \bullet \vec{n}\, d\sigma$$

as an integral over $\mathcal{R}'$. Compare with that arising in Green's theorem and conclude that

$$\oint_C \vec{F} \bullet d\vec{r} = \int\!\!\int_{\mathcal{S}} \left(\nabla \times \vec{F}\right) \bullet \vec{n}\, d\sigma$$

This is a simple version of Stokes' Theorem.

7) Suppose that $f(x,y) = 0$, what does Stokes' theorem become?

If you get here, give yourself a pat on the back.