

## Exam 2 Solutions

1.  $f(x) = \frac{2x+1}{3\sqrt{x}}$

**Solution:** Here we use the quotient rule with  $u = 2x + 1$  and  $v = 3\sqrt{x}$ . So  $u' = 2$ , and  $v' = 3/(2\sqrt{x})$ . So

$$\begin{aligned} f'(x) &= \frac{vu' - uv'}{v^2} \\ &= \frac{\frac{3}{2\sqrt{x}}(2) - 3\sqrt{x}(2x+1)}{(3\sqrt{x})^2}. \end{aligned}$$

2.  $g(x) = \frac{\sin x}{x} + \frac{x}{\sin x}$

**Solution:** Here we have use the sum rule. For each summand, we use the quotient rule. So

$$g'(x) = \frac{x\cos x - \sin x}{x^2} + \frac{\sin x - x\cos x}{\sin^2 x}.$$

3.  $h(t) = (1 + \tan^4(\frac{t}{4}))^3$

**Solution:** Here we have to use the chain rule. So let  $u = \tan(\frac{t}{4})$ . So we can think of  $h(t)$  as the composite of two functions:  $(1 + u^4)^3$  and  $u$ . By the chain rule,

$$\begin{aligned} h'(t) &= \frac{d((1 + u^4)^3)}{du} u'(t) \\ &= 3(1 + u^4)^2(4u^3)\sec^2\frac{t}{4}\left(\frac{1}{4}\right) \\ &= 3\left(1 + \tan^4\frac{t}{4}\right)^2\left(4\tan^3\frac{t}{4}\right)\left(\sec^2\frac{t}{4}\right)\left(\frac{1}{4}\right). \end{aligned}$$

4. Find the absolute maximum and minimum values of the function  $g(x) = \sqrt{1-x^2}$  on the interval  $-1 \leq x \leq 1$ . Also include the  $x$ -coordinates of the

points where the extreme values occur. Note that you must use techniques from Calculus covered in class to do this problem. Other methods will give you no credit even though you get the correct answer.

**Solution:** First we find the critical points. By the chain rule and the rational power rule,

$$g'(x) = \frac{1}{2} \frac{1}{\sqrt{1-x^2}} (-2x) = \frac{-x}{\sqrt{1-x^2}}.$$

So there is only one critical point in the interior of the domain:  $x = 0$ . At points  $x = 1, -1$ , the derivative doesn't exist (and hence may be considered as critical points). But they also happen to be the endpoints of the interval. So we should find values of  $g(x)$  at  $x = 0, -1, 1$ . So we have

$$g(0) = 1, \quad g(1) = 0, \quad g(-1) = 0.$$

Since the function  $g(x)$  is continuous on this interval, the extreme values are from this set. So the absolute maximum value for  $g(x)$  on the given domain is 1 and occurs at  $x = 0$ . The absolute minimum value for  $g(x)$  on the given domain is 0 and occurs at  $x = -1/1$ .

**5.** Sand falls from a conveyor belt at the rate of  $8\text{m}^3/\text{min}$  onto the top of a conical pile. The height of the pile is always two-fifths of the base diameter. How fast is the radius changing when the pile is 2 m high? Recall that the volume of a cone of radius  $r$  and height  $h$  is  $\frac{1}{3}\pi r^2 h$ .

**Solution:** Let  $h$  be the height of the pile and  $r$  be the radius at time  $t$ . Then we are given that

$$h = \frac{2}{5}(2r) = \frac{4}{5}r.$$

So we can write the volume  $V$  of the pile at time  $t$  as

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2 \left(\frac{4}{5}r\right) = \frac{4}{15}\pi r^3.$$

By the chain rule,

$$\frac{dV}{dt} = \frac{4}{15}\pi(3r^2)\frac{dr}{dt} = \frac{4}{5}\pi r^2 \frac{dr}{dt}.$$

We know that  $\frac{dV}{dt} = 8$  and we need to find  $\frac{dr}{dt}$  when  $h = 2$ . But when  $h = 2$ ,  $r = \frac{5}{4}h = \frac{5}{2}$  m. So substituting in the last equation,

$$8 = \frac{4}{5}\pi\left(\frac{5}{2}\right)^2\frac{dr}{dt}.$$

Solving this equation, we get  $\frac{dr}{dt} = \frac{8}{5\pi}$  m/sec.

**6.** Explorers on a small airless planet used a spring gun to launch a ball bearing vertically upward from the surface at a launch velocity of 10 m/sec. Because the acceleration of gravity at the planet's surface was  $a$  m/sec<sup>2</sup>, the explorers expected the ball bearing to reach a height of  $s = 10t - (1/2)at^2$  meters  $t$  seconds later. The ball reached its maximum height 25 seconds after it was launched. What is the value of  $a$ ?

**Solution:** The ball bearing reaches its maximum height when the velocity is zero. Now the velocity at time  $t$  is

$$v(t) = s'(t) = 10 - at.$$

Since the maximum height is reached at  $t = 25$  s, we have

$$0 = 10 - a(25).$$

Solving this equation for  $a$ , we get  $a = \frac{2}{5}$  m/s<sup>2</sup>.

**7.** Find an equation for the tangent line to the graph of the equation  $y^3 + x^2y - y = 1$  at the point  $(1, 1)$ .

**Solution:** Since the slope of the tangent line is the value of the derivative at that point, we should find  $\frac{dy}{dx}$  at  $(1, 1)$ . We use implicit differentiation for this purpose. Differentiating both the sides of the given equation,

$$3y^2\frac{dy}{dx} + (2x)y + x^2\frac{dy}{dx} - \frac{dy}{dx} = 0.$$

Note that we used the product rule for  $x^2y$ . Solving this equation for  $\frac{dy}{dx}$ , we get

$$\frac{dy}{dx} = \frac{-2xy}{3y^2 + x^2 - 1}.$$

Substituting the point  $(1, 1)$  in this equation, we have the slope of the tangent line to be  $\frac{dy}{dx} = \frac{-2}{3}$ . So an equation of the tangent line is

$$y - 1 = -\frac{2}{3}(x - 1).$$

8. Find the value or values of  $c$  that satisfy the equation

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

in the conclusion of the Mean Value Theorem for the function  $f(x) = x + \frac{1}{x}$  on the interval  $[\frac{1}{2}, 2]$ .

**Solution:** Here  $a = \frac{1}{2}$  and  $b = 2$ . So we get

$$\frac{f(b) - f(a)}{b - a} = \frac{f(2) - f(\frac{1}{2})}{2 - \frac{1}{2}} = \frac{(2 + \frac{1}{2}) - (\frac{1}{2} + 2)}{\frac{3}{2}} = 0.$$

So we need to find  $c$  in the domain such that  $f'(c) = 0$ . But  $f'(x) = 1 - \frac{1}{x^2}$ . So we first solve the equation  $1 - \frac{1}{c^2} = 0$ . This has two solutions  $c = \pm 1$ . But  $c = -1$  is not the domain, so the only value that satisfies the conclusion of the Mean Value Theorem is  $c = 1$ .

9. Find the derivative of the function  $k(\theta) = (\sin(\theta + 2))^{4/5}$ .

**Solutions:** Here we have to use the rational power rule and the chain rule. So

$$k'(\theta) = \frac{4}{5}(\sin(\theta + 2))^{-1/5}(\cos(\theta + 2))(1).$$

10. Find the following limit.

$$\lim_{x \rightarrow 4} \frac{\sin(\sqrt{x} - 2)}{x - 4}$$

**Solution:** We use the following limit:

$$\lim_{x \rightarrow a} \frac{\sin(f(x))}{f(x)} = 1,$$

if  $\lim_{x \rightarrow a} f(x) = 0$ . So the given limit is

$$\begin{aligned} & \lim_{x \rightarrow 4} \frac{\sin(\sqrt{x} - 2)}{x - 4} \\ &= \lim_{x \rightarrow 4} \frac{\sin(\sqrt{x} - 2)}{\sqrt{x} - 2} \frac{\sqrt{x} - 2}{x - 4} \\ &= \lim_{x \rightarrow 4} \frac{\sin(\sqrt{x} - 2)}{\sqrt{x} - 2} \lim_{x \rightarrow 4} \frac{\sqrt{x} - 2}{x - 4} \\ &= 1 \left( \lim_{x \rightarrow 4} \frac{\sqrt{x} - 2}{x - 4} \right) \\ &= \lim_{x \rightarrow 4} \frac{\sqrt{x} - 2}{(\sqrt{x} - 2)(\sqrt{x} + 2)} \\ &= \lim_{x \rightarrow 4} \frac{1}{\sqrt{x} + 2} \\ &= \frac{1}{4}. \end{aligned}$$