A Survey of Regularization Methods for First-Kind Volterra Equations

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Abstract. We survey continuous and discrete regularization methods for first-kind Volterra problems with continuous kernels. Classical regularization methods tend to destroy the non-anticipatory (or causal) nature of the original Volterra problem because such methods typically rely on computation of the Volterra adjoint operator, an anticipatory operator. In this survey we pay special attention to particular regularization methods, both classical and nontraditional, which tend to retain the Volterra structure of the original problem. Our attention will primarily be focused on linear problems, although extensions of methods to nonlinear and integro-operator Volterra equations are mentioned when known.

1 Introduction

Consider the Volterra equation of first kind

$$\int_0^t k(t,s)u(s)\,ds = f(t), \quad t \in [0,T], \tag{1}$$

where the kernel k is a continuous function on $[0,T]\times[0,T]$. If k is a convolution kernel, then $k(t,s) = \kappa(t-s)$ for some continuous function κ on $[0,T]$.

We will assume throughout that the data f is such that there exists a unique solution $\bar{u} \in U \equiv L_2(0,T)$ of equation (1), and, in particular, we require that $f(0)=0$. For simplicity we will consider all quantities to be real-valued. Additional regularity on f, k, and $\bar{u}$ may be required in the results which follow.

It will often be useful to write equation (1) in the form

$$Au = f, \tag{2}$$

where the operator A is defined for $u \in U$ by

$$Au(t) := \int_0^t k(t,s)u(s)\,ds, \quad \text{a.e. } t \in [0,T]. \tag{3}$$

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Then $A \in \mathcal{L}(U)$, the space of continuous linear operators from U to U.

If the kernel k is non-degenerate, then the range $\mathcal{R}(A)$ of A is non-closed in U. This means that equation (1) is *ill-posed*, which has serious implications in the usual case where we only have available an approximation $f^\delta \in U$ of f; here f^δ satisfies $\|f - f^\delta\| \le \delta$ for some $\delta > 0$, where throughout $\|\cdot\|$ will denote the usual norm on $U = L_2(0,T)$. The ill-posedness means that the solution u^δ of $Au = f^\delta$ (when such a solution exists) may be arbitrarily far from the solution $\bar{u}$ of the unperturbed problem (1). Therefore, some kind of regularization procedure will be needed to solve the problem in the case of perturbed data f^δ.

1.1 Degree of Ill-Posedness of First-Kind Volterra Problems

For sufficiently smooth f and k, we may differentiate equation (1) with respect to t to obtain

$$k(t,t)u(t) + \int_0^t \frac{\partial}{\partial t} k(t,s)u(s)\, ds = f'(t), \quad t \in [0,T]. \tag{4}$$

If $k(t,t) \neq 0$, for $t \in [0,T]$, division of equation (4) by $k(t,t)$ yields a standard Volterra equation of the second kind which is known to be a *well-posed* problem. In particular, the solution of equation (4) depends continuously on the "data" in (4), which (for the new equation) is the function f'.

If $k(t,t) = 0$, $t \in [0,T]$, then we may repeat the process by differentiating the equation once again, this time resulting in a second-kind Volterra equation (with "data" f'') provided $\partial k(t,t)/\partial t \neq 0$ for $t \in [0,T]$. We generalize this idea with the following where, without loss of generality, we will normalize k.

Definition 1. *We will say that the Volterra operator A is* ν-smoothing *for integer $\nu \ge 1$ if the kernel k is such that $\left(\partial^\ell k/\partial t^\ell\right)(t,t) = 0$, for $0 \le t \le T$ and $\ell = 0, \dots, \nu-2$, and such that $\left(\partial^{\nu-1} k/\partial t^{\nu-1}\right)(t,t) = 1$, for $0 \le t \le T$, with $\partial^\nu k/\partial t^\nu$ continuous on $[0,T] \times [0,T]$.*

We will say that the Volterra problem $Au = f$ is a ν-smoothing problem *if the operator A is a ν-smoothing operator and $f \in C^\nu[0,T]$.*

Thus, if (1) is a ν-smoothing problem, we may differentiate equation (1) ν times to obtain

$$u(t) + \int_0^t \frac{\partial^\nu}{\partial t^\nu} k(t,s)u(s)\, ds = f^{(\nu)}(t), \quad t \in [0,T], \tag{5}$$

an equivalent problem to equation (1) in this case (provided $f(0) = f'(0) = \cdots = f^{(\nu-1)}(0) = 0$).

Definition 2. *We will say that the Volterra operator A is* infinitely-smoothing *if $\left(\partial^\ell k/\partial t^\ell\right)(t,t) = 0$, for $0 \le t \le T$ and all $\ell = 0, 1, 2, \dots$.*

We will say that the Volterra equation $Au = f$ is an infinitely-smoothing problem *if the operator A is a infinitely-smoothing operator.*

Of course, not all equations of the form (1) fall into precisely one of the above classes of problems. Nevertheless, these definitions serve to characterize a wide class of Volterra problems and begin a discussion on the "degree" of ill-posedness of particular first-kind problems.

Among the class of problems defined above, a *one-smoothing problem* is the least ill-posed since it requires only one differentiation of the data (which, in general, is an ill-posed operation itself since perturbations f^δ of f need not be differentiable). In fact, the canonical one-smoothing convolution problem is the "differentiation problem" $u = f'$, or the problem of finding u solving the equation

$$\int_0^t u(s)\,ds = f(t), \quad t \in [0, T]. \tag{6}$$

for sufficiently smooth f. The j^{th} singular value σ_j of the operator A in this case satisfies $\sigma_j = \mathcal{O}(1/j)$ as $j \to \infty$.

Similarly, the canonical ν-smoothing convolution problem is associated with the convolution kernel $\kappa(t) = t^{\nu-1}/(\nu-1)!$, for which the associated Volterra operator has singular values $\sigma_j = \mathcal{O}(1/j^\nu)$ as $j \to \infty$. Thus, the degree of ill-posedness of ν-smoothing problems increases with increasing ν. See [4] for general information about degree of ill-posedness. See also [5, 6] for a discussion of singular values for ν-smoothing Volterra operators and the way in which these singular values are representative (asymptotically) of a class of more general ν-smoothing problems of convolution type.

A classic example of an infinitely-smoothing problem is the inverse heat conduction problem (IHCP), or sideways heat equation, one formulation of which is as follows. Given an insulated semi-infinite bar on the non-negative x-axis, an unknown heat source $u = u(t)$ is applied to the end of the bar at $x = 0$. The goal is to determine the value of u, given measurements $f = f(t)$, $0 \le t \le T$, of the temperature at position $x = 1$ of the bar. The unknown heat source u is the solution of the first-kind equation (1), where the kernel k is of convolution type, $k(t, s) = \kappa(t - s)$, with

$$\kappa(t) = \frac{1}{2\sqrt{\pi}\, t^{3/2}} e^{-1/4t}, \quad t \in [0, T].$$

Because we can rarely obtain f exactly, the solution of the IHCP with perturbed data f^δ is a severely ill-posed problem.

Remark 1. A number of the regularization methods we describe below will be suitable for application to the IHCP. However, the specialized nature of the problem also suggests other particularly effective methods, some taking advantage of its equivalent formulation as an partial differential equation problem with overspecified data. As our focus here will be on methods applicable to more general Volterra equations of the form (1), a discussion of methods specific to the IHCP is beyond the scope of this paper. Instead, the reader is encouraged to consult surveys by Beck, Blackwell, and St. Clair, Jr. [16], Eldén [17] (as well as other papers in the collection [19]), Murio, [81], Alifanov [15], Hào and

Reinhardt [22], Kurpisz and Nowak [25], and the many references found therein. More recent contributions (not likely to be found in the above) include, for example, wavelet-based methods and analyses [27, 31, 32], updates on mollification methods [28–30, 35, 36], as well as the recent papers [18, 20, 21, 23, 24, 26, 33, 34].

In this paper we survey continuous and discrete regularization methods for first-kind Volterra problems of the form of (1). In particular, we pay specific attention to regularization methods which tend to retain the Volterra structure of the original problem. We have tried to pick methods which are especially representative of this idea and which also have a substantial theoretical basis. Certainly there will be errors in omission, particularly so since there is a large body of untranslated Russian literature on this subject.

In addition, although we will primarily be concerned with the linear problem, some of what will be said below also applies to nonlinear equations of the form

$$\int_0^t k(t,s,u(s))\,ds = f(t), \quad t \in [0,T], \tag{7}$$

and to Volterra operator equations of the form (1) where in this case $u(t) \in H$, for some Hilbert space H, and $k(t,s)$ is a two-parameter family of continuous operators from H to H (see, for example, the discussion of Volterra integro-operator equations in Section 6.8 of [9]).

Finally, a discussion of first-kind Volterra problems with weakly singular kernels is beyond the scope of this paper. This is an important class of problems which includes the Abel integral equation as a special case (and for which the Abel integral operator might be considered "half-smoothing" in the terminology of this section). Although a number of the methods we discuss are applicable to the Abel equation, we will not in general attempt a comprehensive study of regularization methods for this problem.

2 Classical Regularization and Volterra Problems

The classical theory of regularization is well-developed for linear ill-posed problems. For example, given the equation $Au = f$ on a Hilbert space U, with $A \in \mathcal{L}(U)$, and given a perturbation f^δ of f, the method of *Tikhonov regularization* determines u_α^δ solving

$$\min_{u \in U} \|Au - f^\delta\|^2 + \alpha\|u\|^2,$$

or, equivalently, as the solution of the normal equations on U,

$$(A^*A + \alpha I)u = f^\delta. \tag{8}$$

where $A^* \in \mathcal{L}(U)$ is the (Hilbert) adjoint operator associated with A. Standard Tikhonov regularization theory (which is applicable to first-kind Volterra problems) gives well-known conditions on the selection of $\alpha = \alpha(\delta)$ so that $u^\delta_{\alpha(\delta)} \to \bar{u}$

in U as $\delta \to 0$. *A posteriori* discrepancy principles facilitate the selection of α for a particular perturbation f^δ and given value of δ. For more information on these topics, see, for example, [4, 8]. We note that the first application of Tikhonov regularization specifically to general first-kind Volterra problems was evidently due to Schmaedeke in 1968 [14].

2.1 Regularization Methods of "Volterra Type"

An observation relevant to the objectives of this paper is the following. The original Volterra operator A in (3) is *non-anticipatory*, or *causal*; as a consequence, if we wish to determine the values of the solution $\bar{u}$ of the original Volterra problem (1) on the interval $[0, \tau]$ (for any $\tau > 0$), we need only make use of the values of data f on the same interval. In contrast, the adjoint operator A^* of the Volterra operator is given by

$$A^* u(t) = \int_t^T k(s,t) u(s)\, ds, \quad \text{a.e. } t \in [0,T], \tag{9}$$

which is an *anticipatory* operator. So if, for any $\tau > 0$, we wish to determine the values of the solution u_α^δ of the regularized equation (8) on the interval $[0, \tau]$, we will need to make use of both past and future values of the data f^δ, i.e., we require knowledge of f^δ on all of $[0, T]$.

This point is especially clear when the two equations (1) and (8) are discretized. Typical numerical realizations of the Volterra operator A lead to a lower-triangular (or nearly lower-triangular) matrix $\mathbf{A}^N$, so that the solution of a discretization of (1) may be handled by efficient, sequential methods (often in near real time). In contrast, $(\mathbf{A}^N)^\top \mathbf{A}^N$ is typically a full matrix and, in general, more poorly conditioned than the original matrix $\mathbf{A}^N$.

Like Tikhonov regularization, numerous classical regularization methods are based on the computation of $g_\alpha(A^*A)$, where g_α is appropriately defined [4]. As a consequence, such methods do not generally retain the Volterra structure of the original problem. In this paper we will focus on regularization methods which specifically preserve the non-anticipatory nature of the original problem, referring to such approaches as *regularization methods of "Volterra type"*. Where possible, we'll indicate convergence results and state whether rates of convergence are known to be order-optimal (see, for example, [4]).

There is, unfortunately, a price associated with limiting our discussion to regularization methods which avoid use of the adjoint operator A^*. While methods based on the operator $g_\alpha(A^*A)$ are generally associated with well-developed convergence theories (because such theories can make use of the spectral properties of the operator A^*A), the same is not true in general for methods based on the Volterra operator A. Quite often this means that theoretical results for regularization methods of "Volterra type" are limited to one-smoothing problems, or to ν-smoothing problems for ν small; other methods may require the assumption that A have special properties (such as the assumption that A is accretive). These are often only theoretical limitations, however, and do not in general mean that the method may not be applicable to a larger class of Volterra problems.

3 Continuous Regularization Methods of "Volterra Type"

3.1 A Singular Perturbation Approach

The early theoretical development of a singular perturbation approach for regularizing first-kind Volterra problems is generally attributed to Sergeev [53] and Denisov [41] in the early 1970's, following the ideas of Lavrent′ev [51]. For this reason, the method is often referred to as *Lavrent′ev's classical method*, or the *small parameter method*. The ideas have seen numerous extensions to vector, nonlinear, integro-operator, and other types of Volterra equations, with contributions made by Asanov, Imanaliev, Imomnazarov, and Magnickiĭ, [39, 44–47, 52], to mention just a few of the names important in the historical development of this approach. More recent applications of this method may be found in [40, 42, 43, 48, 74].

As motivation for the method, we recall from Section 2 that most reasonable finite-dimensional approximations of equation (1) lead to a linear system governed by a lower-triangular matrix $\mathbf{A}^N$. Generally, such a matrix has very small entries along the diagonal (due to the ill-posedness of the original problem), and thus a natural way to stabilize such a system would be to augment the values on the diagonal. In the infinite-dimensional setting, the analog of this process is to add a term of the form $\alpha u(t)$ to the values of $Au(t)$, for $\alpha > 0$ small. Thus, when noisy data f^δ is used in place of f, the approach is to consider a perturbed version of equation (2), namely,

$$(\alpha I + A)u = f^\delta, \tag{10}$$

where I is the identity operator on U. That is, we consider

$$\alpha u(t) + \int_0^t k(t,s)u(s)\,ds = f^\delta(t), \quad t \in [0,T], \tag{11}$$

which is a (well-posed) second-kind Volterra equation and, as such, has a unique solution u_α^δ depending continuously on data.

The regularized convergence theory for this problem is well-understood in the case of one-smoothing kernels. For example, the following result is obtained after making a slight variation in the theoretical arguments found in Section 3.4 of [1].

Theorem 1. [1] **(Lavrent′ev's classical method)** *Let $\bar{u} \in C[0,T]$ be the solution of the original problem (1) associated with data f, and assume that (1) is a one-smoothing problem. Let u_α^δ be the solution of (11) associated with data f^δ, where $|f(t) - f^\delta(t)| < \delta$, $t \in [0,T]$.*

Then if $\alpha = \alpha(\delta)$ *is selected satisfying* $\delta/\alpha(\delta) \to 0$ as $\delta \to 0$, *it follows that*

$$u^\delta_{\alpha(\delta)} \to \bar{u} \quad \text{as} \quad \delta \to 0,$$

uniformly on $[0, T]$ *provided* $\bar{u}(0) = 0$*; the convergence is uniform on* $[a, T]$*, for any* $a > 0$*, in the case of* $\bar{u}(0) \neq 0$.

The lack of uniform convergence near $t = 0$ in the case of $\bar{u}(0) \neq 0$ is unfortunate, but not surprising if one observes that equation (11) gives that $u^\delta_\alpha(0) = f^\delta(0)/\alpha = (f^\delta(0) - f(0))/\alpha$, so that the selection of $\alpha = \alpha(\delta)$ as prescribed in Theorem 1 guarantees $u^\delta_\alpha(0) \to 0 \neq \bar{u}(0)$ as $\delta \to 0$. Thus, in the case of $\bar{u}(0) \neq 0$, there is a boundary layer near $t = 0$ where the solution u^δ_α of (11) must exhibit rapid change for α small [37]. This means that if (11) is to form the basis of a regularization method for the stable solution of (1), one must employ numerical methods for *singularly perturbed* Volterra equations, a class of "stiff" Volterra second-kind equations that has received little attention by the numerical analysis community to date [49]. (Asymptotic analysis of such singularly perturbed equations may be found, for example, in [37, 38, 49], and the papers cited therein.)

To correct for the difficulty of the rapidly-varying nature of the solution u^δ_α of (11), several authors have suggested a modification of the regularization equation as follows (e.g., see [44, 52], and the references therein):

$$\alpha[u(t) - \bar{u}(0)] + \int_0^t k(t,s)u(s)\,ds = f^\delta(t), \quad t \in [0, T]. \tag{12}$$

Although this formulation avoids the singularly perturbed nature of (11) when $\bar{u}(0) \neq 0$, the main drawback is that one must know the value of the true solution $\bar{u}$ at $t = 0$. We note that for one-smoothing problems, equation (4) gives that $\bar{u}(0) = f'(0)$, so we must either have knowledge of the exact value of f' at 0, or else we must perform a differentiation of the perturbed data f^δ (a process itself requiring regularization). In a related paper by Sergeev [53], similar ideas were put forward for general ν-smoothing kernels, but, using the approach taken there, one must have knowledge of $\bar{u}(0)$ as well as higher order derivatives of $\bar{u}(t)$ at $t = 0$ in order to avoid facing the boundary layer phenomenon [50].

We note that although Theorem 1 tells how to select $\alpha = \alpha(\delta)$ asymptotically as $\delta \to 0$, it does not provide a principle for selecting α when we are only given one value of $\delta > 0$ and a particular perturbation f^δ of f. In fact, discrepancy principles do exist for this method in the case of a particular class of Volterra problems, although they are not the classical (Morozov) discrepancy principles. We will postpone a discussion of these modified principles until Section 3.3 where Lavrent′ev's m-times iterated method (a generalization of Lavrent′ev's classical method) is considered; see, in particular, Remark 2 of that section.

3.2 "Local Regularization" Methods

Local regularization methods for Volterra problems share common features with the singular perturbation approach described above in that a second-kind equation similar to (11) is constructed; here, however, equation (11) takes the special form

$$\alpha(t;r)u(t) + \int_0^t \tilde{k}(t,s;r)u(s)\,ds = \tilde{f}^\delta(t;r), \quad t \in [0,T], \tag{13}$$

where $\alpha(\cdot;r)$ is now a prescribed function involving a (new) regularization parameter $r > 0$, and $\tilde{k}(\cdot;r)$, $\tilde{f}(\cdot;r)$ are given r-dependent approximations of k and f^δ (all of which will be defined shortly).

To motivate equation (13), we let $r > 0$ be a small fixed constant and assume that equation (1) holds on an extended domain $[0, T+r]$ (which can always be accomplished by simply decreasing the size of T). Then $\bar{u}$ satisfies

$$\int_0^{t+\rho} k(t+\rho,s)u(s)\,ds = f(t+\rho), \quad t \in [0,T],\ \rho \in [0,r],$$

or, splitting the integral at t and making a change of integration variable,

$$\begin{aligned}\int_0^t k(t+\rho,s)u(s)\,ds + \int_0^\rho k(t+\rho,t+s)u(t+s)\,ds& \\ = f(t+\rho), \quad t \in [0,T],\ \rho \in [0,r].&\end{aligned} \tag{14}$$

For each $t \in [0,T]$, the ρ variable serves to advance the equation slightly into the future. One way to consolidate this future information is to integrate both sides of the equation with respect to $\rho \in [0,r]$, i.e.,

$$\begin{aligned}\int_0^t \int_0^r k(t+\rho,s)\,d\rho\, u(s)\,ds + \int_0^r \int_0^\rho k(t+\rho,t+s)u(t+s)\,ds\,d\rho& \\ = \int_0^r f(t+\rho)\,d\rho, \quad t \in [0,T],&\end{aligned} \tag{15}$$

where we have made a change of order of integration in the first integral above.

We note that the true solution $\bar{u}$ satisfies equation (15); when f is replaced by f^δ, a regularized form of this equation is needed and we obtain this new equation by replacing $u(t+s)$ by $u(t)$ in the second integral term. That is, for fixed t, it is as if u is (temporarily) assumed to be constant on the small *local* interval $[t, t+r]$; the length r of this local interval becomes the regularization parameter. The "local regularization equation" which results is given by (13), where in that equation we make the definitions, for $0 \le s \le t \le T$,

$$\alpha(t;r) := \int_0^r \int_0^\rho k(t+\rho,t+s)\,ds\,d\rho, \tag{16}$$

$$\tilde{k}(t,s;r) := \int_0^r k(t+\rho,s)\,d\rho, \qquad \tilde{f}^\delta(t;r) := \int_0^r f^\delta(t+\rho)\,d\rho. \tag{17}$$

It is clear that this method is similar to the singular perturbation approach discussed earlier because the coefficient $\alpha(t;r)$ of $u(t)$ in (13) can be made small by decreasing r. However, in contrast to the singular perturbation approach, there is never a boundary layer phenomenon for the local regularization equation. That is, one does not require $\bar{u}(0) = 0$ (or knowledge of $\bar{u}(0)$, if nonzero) in order to obtain uniform convergence on $[0,T]$ of the solution $u^\delta(\cdot;r)$ of (13) to $\bar{u}$ as $r, \delta \to 0$ in a coordinated way. Thus, one may apply standard numerical schemes to the solution of the local regularization equation (13) without having to worry about the difficulty of resolving rapidly varying solutions near $t = 0$ (such as was necessary for the singular perturbation approach in the absence of information about $\bar{u}(0)$).

A regularization theory for this method was developed in [56, 57] (for convolution kernels) and [61] (for nonconvolution kernels), and is summarized in the next theorem. An open problem concerns the development of a discrepancy principle for the selection of $r = r(\delta, f^\delta)$ in the case of fixed f^δ, δ.

Theorem 2. [61] **(Local regularization method)** *Let $\bar{r} > 0$ be small and assume that $\bar{u} \in C^1[0, T + \bar{r}]$ satisfies (1) on $[0, T + \bar{r}]$ where it is assumed that (1) is a one-smoothing problem. For $0 < r \le \bar{r}$, let $u^\delta(\cdot;r)$ denote the solution of (13) associated with f^δ, where $|f(t) - f^\delta(t)| \le \delta$ for $t \in [0, T + \bar{r}]$.*

Then if $r = r(\delta)$ is selected satisfying $\delta/r(\delta) \to 0$ as $\delta \to 0$, we have

$$u^\delta(\cdot; r(\delta)) \to \bar{u} \;\text{ as }\; \delta \to 0$$

uniformly on $[0,T]$, with optimal convergence rate $\delta^{1/2}$ for $r(\delta) = C\delta^{1/2}$, $C > 0$.

The local regularization theory described above may be generalized so that integration with respect to ρ in (16)–(17) is replaced by integration with respect to a suitable Borel measure η_r on $[0, r]$ [56, 57, 61]. In particular, a choice of a discrete η_r (such as $\int_0^r g(\rho)\, d\eta_r(\rho) = \sum_{i=1}^K s_i g(\tau_i)$, for $s_i > 0$, $i = 1, \ldots, K$, and $0 \le \tau_1 < \tau_2 < \ldots < \tau_K = r$) is particularly useful in numerical implementations of this method. This idea will be revisited in Section 4.3.

We note that, although the above theorem is stated for one-smoothing problems, conditions guaranteeing convergence for general ν-smoothing problems may be found in [57] for convolution kernels. Verification of these conditions is not easy, but convergence has been demonstrated in the case of $1 \le \nu \le 4$ for classes of Borel measures η_r of practical use [57]. The theory for the local regularization method described above does not yet extend to infinitely-smoothing problems, but, as will be discussed in Section 4.3, a particular discretized version of (13) (with a fixed discrete measure η_r) has been used for over 30 years in practical solution of the infinitely-smoothing IHCP. Indeed, it was this numerical method due to Beck [16] for the IHCP that motivated the development of the local regularization methods described in this section and in Section 4.3. (As a final comment regarding local regularization and infinitely smoothing problems, we note that it was shown in [58] that a particular variation of the method described above can be shown to converge for infinitely-smoothing problems, however this alternate approach can no longer be considered a regularization method of "Volterra type".)

Theorem 2 was extended in [61] to include the possibility of a variable regularization parameter ($r = r(t)$), a generalization which allows for *variable* local regularization of equation (1) (i.e., more regularization in some regions of $[0,T]$ and less in others). Variable regularization has been shown to be helpful in numerical examples where the true solution $\bar{u}$ is not smooth.

A disadvantage of the local regularization method described above is the need for data slightly beyond the initial interval $[0,T]$, or, equivalently, one must be satisfied with approximating $\bar{u}$ on a slightly smaller interval $[0,T-\bar{r}]$. However, this is not a serious shortcoming as it is well-known that one cannot expect to adequately determine solutions of Volterra problems near the end of the interval $[0,T]$ when using data on the same interval.

3.3 Lavrent′ev's m-times iterated method.

For fixed integer $m \geq 1$ and given regularization parameter $\alpha > 0$, Lavrent′ev's m-times iterated method determines $u_\alpha^\delta \in U$ via

$$(\alpha I + A)u_n = \alpha u_{n-1} + f^\delta, \quad n = 1, 2, \ldots m, \tag{18}$$

$$u_\alpha^\delta := u_m, \tag{19}$$

starting from $u_0 = 0$. For $m = 1$, it is clear that the method (18)–(19) reduces to Lavrent′ev's classical method of Section 3.1; for $m > 1$, corrections are applied to further stabilize the problem. Because typical discretizations of the operator $(\alpha I + A)$ lead to lower triangular matrix representations, the calculation of u_α^δ in (18)–(19) is easily accomplished via backward substitution in (18); as before, the addition of the term αu_n (for $\alpha > 0$ small) serves to stabilize the numerical process. We note that because $m \geq 1$ is fixed, the approach is considered a parametric regularization method and not an iterative one.

In contrast to Lavrent′ev's classical method for first-kind Volterra equations, where the theoretical development is available for one-smoothing problems (and for general ν-smoothing problems using the approach of Sergeev [53]), the theoretical analysis of Lavrent′ev's m-times iterated method for Volterra problems appears to be limited at present to *accretive* operators, i.e., to those operators $A \in \mathcal{L}(U)$ (for a general Hilbert space U, here over a complex or real scalar field) satisfying

$$\Re\langle Au, u\rangle \geq 0, \quad u \in U,$$

where $\langle\cdot,\cdot,\rangle$ denotes the usual inner product on U, and $\Re z$ denotes the real part of $z \in \mathbb{C}$.

Accretive Volterra operators include those with convolution kernels $k(t,s) = \kappa(t-s)$ that are completely monotone, i.e.,

$$\Re L[\kappa](z) \geq 0, \quad \Re z > 0,$$

where $L[\kappa]$ denotes the Laplace transform of κ. (For more details about completely monotone kernels, see, for example, Gripenberg, Londen, and Staffans [7], or Nohel and Shea [12].) Among this class of convolution kernels are κ which are positive, decreasing, and convex [12]. Another example of an accretive Volterra operator is the generalized Abel integral operator (where U must be interpreted to denote $L_2(0,T)$ with an associated weighted inner product). Further, the Volterra operator associated with the IHCP is evidently not accretive in $U = L_2(0,T)$, without the use of a weighted inner product [67].

We describe here some of the basic theoretical results for Lavrent′ev's m-times iterated method, assuming throughout the remainder of this section that A is an accretive Volterra operator. In [63], Plato showed that the method is convergent in the case of exact data f in (18)–(19), i.e.,

$$u_\alpha \to \bar{u} \text{ as } \alpha \to 0.$$

In the case of perturbed data f^δ, the definition of a classical discrepancy principle provides a way of selecting $\alpha = \alpha(\delta)$ so that regularized convergence is obtained as $\delta \to 0$. The following theorem involves the use of a classical discrepancy principle, but is valid only for the case of $m \geq 2$.

Theorem 3. [64,65] **(Lavrent′ev's m-times iterated method, I)**
Let $A \in \mathcal{L}(\mathcal{U})$ be an accretive Volterra operator and let $\bar{u} \in \overline{\mathcal{R}(A)}$ solve the original equation $Au = f$. Let u_α^δ denote the solution of the Lavrent′ev's m-times iterated method (18)–(19) for $m \geq 2$, with f^δ in place of f and $\|f - f^\delta\| < \delta$ for some $\delta > 0$.

Then if $\alpha = \alpha(\delta) > 0$ is selected satisfying the discrepancy principle,

$$b_0\delta \leq \|Au_\alpha^\delta - f^\delta\| \leq b_1\delta \tag{20}$$

(for fixed constants b_0, b_1, with $b_1 \geq b_0 > 1$, $b_1\delta < \|f^\delta\|$), it follows that $\delta/\alpha(\delta) \to 0$, and

$$\|u_{\alpha(\delta)}^\delta - \bar{u}\| \to 0,$$

as $\delta \to 0$. Moreover, under additional smoothness assumptions on $\bar{u}$ (and with some restrictions), order-optimal convergence rates are obtained.

Remark 2. Plato showed that the (classical) discrepancy method given by (20) in Theorem 3 is *divergent* for the case of $m = 1$ (i.e., for Lavrent′ev's classical method considered Section 3.1) when applied to accretive Volterra operators with nondegenerate kernels (cf. Prop. 3.2.4 of [63]). Although this result is restricted to accretive operators, it matches a similar finding for symmetric, positive definite operators in $\mathcal{L}(U)$ and thus does not give much hope that a discrepancy principle of this type will be useful for general ν-smoothing Volterra problems.

Fortunately, in the accretive operator case there are modified discrepancy principles that are convergent (and lead to "pseudo-optimal" choices of $\alpha = \alpha(\delta)$, in the terminology of [69]) for $m \geq 1$. Thus, provided that A is accretive, these modified discrepancy principles apply to Lavrent′ev's classical method described in Section 3.1. One such modified discrepancy principle is given below.

Theorem 4. [64,69] **(Lavrent′ev's m-times iterated method, II)**
Let $A \in \mathcal{L}(\mathcal{U})$ be an accretive Volterra operator and let $\bar{u} \in \overline{\mathcal{R}(A)}$ solve the original equation $Au = f$. Let u_α^δ denote the solution of the Lavrent′ev's m-times iterated method (18)–(19) for any fixed $m \geq 1$, with f^δ in place of f, $\|f - f^\delta\| < \delta$.

Then if $\alpha = \alpha(\delta) > 0$ is selected satisfying the discrepancy principle,

$$b_0\delta \leq \|\alpha(A + \alpha I)^{-1}(Au^\delta_\alpha - f^\delta)\| \leq b_1\delta \tag{21}$$

(for fixed constants b_0, b_1, with $b_1\delta < \|f^\delta\|$, $b_1 \geq b_0 > \tau_0$, for suitable $\tau_0 > 0$), it follows that $\delta/\alpha(\delta) \to 0$, and

$$\|u^\delta_{\alpha(\delta)} - \bar{u}\| \to 0,$$

as $\delta \to 0$, with order-optimal convergence rates under additional smoothness assumptions on $\bar{u}$.

Plato and Hämerik considered a second convergent parameter choice in [69] for the m-times iterated method in the case of $m \geq 1$. Both modified discrepancy principles extend (to accretive Volterra operators) analogous results for symmetric operators on a Hilbert space [13] and similar results for normal equations [3]. It was also shown in [69] that an alternate parameter selection method based on an extrapolation strategy is convergent for $m = 1$ and $m = 2$; the question is unresolved for $m > 2$.

3.4 Iterative Methods of "Volterra Type"

Classical iterations such as Landweber, conjugate gradient, ν-methods, and iterated Tikhonov regularization, generally rely on the computation of the anticipatory operator A^* and thus will not be considered here. For a discussion of classical iterative regularization, see, for example, [4].

A few simple iterations do, however, fit our notion of a regularization method of "Volterra type".

Richardson Iteration. Starting from an initial guess of u_0, the Richardson iteration is a simple explicit method which defines a sequence $\{u^\delta_n\}$ of functions satisfying

$$u_n = u_{n-1} - \beta_n(Au_{n-1} - f^\delta), \quad n = 1, 2 \ldots, \tag{22}$$

starting from a suitable initial guess $u_0 \in U$.

In the case of noise-free data ($f^\delta = f$, $u^\delta_n = u_n$), Vasin obtained the following result concerning convergence of the iteration under quite general conditions on the original problem (1) (in particular, equation (1) need not be ν-smoothing), with results applicable to certain nonlinear Volterra problems [70, 71].

Theorem 5. [70, 71] **(Richardson iteration with noise-free data)**
Assume the kernel $k \in C^1$ in (1) is such that $k(t,s) \geq 0$ and $\frac{\partial}{\partial t}k(t,s) \geq 0$ for $0 \leq s, t \leq T$. Let u_n denote n^{th} Richardson iterate defined by (22) for noise-free data f. Then, for sufficiently smooth initial guess u_0 and for $\beta_n \geq \beta > 0$ sufficiently small, the iterates u_n converge to $\bar{u}$ as $n \to \infty$.

The convergence of the Richardson iteration in the case of noisy data is a more difficult problem as this particular iteration seems quite sensitive to noise in the data, even for ν-smoothing problems with ν small. Of course, when perturbed data f^δ is used in place of f in (22), we cannot expect convergence as $n \to \infty$. Instead (as with all iterative methods for ill-posed problems), we expect to see "semiconvergence" (see, e.g., [4] or [11]), meaning that the error $\|u_n^\delta - \bar{u}\|$ decreases for some initial iteration steps, but eventually begins to increase. Therefore, in the presence of noisy data one must determine a stopping criterion, or method of selecting a stopping point $n = n(\delta)$ in the iteration process, for which we have regularized convergence of the iterates as $\delta \to 0$. That is, we require that $n(\delta) \to \infty$ and

$$\|u^\delta_{n(\delta)} - \bar{u}\| \to 0,$$

as $\delta \to 0$.

Vasin discussed a stopping criteria in [70], however simple numerical examples suggest that the Richardson method may be most effective for only moderately ill-posed problems. For example, Plato and Hämarik [69] provided a thorough analysis of a stopping criteria for the Richardson iteration based on classical discrepancy principles, but the theory they provided was limited to a restricted class of operators (strictly sectorial operators) which include the classical Abel operator. The Abel problem is only moderately ill-posed and, indeed, it is an open question as to whether the Richardson iteration is better suited for such problems than for general ν-smoothing (or infinitely-smoothing) problems. In addition, in [64], Plato noted that generally Lavrent′ev's m-times iterated method is superior to Richardson iteration (and also superior to an implicit scheme given in (23) below) for strictly sectorial operators.

Other Iterative Methods There are additional iterative methods, which when applied to equation (1), take full advantage of the Volterra nature of the problem. Among these we mention an implicit iteration,

$$(I + \beta A)u_n = u_{n-1} + \beta f^\delta, \quad n = 1, 2 \ldots, \tag{23}$$

and an alternating directions iteration,

$$u_{n-1/2} = u_{n-1} - \frac{\beta}{2}(Au_{n-1} - f^\delta),$$
$$(I + \frac{\beta}{2}A)u_n = u_{n-1/2} + \frac{\beta}{2}f^\delta, \quad n = 1, 2, \ldots,$$

both for fixed $\beta > 0$ and for suitable initial $u_0 \in U$. Rigorous analysis of these iterations (and their connections to Cauchy's method), along with theoretically-sound stopping criteria, may be found in Plato [63, 65, 66] and Plato and Hämerik [69]. However, the convergence theory for these iterations as applied to Volterra problems is apparently limited at present to only moderately ill-posed equations such as the classical Abel equation (and other Volterra equations with strictly sectorial operator A), so an open question concerns their applicability to the more general problems under consideration in this paper.

3.5 Differentiation and Mollification

For ν-smoothing problems, we know that the original first-kind equation (1) is equivalent (via ν differentiations) to a second-kind Volterra equation (5). However, since we typically only have available a non-smooth perturbation f^δ of f, the differentiation of f^δ is not a well-posed process. In the usual case of perturbed data, one approach is to replace (5) by

$$u(t) + \int_0^t \frac{\partial^\nu}{\partial t^\nu} k(t,s) u(s)\, ds = L_\alpha f^\delta(t), \quad t \in [0,T], \tag{24}$$

where the operator L_α is constructed to satisfy $L_\alpha f^\delta \to f^{(\nu)}$ (in an appropriate sense) as $\delta \to 0$, provided the selection of α is coordinated with that of δ.

There are many ways of selecting L_α, with different theoretical arguments required for each, so we will only indicate some of the possibilities here. For example, Murio [81] considered the one-smoothing differentiation problem (6) and took as $L_\alpha f^\delta$ the approximate derivative of a particular mollification (a convolution with a Gaussian kernel) of f^δ. Magnicki [80] and Srazhidinov [82] considered the general ν-smoothing problem and viewed L_α as a type of mollified differentiation operator, while Kabanikhin [74] employed a difference operator approach. (In both [82] and [74], theoretical results were given for nonlinear Volterra equations.) A further variation on these ideas may be found in Sergeev [53]. We note that although the above approaches are simple, they suffer if k is not known precisely (since derivatives of k must also be taken); further, the method cannot be extended to infinitely-smoothing problems.

Murio (cf. [81] and the references therein) considered specific mollification approaches for solving a number of applied problems. For example, he used mollification to solve a Volterra equation of Abel type, in this case applying the inverse Abel transform (which requires the differentiation of data, under an integral) to a construction of $L_\alpha f^\delta$ similar to that described above for the differentiation problem. He also applied mollification techniques to the IHCP.

Louis and Maaß [79] developed an abstract formulation of the mollification problem for bounded linear operators A on Hilbert spaces, and Louis extended the overall analysis further in [75–78]. The ideas in [77] will be discussed more fully in the next paragraph. Háo took a general approach through the use of a one-parameter class of mollification operators defined on Banach spaces. His treatment included applications to numerical differentiation and the ICHP, generalizing a number of existing theories (see [72] and the references therein). Hegland and Anderssen [73] provided a general Hilbert space analysis, defining "range mollifications" and "domain mollifications" for an operator A. In applications to numerical differentiation and the Abel equation, they made use of translation operator representations to give estimates of the mollification error.

In what follows we sketch the approach taken by Louis in [77] as it pertains to a first-kind Volterra problem of the form (1); for more complete details, see [76, 77]. The idea is to define a one-parameter family $\{e_\alpha\}$ of mollifiers, $e_\alpha = e_\alpha(t,s)$, for which $E_\alpha u \in U$ given by

$$E_\alpha u(t) := \int_0^T e_\alpha(t,s)u(s)\,ds, \quad t \in [0,T],$$

is a suitable approximation of $u \in U$ for $\alpha > 0$ small. The initial task is to find $\psi_\alpha = \psi_\alpha(t,s)$ satisfying $\psi_\alpha(t,\cdot) \in U$ and

$$A^*\psi_\alpha(t,\cdot) \simeq e_\alpha(t,\cdot), \quad t \in [0,T], \tag{25}$$

a computation which can be made prior to receiving any data. Using the perturbed data f^δ of f, the mollified approximation u_α^δ to $\bar{u}$ is then given by

$$u_\alpha^\delta(t) = \int_0^T \psi_\alpha(t,s)f^\delta(s)\,ds, \quad t \in [0,T], \tag{26}$$

and, under suitable assumptions, it can be shown that $\alpha = \alpha(\delta)$ may be selected guaranteeing $u_\alpha^\delta \to \bar{u}$ as $\delta \to 0$, with order-optimal rate of convergence, under additional conditions (cf. [76, 77]).

Although the adjoint operator A^* plays a role in this regularization method, in the case of certain specific Volterra operators A (in particular, those for which $(A^*)^{-1}$ is easily evaluated), it is possible to pick e_α so that ψ_α determined by (25) has support which is optimal with regard to the Volterra problem, i.e., so that this particular ψ_α leads to a reconstruction of u_α^δ in (26) which essentially reduces to

$$u_\alpha^\delta(t) = \int_0^{t+\varepsilon(t)} \psi_\alpha(t,s)f^\delta(s)\,ds, \quad t \in [0,T], \tag{27}$$

for some $\varepsilon = \varepsilon(t,\alpha) \geq 0$ small. Louis gave an example of such an e_α for the derivative problem (6) in [77] and for a Volterra problem of Abel type in [78], which, for both examples, leads to the “Volterra-type” construction (27). (We note that the constructions of u_α^δ for these two particular cases are also related to the “local regularization” ideas discussed in Section 3.2.) The goal of using these mollification ideas to obtain a Volterra-type regularization method for general equations of the form (1) is considerably more difficult because $(A^*)^{-1}e_\alpha$ is not so easily evaluated in the general case.

4 Discrete Regularization Methods of “Volterra Type”

Discrete approximation methods provide another way to regularize the original problem (1). In this case the regularization parameter is the discretization parameter (or stepsize), and a coordination between this parameter and the amount of noise δ in the problem is required in order to obtain good approximations in the presence of noise. This is sometimes known as the “self-regularization” property of discretizations. An important issue is whether self-regularization is sufficient for the kinds of problems we consider here. In the general case it can be said that further regularization will be required in order to obtain acceptable approximations in the presence of perturbed data.

In this section we describe numerical methods that have been developed over the past few decades for solution of the first-kind Volterra problem (1), focusing in particular on those methods that are of "Volterra type" and for which something can be said about the "self-regularization" properties of the algorithms. We note that most often the theory is limited to one-smoothing problems. We begin the discussion with the case of exact data.

4.1 Standard Numerical Discretizations with Exact Data f

As described in survey papers by Brunner [85, 88], the idea of approximating Volterra equations (admittedly, of second kind) in a finite-dimensional setting actually originated with Volterra himself in the late 1800's. Evidently the first application of similar ideas to first-kind equations was given by Huber [101] in 1939 [88], and it was in the 1960's that researchers (notably Jones [103], Kobayasi [105], and Linz [108, 109]) began to look closely at the theoretical issues associated with adapting traditional numerical integration techniques to the approximate solution of first-kind Volterra equations [88, 114].

Discretizations Based on Numerical Integration Rules. The idea behind numerical methods based on integration rules is quite straightforward. The integral in (1) is replaced by a numerical quadrature, and then it is required that the resulting equation be satisfied exactly at a finite number of points in $[0, T]$.

For example, consider the *rectangular* integration rule, given for integer $N \geq 1$ and $h = T/N$ by

$$\int_0^{t_i} \varphi(t)\, dt \approx h \sum_{j=0}^{i-1} \varphi(t_j),$$

$i = 1, \ldots N$, for continuous φ. Throughout we will make the definition $t_j = jh$, for $j = 0, 1, \ldots N$. Replacing the integral in equation (1) by a sum such as given above, one obtains the *Euler method* for approximate solution of (1),

$$h \sum_{j=0}^{i-1} k(t_i, t_j) u_j = f(t_i), \quad i = 1, 2, \ldots, N, \tag{28}$$

where u_i is an approximation for $u(t_i)$. Equivalently, making the definitions $\mathbf{f}^N = (f(t_1), \ldots, f(t_N))^\top$, $\mathbf{u}^N = (u_0, \ldots, u_{N-1})^\top \in \mathrm{IR}^N$, we obtain the matrix equation

$$\mathbf{A}^N \mathbf{u}^N = \mathbf{f}^N \tag{29}$$

where $\mathbf{A}^N$ is a lower-triangular $N \times N$ matrix with entries depending on h and the kernel k. It is easy to see that, under one-smoothing assumptions, the diagonal entries of $\mathbf{A}^N$ are nonzero for all $h > 0$ sufficiently small; thus the Euler approximation algorithm may be solved sequentially for a unique approximation vector $\mathbf{u}^N$.

Theorem 6. [107] **(Euler Method)** *Let $\bar{u}$ be the solution of equation (1), where it is assumed that (1) is a one-smoothing problem, with k, f, $\bar{u}$ sufficiently smooth. Let $\mathbf{u}^N = (u_0, \ldots, u_{N-1})^\top \in \mathbb{R}^N$ be determined by the Euler method (28) using exact data f.*

Then

$$\max_{0 \leq i \leq N-1} |\bar{u}(t_i) - u_i| \to 0 \quad \text{as} \quad h \to 0,$$

with order of discrete convergence equal to 1; that is, $p = 1$ is the largest such number for which

$$\max_{0 \leq i \leq N-1} |\bar{u}(t_i) - u_i| \leq M h^p$$

for some M independent of h.

Similarly, the composite *midpoint* integration rule,

$$\int_0^{t_i} \varphi(t)\, dt \approx h \sum_{j=0}^{i-1} \varphi(t_{j+1/2}),$$

$i = 1, \ldots, N$, where $t_{j+1/2} = t_j + h/2$, leads to the *midpoint method* for the approximate solution of (1),

$$h \sum_{j=0}^{i-1} k(t_i, t_{j+1/2})\, u_{j+1/2} = f(t_i), \quad i = 1, 2, \ldots, N, \tag{30}$$

also generating a triangular matrix system of the form (29), uniquely solvable for all $h > 0$ sufficiently small.

Theorem 7. [107] **(Midpoint Method)** *Let $\bar{u}$ be the solution of equation (1), where (1) is assumed to be a one-smoothing problem and k, f, and $\bar{u}$ are sufficiently smooth. Let $\mathbf{u}^N = (u_{1/2}, \ldots, u_{N-1/2})^\top \in \mathbb{R}^N$ be determined by the midpoint method (30) using exact data f.*

Then

$$\max_{0 \leq i \leq N-1} |\bar{u}(t_i + h/2) - u_{i+1/2}| \to 0 \quad \text{as} \quad h \to 0,$$

with order of discrete convergence equal to 2.

Obviously, one could continue applying higher order integration rules to the integral in (1), but the fact is that the midpoint and Euler methods are the only reasonable approaches that result from the use of standard quadrature rules. For example, the trapezoidal rule results in a method that is convergent of order 2, but which is also considered to be numerically unstable [106, 107]. (We note that the notion of numerical instability is separate from that of ill-conditioning. Ill-conditioning is an inherent feature of the original problem, while numerical instability refers to undesirable aspects of the numerical difference scheme. The idea of numerical stability for Volterra equations is similar to that used in analyzing numerical methods to solve ordinary differential equations.)

It was first observed by Linz [111] (for some representative quadrature rules) and later confirmed by Gladwin and Jeltsch [100] that all of the standard higher order Gregory methods and Newton-Cotes integration rules (such as Simpson's rule and the three-eights rule) lead to unstable numerical methods for solving equation (1) [106]. This is in sharp contrast to numerical approximation of second-kind Volterra equations (where all "reasonable" integration rules lead to "reasonable" numerical approximation methods for (1), with complications due mainly to "stiffness" associated with the particular case of singularly perturbed problems), and is an example of the way in which the ill-posedness of (1) influences numerical implementation even when one is using perfect data. [107]

In summary, of the standard integration-type methods the midpoint method is generally the best choice for the approximate solution of (1) in the presence of noise-free data; we note also that the accuracy of the midpoint method may be improved further using Richardson extrapolation [107].

Collocation-Based Discretization Methods. In spite of the undesirable situation for discretizations based on standard integration rules, in fact arbitrarily high order numerical methods for the solution of equation (1) do exist (under conditions of additional smoothness on the kernel k and true solution $\bar{u}$). There are several different types of these methods (see the end of this section), and we describe here the class of collocation-based methods. As in the case of methods based on integration rules, the theoretical analysis for collocation methods is typically restricted to one-smoothing problems.

One commonly seen implementation of collocation is as follows. Given intervals $\sigma_i = (t_{i-1}, t_i]$, $i = 2, \ldots, N$, $\sigma_1 = [t_0, t_1]$, we seek an approximate solution u^N in the space of piecewise polynomials $S_{m-1}^{(-1)}(N)$,

$$S_{m-1}^{(-1)}(N) = \{u \,:\, u|_{\sigma_i} \in \Pi_{m-1},\ i = 1, \ldots N\},$$

where Π_k denotes the space of real polynomials of degree not exceeding k. Since $\dim S_{m-1}^{(-1)}(N) = Nm$, one must impose the same number of conditions on the approximate solution in order to obtain the solution uniquely. To this end we define Nm collocation points,

$$X_m(N) = \{t_{i,j} = t_{i-1} + c_j h,\ j = 1, \ldots m,\ i = 1, \ldots N\},$$

where the $\{c_j\}$ are "collocation parameters" selected satisfying

$$0 < c_1 < \ldots < c_m \leq 1,$$

and we impose the condition that $u^N \in S_{m-1}^{(-1)}(N)$ must satisfy equation (1) precisely at each collocation point in $X_m(N)$, i.e.,

$$Au^N(t_{i,j}) = g(t_{i,j}), \quad t_{i,j} \in X_m(N). \tag{31}$$

This leads to a system of equations of the form (29) where now the Nm-square matrix $\mathbf{A}^N$ is block-triangular (with N diagonal $m \times m$ blocks, each nonsingular for $h > 0$ sufficiently small provided the problem is one-smoothing), leading to a block-sequential solution method. The following convergence result is due to Brunner [87].

Theorem 8. [87–90] **(Method of Collocation)** *Let the original problem (1) be one-smoothing and assume that k and f are sufficiently smooth to guarantee that $\bar{u} \in C^m$. Let u^N be given by the collocation procedure described above using exact data f.*

Then u^N converges uniformly to $\bar{u}$ (with discrete convergence of order m) if and only if the quantity $\xi_m = \prod_{j=1}^{m} \frac{1-c_j}{c_j}$ satisfies

$$\xi_m \leq 1, \ \text{ if } m \text{ is odd}, \quad \text{ and } \ \xi_m < 1, \ \text{ if } m \text{ is even}.$$

In addition, "local superconvergence" at certain discrete interior points of σ_i is achievable under additional conditions.

Thus, even though convergence does not occur for all choices of collocation parameters, the particular choice of $c_m = 1$ (i.e., collocation occurs at each of the original gridpoints t_j, $j = 1, \ldots N$) assures uniform convergence of order m.

In addition, Eggermont showed how the superconvergence (at selected points) indicated in the above theorem may be improved via a postprocessing of the collocation solution. See, for example, [97, 99]. Finally, Kauthen and Brunner considered analogous results for the case of *continuous* collocation approximation spaces $S_{m-1}^{(0)}(N)$ in [104]; methods based on splines with full continuity and of degree exceeding one are known to be divergent [102].

To illustrate the simplicity of the collocation algorithm, we briefly consider collocation in the space $S_0^{(-1)}(N)$ of piecewise constants. In this case, for $i = 1, \ldots, N$, let χ_i be the usual characteristic function on σ_i (i.e., $\chi_i(t) = 1$, for $t \in \sigma_i$, and $\chi_i(t) = 0$, otherwise). Then the collocation solution u^N is given by

$$u^N(t) = \sum_{i=1}^{N} u_i \chi_i(t), \quad t \in [0, T],$$

where the coefficients $u_1, \ldots, u_N$ satisfy

$$\sum_{j=1}^{i} u_j \int_{t_{j-1}}^{t_j} k(t_i, s)\, ds = f(t_i), \quad i = 1, \ldots, N \tag{32}$$

(again, a system of the form (29) with $\mathbf{A}^N$ lower-triangular). From the last theorem we know that equation (32) generates a piecewise constant u^N that converges uniformly to $\bar{u}$ at a rate of $\mathcal{O}(h)$.

For simple kernels k, the moment integrals $\int_{t_{j-1}}^{t_j} k(t_i, s)\, ds$ appearing in (32) (and in the more general problem (31)) may be evaluated analytically, however, more commonly they will need to be evaluated using a discrete quadrature rule. In [87], Brunner provided an analysis of the coordination of the particular integration rule with m so that the overall order of discrete convergence is not lost.

In addition, Brunner showed that a number of other high order methods developed independently for use in solving first-kind Volterra equations (methods such as the block-by-block approach considered in the early 1970's by de Hoog and Weiss [92, 93] and some product integration methods studied, for example, by Young [124], Weiss and Anderssen [121], Linz [110], and McAlevey [113]), can be obtained via collocation if a particular numerical quadrature scheme is used to approximate the moment integrals [86]. Further, collocation appears to offer some flexibility in the approximation of (1) in that specially-selected quadrature rules can be used to construct the moment integrals in the case of a singular, nonsmooth, or rapidly-varying kernel.

Other Higher Order Methods. A comprehensive discussion of higher order methods for first-kind Volterra equations would take us beyond the scope of this paper. Instead we will only mention a few of the many methods available, including linear multistep methods (see, for example, McKee [115] and the references in his survey [114]), generalized linear multistep methods (van der Houwen and te Riele [120]), inverted differentiation formulas (Taylor [119]), reducible quadrature (Wolkenfelt [123]), product integration rules (e.g., see the references given above), and multilag methods [122]. A comprehensive list of references is not possible here. A "unified" convergence theory was given for many of these methods (including collocation) in the case of both linear and nonlinear Volterra problems of the first kind in papers by Scott (née Dixon), McKee, and Jeltsch (see, e.g., [94, 95, 117, 118]).

Discretizations of ν-Smoothing Problems for $\nu > 1$. Those numerical methods described above which rely heavily on the use of discrete evaluations of the kernel along the "diagonal" (i.e., $k(t,t)$) will no longer be of use for ν-smoothing problems for $\nu > 1$ since, in this case, $k(t,t) = 0$. Further, as mentioned above, the vast amount of theoretical analysis for standard methods is restricted to one-smoothing Volterra problems. Among the exceptions are a convergence analysis for collocation methods (Eggermont, [98]) and for linear multistep methods (Andrade, Franco, and McKee [83]), in the case of *two-smoothing* Volterra problems. Of interest in this regard is a comment in [83] that the original goal of the authors was to consider ν-smoothing problems for general $\nu \geq 2$. In fact, they were successful only in the case of $\nu = 2$, and found that even in this case, the stability properties of the method were quite different from the stability of the same methods applied to one-smoothing problems. Thus, from a theoretical point of view, the analysis of standard discretization methods when applied to general ν-smoothing Volterra problems (in the case of ν bigger than one or two) remains an open problem.

For a more comprehensive view of numerical methods for first-kind Volterra equations in the presence of noise-free data, see, for example, [84, 88, 89, 91, 106, 107, 114], survey papers written at different times over the last 25 years of study on this subject. In recent years, many of these numerical methods have been extended to nonlinear first-kind Volterra problems.

4.2 Standard Numerical Discretizations with Perturbed Data f^δ: "Self-Regularization"

We now consider the use of one of the above numerical methods for the solution of equation (1) in the case where f has been replaced by f^δ. In this situation we are interested in the convergence of numerical approximations (constructed using f^δ) to the true solution $\bar{u}$ as both the amount δ of noise in the data and the discretization parameter h go to zero. As $\delta \to 0$ we may let $h \to 0$, but, in general, we cannot let h get as small as we might like for a given value of $\delta > 0$.

Natterer [10] gave a general mathematical context in which to evaluate the regularizing properties of numerical methods such as those given the last section, in particular, those methods which can be formulated as projection methods. In keeping with the ideas of [10], we will say that a discrete numerical method is "robust" for one-smoothing problems if the magnification of data error by the method is of order δ/h as $h \to 0$. (Compare, for example, with Richter [116].)

Linz [107, 112] investigated the handling of error for the midpoint method and found that the method was robust for one-smoothing problems. Eggermont extended the analysis to higher-order numerical methods (such as cyclic linear multistep and reducible quadrature methods) and, using a projection-type analysis, showed that under reasonable circumstances these methods were also robust for one-smoothing problems [96]. In addition, the same results hold for one-smoothing problems when collocation is coupled with certain quadrature methods for moment integrals [96, 99].

Thus, for any of the above methods, we are assured of optimal error handling of the method provided that the original problem is one-smoothing. A very important question is whether this error handling is sufficient when the same method is applied to a ν-smoothing problem for $\nu > 1$, or to an infinitely-smoothing problem such as the IHCP. Practical experience indicates that standard numerical methods alone (of the form described in the last section) are *not* sufficient to handle the more ill-conditioned problems (and, in fact, often give meaningless results) without the use of additional regularization techniques.

4.3 Combined "Volterra-Type" Discretization and Regularization

Effective discretized regularization techniques for Volterra problems involve the pairing of a discretization method of "Volterra type" (such as one of the methods described in Section 4.1) with a continuous regularization method which also preserves the Volterra structure of the original problem (cf. Section 3).

Discretization and Mollification. An approach taken by Linz [112] was to smooth or filter the data first before applying a standard numerical method to (1). In [73], Hegland and Anderssen combined mollification with projection methods. Their analysis provided for theoretical error estimates for mollified finite difference techniques as applied to the differentiation problem and the Volterra equation of Abel type.

Discretization of Lavrent′ev's m-times iterated method. In [67, 68], Plato combined a Galerkin method for approximating (1) with Lavrent′ev's m-times iterated method in the case of an accretive Volterra operator A (see Section 3.3). Plato showed that the numerical realization $\mathbf{A}^N$ is lower Hessenberg; efficient solution methods exist for equations governed by such matrices. Order-optimal convergence rates for the combined discretization-regularization method were obtained.

Discretization of the Local Regularization Method. The analysis of a collocation method paired with the ideas of "local regularization" (cf. Section 3.2) was first considered in [56] where it was shown that collocation of the second-kind local regularization equation (13) (with a specific choice of discrete measure η_r) over the space of piecewise constant functions leads to a particularly simple discrete regularization procedure (known as "Beck's method" when applied to the IHCP). See [16] and the references therein for practical application of this procedure to the IHCP. We briefly describe this discrete process in what follows.

Let $h = T/N$ and let $r = \gamma h$ denote the discrete regularization parameter (indicating the length of the local regularization interval) for a fixed integer $\gamma \geq 1$. Then, given $u_1, \ldots, u_{i-1}$ (corresponding to the regularized solution u at $t_1, \ldots, t_{i-1}$), we determine u_i by first "predicting" an optimal constant-valued (i.e., *over*-regularized) solution $\hat{u}$ on the interval $(t_{i-1}, t_{i-1} + \gamma h]$, where $\hat{u}$ is constructed via a least squares fitting to the data at points $t_i, t_{i+1}, \ldots, t_i + \gamma h$. We next "correct" for over-regularization by retaining only the value of $\hat{u}$ at the position t_i; i.e., we set $u_i := \hat{u}$. Then the procedure is repeated, until all u_i have been determined in this sequential process. For obvious reasons, the discretized local method is often called a *predictor-corrector* regularization method.

In [55], a convergence theory was given for this discrete method for the one-smoothing convolution problem, and convergence was shown to be of optimal order. An analysis of the conditioning of the discretized ν-smoothing problem (and the dependence of condition numbers on ν, γ, and h) was discussed in [57]. More recently these regularization ideas were extended in [62] to variable r (i.e., variable γ) and nonconvolution problems, with the addition of an optional penalty term. The ideas were generalized in a different direction in [54], where there one seeks an optimal degree-d polynomial, for integer $d \geq 0$, in the "prediction" step.

In [60], the discrete local regularization approach was modified further, forming the basis for the method of *sequential Tikhonov regularization* for Volterra convolution problems. In this case, a local Tikhonov regularization is performed at each sequential step and, again, only the first component of the local solution is retained at each step. The cost of each local Tikhonov problem is reduced substantially using an efficient algorithm of Eldén [2], one which employs orthogonal transformations and takes advantage of the Toeplitz structure of discretizations of Volterra convolution problems. Although the convergence theory is again limited to one-smoothing problems (with optimal convergence rates), numerical examples in [60] were used to illustrate that the method works well when applied to the (infinitely-smoothing) IHCP. In addition, a preliminary numerical study of discrepancy principles to pick local Tikhonov regularization parameters was undertaken in [59], where it appears that *variable* regularization of solutions (effectively finding steep/sharp areas of solutions) is possible using these ideas.

5 Conclusion

We have reviewed some representative continuous and discrete regularization methods for first-kind Volterra problems with continuous kernels, paying particular attention to those methods which which tend to retain the Volterra structure of the original problem. As seen in the previous sections, there are many interesting open problems in this research area. In particular, the extension of methods and theoretical results to problems which are infinitely-smoothing, or even ν-smoothing for large ν, remains an important issue.

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