Final Exam Spring 2001 Solutions

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1. Compute the following limits.

(a)

$$\begin{aligned}\lim_{x\to -3}\frac{x+3}{x^2+7x+12} &= \lim_{x\to -3}\frac{x+3}{(x+3)(x+4)}\\ &= \lim_{x\to -3}\frac{1}{(x+4)}\\ &= 1.\end{aligned}$$

(b)

$$\begin{aligned}\lim_{x\to 9}\frac{\sqrt{x}-3}{x-9} &= \lim_{x\to 9}\frac{(\sqrt{x}-3)(\sqrt{x}+3)}{(x-9)(\sqrt{x}+3)}\\ &= \lim_{x\to 9}\frac{x-9}{(x-9)(\sqrt{x}+3)}\\ &= \lim_{x\to 9}\frac{1}{(\sqrt{x}+3)}\\ &= \frac{1}{6}\end{aligned}$$

(c)

$$\begin{aligned}\lim_{x\to 0}\frac{x}{\sin 5x} &= \lim_{x\to 0}\frac{\frac{x}{x}}{5\frac{\sin 5x}{5x}}\\ &= \lim_{x\to 0}\frac{1}{5\frac{\sin 5x}{5x}}\\ &= \frac{1}{5\lim_{x\to 0}\frac{\sin 5x}{5x}}\\ &= \frac{1}{5\cdot 1}\\ &= \frac{1}{5}.\end{aligned}$$

(d)

$$\lim_{x\to 2^+}\frac{|x-2|}{2-x} = \frac{x-2}{2-x} = -1.$$

Here we used the fact that as $x \to 2^+$, $x \geq 2$ and so $|x-2| = (x-2)$.

2. Find the following derivatives.

(a)

$$\begin{aligned}\frac{d}{dx}(x^2\sqrt{\sin x}) &= x^2\frac{d}{dx}\sqrt{\sin x} + \sqrt{\sin x}(2x) \\ &= x^2\frac{1}{2\sqrt{\sin x}}\cos x + \sqrt{\sin x}(2x)\end{aligned}$$

(b)

$$\begin{aligned}\frac{d}{dx}\frac{x^2+x+1}{\sqrt{x^2+1}} &= \frac{\sqrt{x^2+1}(2x+1) - (x^2+x+1)\frac{d}{dx}\sqrt{x^2+1}}{(\sqrt{x^2+1})^2} \\ &= \frac{\sqrt{x^2+1}(2x+1) - (x^2+x+1)\frac{1}{2\sqrt{x^2+1}}(2x)}{(\sqrt{x^2+1})^2}\end{aligned}$$

(c)

$$\begin{aligned}\frac{d}{dx}\int_1^{\sqrt{x}}\sin^5 t\,dt &= \sin^5(\sqrt{x})\frac{d}{dx}\sqrt{x} \\ &= \sin^5(\sqrt{x})(\frac{1}{2\sqrt{x}}).\end{aligned}$$

Here in the first step we used the fundamental theorem of Calculus.

3. Use the definition of derivatives to compute $f'(2)$ where $f(x) = \frac{1}{3x}$.

The derivative of a function $y = f(x)$ at $x = x_0$ is given by the limit

$$f'(x_0) = \lim_{h\to 0}\frac{f(x_0+h) - f(x_0)}{h}$$

provided the limit exists. Using this definition in our case,

$$\begin{aligned}
f'(2) &= \lim_{h\to 0}\frac{f(2+h)-f(2)}{h}\\
&= \lim_{h\to 0}\frac{\frac{1}{3(2+h)}-\frac{1}{6}}{h}\\
&= \lim_{h\to 0}\frac{\frac{6-3(2+h)}{6(3(2+h))}}{h}\\
&= \lim_{h\to 0}\frac{\frac{-3h}{6(3(2+h))}}{h}\\
&= \lim_{h\to 0}\frac{-3h}{6(3(2+h))h}\\
&= \lim_{h\to 0}\frac{-3}{18(2+h)}\\
&= -\frac{1}{12}.
\end{aligned}$$

4. Evaluate the following indefinite integrals.

(a)

$$\int x^2\sqrt{1+10x^3}dx.$$

First we make a substitution $u = 1+10x^3$. Then

$$\frac{du}{dx} = 30x^2.$$

So $x^2dx = \frac{1}{30}du$. So

$$\begin{aligned}
\int x^2\sqrt{1+10x^3}dx &= \int \sqrt{u}\frac{1}{30}du\\
&= \frac{1}{30}\frac{u^{3/2}}{3/2}+c\\
&= \frac{2}{90}(1+10x^3)^{3/2}+c.
\end{aligned}$$

(b)

$$\int \frac{x}{\sqrt{x+1}}dx.$$

Once again, we need to make a substitution. Let $u = x + 1$. Then $\frac{du}{dx} = 1$ and so $dx = du$. Also note that $x = u - 1$. So

$$\begin{aligned}
\int \frac{x}{\sqrt{x+1}} dx &= \int \frac{u-1}{\sqrt{u}} du \\
&= \int (\sqrt{u} - \frac{1}{\sqrt{u}}) du \\
&= \frac{2}{3} u^{3/2} - 2\sqrt{u} + c \\
&= \frac{2}{3}(x+1)^{3/2} - 2\sqrt{x+1} + c.
\end{aligned}$$

5. Evaluate the following definite integrals.

(a)

$$\begin{aligned}
\int_0^1 (x^2+1)(3x-2)dx &= \int_0^1 (3x^3 + 3x - 2x^2 - 2)dx \\
&= 3\int_0^1 x^3 dx + 3\int_0^1 x dx - 2\int_0^1 x^2 dx - 2\int_0^1 dx \\
&= 3(\frac{1^4 - 0^4}{4}) + 3(\frac{1^2 - 0^2}{2}) - 2(\frac{1^3 - 0^3}{3}) - 2(1-0) \\
&= \frac{3}{4} + \frac{3}{2} - \frac{2}{3} - 2 \\
&= -\frac{5}{12}.
\end{aligned}$$

(b)

$$\int_0^\pi \sin x \cos^2 x dx.$$

First we make the substitution $u = \cos x$. Then $\frac{du}{dx} = -\sin x$ and so $\sin x dx = -du$. Also when $x = 0$, $u = 1$ and when $x = \pi$, $u = -1$. So

$$\begin{aligned}
\int_0^\pi \sin x \cos^2 x dx &= \int_1^{-1} u^2(-du) \\
&= -\int_1^{-1} u^2 du \\
&= \int_{-1}^1 u^2 du \\
&= \frac{1^3 - (-1)^3}{3} \\
&= \frac{2}{3}.
\end{aligned}$$

6. Find the equation of the tangent line to the curve $x^3 + y^3 = 9xy$ at the point $(4, 2)$.

We need to compute the slope of the tangent line. In order to do that, we compute the derivative using implicit differentiation. So differentiating implicitly the given equation, we get

$$3x^2 + 3y^2y' = 9(xy' + y).$$

Substituting $x = 4,\ y = 2$ in this equation, we get

$$48 + 12y' = 9(4y' + 2).$$

Solving this equation for y', we get $y' = 5/4$. So the slope of the tangent line is $5/4$ and it passes through $(4, 2)$. So the equation is

$$y - 2 = \frac{5}{4}(x - 4)$$

which simplifies to $5x - 4y = 12$.

7. 100 m^3 of oil is spilled when a tanker collides with a tuna boat. The resulting oil slick forms a right circular cylinder on the surface of the water. If the thickness (h) of the slick is decreasing at a rate of 0.001 m/sec, how fast is the radius (r) increasing when the slick is 0.01 m thick? Note that $V = \pi r^2 h$.

Note that the volume of the cylinder is always constant, it being the volume of the spilled oil. So we get

$$100 = \pi r^2 h$$

which can be solved to get

$$h = \frac{100}{\pi r^2}.$$

Now we are given that $\frac{dh}{dt} = -0.001$ (negative sign means it's decreasing). From the last equation

$$\frac{dh}{dt} = \frac{100}{\pi}\frac{-2}{r^3}\frac{dr}{dt}.$$

We are trying to find $\frac{dr}{dt}$ for a given h. But again using the constant volume, we can get r for a given value of h. So when $h = 0.01$m, we have

$$100 = \pi r^2(0.01),$$

which gives

$$r = \frac{100}{\sqrt{\pi}}.$$

Substituting these values in the equation for $\frac{dh}{dt}$, we get

$$-0.001 = \frac{100}{\pi}\frac{-2}{(\frac{100}{\sqrt{\pi}})^3}\frac{dr}{dt}$$

which can be solved for $\frac{dr}{dt}$. This gives

$$\frac{dr}{dt} = (0.0005)(\frac{100}{\sqrt{\pi}})^3\frac{\pi}{100} = \frac{5}{\sqrt{\pi}}\ \text{m/sec.}$$

Note that the positive sign means that r is increasing at this time.

Problem 8. Let ℓ, h denote the length, and height of the box (the base is a square, so the length is the same as the width). The required areas are

$$\text{Area(bottom square)} = \ell^2$$

$$\text{Area(a side face)} = \ell h$$

The total cost will be (one bottom square for $\$2/ft^2$ and four side faces for $\$1.5/ft^2$)

$$C = \ell^2 \cdot 2 + 4\ell h \cdot 1.5 = 2\ell^2 + 6\ell h$$

The volume of the box is

$$\text{Volume(box)} = 18 = \ell^2 h,$$

hence $h = 18/\ell^2$, which shows that the cost can be expressed as a function of the variable ℓ:

$$C = 2\ell^2 + 6\ell \cdot 18/\ell^2 = 2\ell^2 + 108/\ell$$

The first derivative of this function with respect to ℓ is

$$\frac{dC}{d\ell} = 4\ell - 108/\ell^2$$

The derivative is zero when

$$4\ell - 108/\ell^2 = 0, \quad \text{or} \quad \ell^3 = 27, \quad \text{or} \quad \ell = 3$$

Correspondingly, the minimal cost is

$$C = 2 \cdot 3^2 + 108/3 = \$54$$

Note that the cost is minimal at $\ell = 3$ due to the fact that $\frac{d^2C}{d\ell^2} > 0$ when $\ell = 3$. ($\frac{d^2C}{d\ell^2} = 4 + 108(2/\ell^3)$ and so is positive.)

Problem 9. First note that $a = 0$ and $b = 1$ in this problem. Also $n = 4$ and so $\frac{b-a}{n} = 1$. The interval $[0, 4]$ is partitioned as follows:

$$[0, 4] = [0, 1] \cup [1, 2] \cup [2, 3] \cup [3, 4]$$

The right end points of these sub-intervals are $1, 2, 3, 4$. So $c_1 = 1,\ c_2 = 2,\ c_3 = 3,\ c_4 = 4$. Since the function is $f(x) = x^3$, the Riemann sum is

$$\frac{b-a}{n}(f(c_1) + f(c_2) + f(c_3) + f(c_4)) = 1^3 \cdot 1 + 2^3 \cdot 1 + 3^3 \cdot 1 + 4^3 \cdot 1$$

(1 is the length of each sub-interval).

Problem 10. a)

$$\int_0^5 x^2 - 4x + 3 = (\frac{1}{3}x^3 - 2x^2 + 3x)|_0^5 = \frac{20}{3}$$

b) First we determine the intersection points of the graph with the x-axis by solving the equation

$$x^2 - 4x + 3 = 0$$

The two solutions are $x = 1, 3$ and the equation of the x-axis is $y = 0$, so the shaded area is

$$\int_0^1 [(x^2 - 4x + 3) - 0]dx + \int_1^3 [0 - (x^2 - 4x + 3)]dx + \int_3^5 [(x^2 - 4x + 3) - 0]dx =$$

$$= (\frac{1}{3}x^3 - 2x^2 + 3x)|_0^1 - (\frac{1}{3}x^3 - 2x^2 + 3x)|_1^3 + (\frac{1}{3}x^3 - 2x^2 + 3x)|_3^5 = \frac{28}{3}$$

Problem 11.

a) The function f is not continuous at 3 (the denominator vanishes at 3).

b) f' vanishes at 0 and is not defined at 3. Notice that f' is negative to the left of 0 and to the right of 3 and it is positive between 0 and 3, so

$$f(x) \text{ decreases on } (-\infty, 0) \cup (3, \infty)$$

$$f(x) \text{ increases on } (0, 3)$$

c) f'' vanishes at $-3/2$ and is not defined at 3. Notice that f'' is negative to the left of $-3/2$ and it is positive to the right of $-3/2$, so

$$f(x) \text{ is concave up on } (-3/2, 3) \cup (3, \infty)$$

$$f(x) \text{ is concave down on } (-\infty, -3/2)$$

d) According to b), the only possibility is $x = 0$. (We ignore $x = 3$ because the function is not continuous there.) $x = 0$ is a local minimum since f' is negative to the left of 0 and positive to right of 0.

e) Inflection point: $x = -3/2$. First note that f'' is zero at $x = -3/2$ and does not exist at $x = 3$. But $x = 3$ is not in the domain and so we ignore it. Next note that $f(x)$ is concave up to the right of this point and concave down to the left. So this is a point of inflection.

f) Vertical assymptote: $x = 3$ (the denominator of f vanishes at 3).

Horizontal assymptote: $y = 1$ due to the fact that

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{x^2}{(x-3)^2} = \lim_{x\to\pm\infty} \frac{x^2}{x^2-6x+9} = 1.$$

Problem 12.

$$y = \int (3\sin 2x + 6)dx = 3\int \sin 2x dx + 6\int dx = -\frac{3}{2}\cos 2x + 6x + C$$

If $y(0) = 1$, then

$$-\frac{3}{2}\cos 0 + 6\cdot 0 + C = 1$$

Notice that $\cos 0 = 1$, so solving for C we get $C = \frac{5}{2}$. Thus

$$y(x) = -\frac{3}{2}\cos 2x + 6x + \frac{5}{2}.$$

Problem 13. a) The Mean Value Theorem is applicable to f due to the fact that f is continuous.

b) The slope of the line containing the points $(0, -9)$ and $(3, 33)$ is

$$m = \frac{33-(-9)}{3-0} = \frac{42}{3} = 14$$

The slope of the line tangent to the graph of f at the point $(c, f(c))$ is equal to

$$f'(c) = 6c + 5$$

Setting these two slopes equal we get

$$6c + 5 = 14$$

Solve for c:

$$6c = 9, \quad \text{so} \quad c = \frac{3}{2}.$$

Notice that the solution is in the interval $(0, 3)$ as desired.