Fall 2002 **MATH 340, Section 2**

Exam 1 Solutions

1. $\lim_{x\to 1} \frac{x^2+3x-4}{x-1}$

Solution:

$$\begin{aligned} & \lim_{x\to 1} \frac{x^2+3x-4}{x-1} \\ = & \lim_{x\to 1} \frac{(x-1)(x+4)}{x-1} \\ = & \lim_{x\to 1} x+4 \\ = & 1+4 \\ = & 5. \end{aligned}$$

So the given limit is 5.

2. $\lim_{x\to -2^+} \frac{x+1}{|x+2|}$

Solution: Note that as $x \to -2$ from the right side, $|x+2|$ approaches 0 from the right side as well. The numerator $x+1$ approaches -1 (from the right side as well, but it doesn't matter). So the quotient is always negative and increasing in its absolute value. So

$$\lim_{x\to -2^+} \frac{x+1}{|x+2|} = -\infty.$$

3. $\lim_{x\to 1} \frac{x-\sqrt{-3x+4}}{x-1}$

Solution: If we plug in $x=1$, then both the numerator and the denominator are zero. So we should try to cancel a factor of $x-1$. We have

$$\begin{aligned}
& \lim_{x\to 1} \frac{x-\sqrt{-3x+4}}{x-1} \\
= & \lim_{x\to 1} \frac{x-\sqrt{-3x+4}}{x-1}\left(\frac{x+\sqrt{-3x+4}}{x+\sqrt{-3x+4}}\right) \\
= & \lim_{x\to 1} \frac{x^2+3x-4}{(x-1)(x+\sqrt{-3x+4})} \\
= & \lim_{x\to 1} \frac{(x-1)(x+4)}{(x-1)(x+\sqrt{-3x+4})} \\
= & \lim_{x\to 1} \frac{x+4}{x+\sqrt{-3x+4}} \\
= & \frac{1+4}{1+\sqrt{1}} \\
= & \frac{5}{2}.
\end{aligned}$$

4. Given a function $f(x)$, state the definition of the derivative $f'(x)$ using limits.

Definition: The value of the derivative of the function $f(x)$ at $x = x_0$ is given by

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$$

provided the limit exists.

5. Consider the function

$$f(x) = \begin{cases} x+1 & -1 \le x < 0 \\ 2x & 0 < x < 1 \\ 1 & x = 1 \\ -2x+4 & 1 < x < 2 \\ 0 & 2 < x < 3 \end{cases}$$

For this function, answer the following questions: a) Does $f(1)$ exist? b) Does $\lim_{x\to 1} f(x)$ exist and if so, what is it? c) Does $\lim_{x\to 1} f(x) = f(1)$? d) Is f continuous at $x = 1$?

Solution: a) From the given definition of $f(x)$, $f(1) = 1$. So $f(1)$ exists and is equal to 1.
b) Again from the given definition,

$$\begin{aligned} & \lim_{x \to 1^-} f(x) \\ = & \lim_{x \to 1^-} 2x \\ = & 2. \end{aligned}$$

The limit from the other side is

$$\begin{aligned} & \lim_{x \to 1^+} f(x) \\ = & \lim_{x \to 1^+} -2x + 4 \\ = & 2. \end{aligned}$$

Since these two limits are equal, we have that $\lim_{x \to 1} f(x)$ exists and is equal to 2.
c) From parts a) and b), $f(1) = 1$ and $\lim_{x \to 1} f(x) = 2$ and so they are not equal.
d) By the test for continuity at $x = 1$, we need to have the values of the limit $\lim_{x \to 1} f(x)$ and $f(1)$ to be equal. Since they are not by c), the function $f(x)$ is not continuous at $x = 1$.

6. Find the limit

$$\lim_{x \to 0.5^-} \sqrt{\frac{x^2 + 2}{x + 1}}.$$

Solution: Here we just plug in $x = 0.5$ to see if we get a number and we do. So

$$\lim_{x \to 0.5^-} \sqrt{\frac{x^2 + 2}{x + 1}} = \sqrt{\frac{2.25}{1.5}} = \sqrt{1.5}.$$

7. Use the definition of the derivative of a function at a point to calculate the derivative of $f(x) = 1 - x^2$ at $x = 1$. Note that you have to use the definition, you will not get any credit if you use the rules of differentiation.

Solution: By using the definition of the derivative (see answer 4), we have the derivative at $x = 1$:

$$\begin{aligned} & \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} \\ = & \lim_{h\to 0} \frac{(1-(1+h)^2) - (1-(1)^2)}{h} \\ = & \lim_{h\to 0} \frac{-2h - h^2}{h} \\ = & \lim_{h\to 0} -2 - h \\ = & -2. \end{aligned}$$

So the derivative of $f(x)$ at $x = 1$ is -2. This can be checked by the rules of differentation. The derivative of $f(x)$ by these rules is $-2x$ and hence at $x = 1$ is -2.

8. Find an equation for the tangent line to the graph of the function $f(x) = 3x^3 - 2x - 7$ at a point whose x coordinate is 1. You can use the fact that the derivative of the function $f(x)$ is $f'(x) = 9x^2 - 2$.

Solution: Recall that the slope of the tangent line to the graph of a function (if it exists) is the value of the derivative of the function at that point. Since we are given the derivative $f'(x)$, we get that the slope of the tangent line at $x = 1$ is $9(1)^2 - 2 = 7$. The y-coordinate of the point at $x = 1$ is $3(1)^3 - 2(1) - 7 = -6$. So the equation of the line is $y - (-6) = 7(x - 1)$, which simplifies to $y = 7x - 13$.

9. Using the Intermediate Value Theorem, show that the equation, $x^3 - 2x^2 - 3x + 1 = 0$, has at least one solution between $x = 0$ and $x = 1$. Justify your answer.

Solution: Let $f(x) = x^3 - 2x^2 - 3x + 1$. Then $f(x)$ is continuous at all x (as it's a polynomial function). Also $f(0) = 1$ and $f(1) = -3$. Since $-3 < 0 < 1$, by the Intermediate Value Theorem, there exists some point x_0 such that $0 < x_0 < 1$ and $f(x_0) = 0$. Such an x_0 gives a solution of the given equation.

10. Recall that we say $\lim_{x\to x_0} f(x) = L$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$0 < |x - x_0| < \delta \implies |f(x) - L| < \epsilon.$$

Let $\epsilon = \frac{1}{2}$, $f(x) = 3x - 2$, and $x_0 = 1$. Find L and the corresponding maximal δ.

Solution: Using the rules of limits from class, we see that $L = 3(1) - 2 = 1$ (since $f(x)$ is a polynomial). So we start with $|f(x) - L| < \varepsilon$. We have

$$\begin{aligned} & |f(x) - L| < \varepsilon \\ \text{so} \quad & |(3x-2) - 1| < \varepsilon \\ \text{so} \quad & |3x - 3| < \varepsilon \\ \text{so} \quad & 3|x - 1| < \varepsilon \\ \text{so} \quad & |x - 1| < \frac{\varepsilon}{3} \\ \text{so} \quad & |x - 1| < \frac{0.5}{3} = \frac{1}{6}. \end{aligned}$$

Since we are trying to find δ such that $|x - 1| < \delta$ implies $|f(x) - L| < \varepsilon$, the maximal value of δ is $1/6$.