Radial weak solutions for the Perona-Malik equation as a differential inclusion

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Abstract

The Perona-Malik equation is an ill-posed forward-backward parabolic equation with some application in image processing. In this paper, we study the Perona-Malik type equation on a ball in an arbitrary dimension n and show that there exist infinitely many radial weak solutions to the homogeneous Neumann boundary problem for smooth nonconstant radially symmetric initial data. Our approach is to reformulate the n-dimensional equation into a one-dimensional equation, to convert the one-dimensional problem into an inhomogeneous partial differential inclusion problem, and to apply a Baire's category method to the differential inclusion to generate infinitely many solutions.

Keywords: Perona-Malik type equation; Infinitely many radial weak solutions in all dimensions; Partial differential inclusion; Baire's category method.

1. Introduction

In this paper we investigate the existence of radial weak solutions for an n-dimensional Perona-Malik type equation under the homogeneous Neumann boundary condition and radially symmetric initial data:

$$\begin{cases} u_t = \operatorname{div}(a(|Du|^2)Du) & \text{in } \Omega_T := \Omega \times (0,T) \\ \frac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega \times (0,T) \\ u(x,0) = u_0(x) & \text{for } x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega := B_R(0)$ is the open ball in $\mathbb{R}^n$ ($n \geq 1$) with center 0 and radius $R > 0$, $T > 0$ is a given time, $\mathbf{n}$ is outward unit normal to $\partial\Omega$, $u_0 \colon \Omega \to \mathbb{R}$ is a radially symmetric initial function, and $a \in C^{2,\alpha}([0,\infty))$, for some $\alpha \in (0,1)$, is a positive function satisfying the following:

$$2p\, a'(p) + a(p) \begin{cases} > 0 & \text{for } 0 \leq p < 1 \\ = 0 & \text{for } p = 1 \\ < 0 & \text{for } p > 1, \end{cases} \qquad \text{and} \qquad \lim_{p\to\infty} \sigma(p) = 0, \tag{1.2}$$

with $\sigma(p) = a(p^2)p$ for $p \in \mathbb{R}$. We can relax the function a in (1.2) to $a \in C^{2,\alpha}([0,1)) \cap C([0,\infty))$ with σ strictly decreasing on $[1,\infty)$ without affecting the result of this paper. The notation and assumptions in this paragraph will be kept throughout the paper unless otherwise stated.

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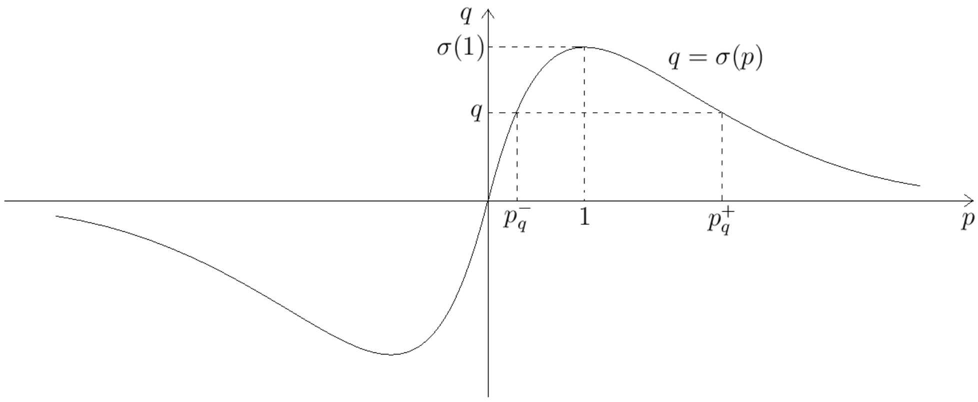

Figure 1: The graph of a typical function $q = \sigma(p)$

In the original paper of Perona & Malik [27], they proposed an anisotropic diffusion model (1.1) for denoising and edge enhancement of a computer vision, where $\Omega \subset \mathbb{R}^2$ is a square and $a(p)$ is given as

$$\text{either} \quad a(p) = \frac{1}{1 + \frac{p}{k^2}} \quad \text{or} \quad a(p) = \exp\left(-\frac{p}{2k^2}\right),$$

with the fixed threshold $k > 0$ according to some experimental purposes. In our case we have chosen $k = 1$ for simplicity, and the class of functions a contains all these functions.

For a general discussion, let us assume for the moment that $\Omega \subset \mathbb{R}^n$ is a bounded C^1 domain and that $a(p) = (1 + p)^{-1}$. Given a point $x \in \bar{\Omega}$, we say that the initial condition $u_0 \in C^1(\bar{\Omega})$ is *subcritical* at x if $|Du_0(x)| < 1$, *supercritical* at x if $|Du_0(x)| > 1$, and *critical* at x if $|Du_0(x)| = 1$. The initial condition u_0 is *transcritical* in Ω if there are two points $x, y \in \Omega$ with $|Du_0(x)| < 1$ and $|Du_0(y)| > 1$. Existence of global or local classical solutions to problem (1.1) depends heavily on the initial condition u_0. Kawohl & Kutev [17] showed that a global classical solution exists in any dimension if u_0 is subcritical in $\bar{\Omega}$. They also proved that the problem cannot admit a global classical solution for $n = 1$ if u_0 is transcritical in Ω under some technical assumptions, and these assumptions were completely removed later by Gobbino [14]. Concerning the Perona-Malik type equation, it had been the general belief that classical solutions can only exist if the initial data are smooth, even analytic, at supercritical points; this was formally streamlined in Kichenassamy [18]. As regards the class of suitable initial conditions for classical solutions of (1.1), Ghisi & Gobbino [11] has recently established that for $n = 1$, the set of initial conditions for which problem (1.1) has a local classical solution is dense in $C^1(\bar{\Omega})$.

The situation concerning the existence of a global classical solution to (1.1) with a transcritical initial condition for $n \geq 2$ turns out to be quite different from the case $n = 1$. The first existence result of global classical solutions with transcritical u_0 for $n \geq 2$ was obtained by Ghisi & Gobbino [12], where they constructed a class of global radial $C^{2,1}$ solutions with suitably chosen radial initial data transcritical on an annulus centered at the origin; these solutions also have the property of finite-time extinction of supercritical region. In contrast to the one-dimensional result of [14, 17] mentioned above, their result showed a quite different feature of the higher dimensional problem.

On the other hand, in the radial case, Ghisi & Gobbino [13] also proved that a global C^1 solution cannot exist if the gradient of initial condition u_0 is very large at a point. Therefore, requirement of regularity of the solution (e.g., classical or C^1) would prevent the existence of such a solution if the initial data should be arbitrarily given and transcritical.

When the initial condition u_0 is any given smooth function (satisfying certain compatibility condition on $\partial\Omega$), it seems natural to lower the expectation on the regularity of solutions by finding plausible weak solutions to (1.1). Even under the lowering of regularity have enormous difficulties occurred in the existence of weak solutions. Among many different approaches and attempts in this direction, e.g., the Γ-limit method in Bellettini & Fusco [3], the Young measure solutions in Chen & Zhang [4], and numerical scheme analyses in Esedoglu [9] and Esedoglu & Greer [10], to our best knowledge, Zhang [30] was the first to successfully prove that, for $n = 1$, there are infinitely many Lipschitz weak solutions to (1.1) for any given smooth nonconstant initial data u_0; his method uses the variational technique of partial differential inclusion together with the so called in-approximation method or convex integration. In this paper, we generalize Zhang's method to the case of radial weak solutions to problem (1.1) in all dimensions. Our generalization can also deal with other ill-posed forward-backward diffusion problems (see, e.g., the pioneering work of Höllig [16] and its recent generalization by Zhang [31]), but we will not include the results in those directions in this paper.

For $\alpha \in (0,1)$, we use $C^{3+\alpha,1+\alpha/2}(\bar{\Omega}_T)$ to denote the parabolic Hölder space of functions $u \in C^0(\bar{\Omega}_T)$ such that $u_t, u_{x_i t}, u_{x_i}, u_{x_i x_j}, u_{x_i x_j x_k} \in C^0(\bar{\Omega}_T)$ and that the quantities

$$\sup_{\substack{x\in\Omega\\ s,t\in(0,T),s\neq t}} \frac{|u_{x_i t}(x,s) - u_{x_i t}(x,t)|}{|s-t|^{\alpha/2}}, \quad \sup_{\substack{x\in\Omega\\ s,t\in(0,T),s\neq t}} \frac{|u_{x_i x_j x_k}(x,s) - u_{x_i x_j x_k}(x,t)|}{|s-t|^{\alpha/2}},$$

$$\sup_{\substack{x,y,\in\Omega,x\neq y\\ t\in(0,T)}} \frac{|u_{x_i t}(x,t) - u_{x_i t}(y,t)|}{|x-y|^{\alpha}}, \quad \sup_{\substack{x,y,\in\Omega,x\neq y\\ t\in(0,T)}} \frac{|u_{x_i x_j x_k}(x,t) - u_{x_i x_j x_k}(y,t)|}{|x-y|^{\alpha}}$$

are all finite, where $i, j, k \in \{1, \dots, n\}$.

We state the main result of this paper in the following theorem.

Theorem 1.1. *Let $u_0 \in C^{3,\alpha}(\bar{\Omega})$ be a radially symmetric function with $u_0(x) = v_0(|x|)$ and*

$$M := \max_{\bar{\Omega}} |Du_0| = \max_{s\in[0,R]} |v_0'(s)| > 0$$

such that the compatibility conditions hold:

$$v_0'(R) = 0, \quad v_0'''(R) + \frac{n-1}{R} v_0''(R) = 0. \tag{1.3}$$

Then the forward-backward Neumann problem (1.1) admits infinitely many radial weak solutions $u \in W^{1,\infty}(\Omega_T)$ satisfying the following:

(a) For every $\xi \in C_0^1(\Omega_T)$,

$$\int_{\Omega_T} (u_t \xi + a(|Du|^2) Du \cdot D\xi) dxdt = 0. \tag{1.4}$$

(b) *The solutions u are (uniformly and locally) classical near* $\{\partial\Omega \cup \{0\}\} \times [0,T]$ *in the sense that there exists a constant* δ_0 *with* $0 < \delta_0 < R/2$, *independent of u, such that*

$$\begin{cases} u \in C^{3+\alpha,1+\alpha/2}\left(\{\overline{B_{\delta_0}(0)} \cup \overline{(B_R(0) \setminus B_{R-\delta_0}(0))}\} \times [0,T]\right), \\ u_t = \operatorname{div}(a(|Du|^2)Du) \text{ pointwise in } \{B_{\delta_0}(0) \cup (B_R(0) \setminus \overline{B_{R-\delta_0}(0)})\} \times (0,T). \end{cases} \tag{1.5}$$

(c) *The initial condition holds:*

$$u(x,0) = u_0(x) \quad \forall\, x \in \bar{\Omega}.$$

(d) *The boundary condition is satisfied:*

$$\frac{\partial u}{\partial \mathbf{n}}(x,t) = 0 \quad \forall (x,t) \in \partial\Omega \times [0,T]. \tag{1.6}$$

(e) *The almost gradient maximum principle holds when* u_0 *is critical or supercritical at some point in* Ω; *that is, if* $M \geq 1$, *then, given any* $\epsilon > 0$, *we can choose the solutions u to satisfy the following:*

$$\|Du\|_{L^\infty(\Omega_T;\mathbb{R}^n)} \leq M + \epsilon. \tag{1.7}$$

(f) *The conservation of mass:*

$$\int_\Omega u(x,t)dx = \int_\Omega u_0(x)dx \quad \forall t \in [0,T]. \tag{1.8}$$

The proof of this theorem will be given in Section 4.

Observe that the condition $u_0 \in C^{3,\alpha}(\bar{\Omega})$ requires $v_0'(0) = v_0'''(0) = 0$. The initial $C^{3,\alpha}(\bar{\Omega})$-regularity and the compatibility condition (1.3) are suitable for the $C^{3+\alpha,1+\alpha/2}$-regularity of u^* in Theorem 2.2 below, which is only needed in the verification of gradient maximum principle. When $n = 1$ and $a(p) = (1+p)^{-1}$, our main theorem is equivalent to the main theorem of Zhang [30] in the following sense. When $n = 1$, the second compatibility condition in (1.3) may not be needed for the $C^{2+\alpha,1+\alpha/2}$-regularity and the gradient maximum principle of the solution u^*, as stated in [30]; in this case, we may lower the regularity of the initial condition as $u_0 \in C^{2,\alpha}(\bar{\Omega})$ and drop the second condition in (1.3), then even-extend the initial function in Zhang's case to obtain an initial function on $[-R,R]$ satisfying the conditions of our initial function u_0, and finally restrict our solutions $u(x,t)$ to $[0,R] \times [0,T]$ to obtain weak solutions to Zhang's problem.

Let us explain our main approach and the major difficulty that arises if $n > 1$. One can easily reformulate the equation in (1.1) for radial functions $u(x,t)$ into the one-dimensional equation:

$$v_t = (a(v_s^2)v_s)_s + a(v_s^2)v_s\frac{n-1}{s} \quad \text{in } (0,R) \times (0,T), \tag{1.9}$$

where $s = |x|$ is the radial variable and $v(s,t) = u(x,t)$. Using the flux function $\sigma(p) = a(p^2)p$ and overlooking the singularity at $s = 0$, this equation can be recast as

$$(s^{n-1}v)_t = (s^{n-1}\sigma(v_s))_s \quad \text{in } (0,R) \times (0,T).$$

Introduce a stream function φ with $\varphi_s = s^{n-1}v$, $\varphi_t = s^{n-1}\sigma(v_s)$, and let $\Phi = (v, \varphi)$. Then, to solve the equation (1.9) in a weak form, it is sufficient to find a function $\Phi = (v, \varphi) \in W^{1,\infty}((0, R)\times(0, T); \mathbb{R}^2)$ with the Jacobian matrix $\nabla\Phi(s, t) = \begin{pmatrix} v_s & v_t \\ \varphi_s & \varphi_t \end{pmatrix}$, such that

$$\nabla\Phi(s, t) \in \Sigma(s, v(s, t)) \quad \text{for a.e. } (s, t) \in (0, R) \times (0, T), \tag{1.10}$$

where, for each $s > 0$ and each $v \in \mathbb{R}$, the set $\Sigma(s, v)$ is defined by

$$\Sigma(s, v) := \left\{ \begin{pmatrix} p & l \\ s^{n-1}v & s^{n-1}\sigma(p) \end{pmatrix} \in \mathbb{R}^{2\times 2} : p,\ l \in \mathbb{R} \right\}.$$

If $n = 1$, the partial differential inclusion (1.10) is the same as in [30] since $s^{n-1} = 1$, with the set $\Sigma(s, v)$ independent of s. But the presence of the term s^{n-1} for $n \geq 2$ enormously affects the inclusion problem by making it essentially inhomogeneous in the variable s. In the fulfillment of the density result, Theorem 3.1, for applying a Baire's category method in Subsection 2.1, we have to construct some auxiliary functions as in [30]. Rather substantial difference occurs in the way of defining these functions in Section 5 as the equation $\varphi_s = s^{n-1}v$ should be kept in every gluing process and the term s^{n-1} makes the functions necessarily depend on the position s where they are glued. Accordingly, auxiliary functions are piecewise C^1 with proper s-derivatives on the regions that are separated by *nonlinear* C^1 curves.

The study of inhomogeneous partial differential inclusions of the type (1.10) stems from the successful understandings of homogeneous inclusion of the form $Du(x) \in K$ first encountered in the study of crystal microstructure by Ball & James [1, 2] and Chipot & Kinderlehrer [5]. Subsequent developments including some important applications and the generalization to inhomogeneous differential inclusions of the form $Du(x) \in K(x, u(x))$ have been extensively explored; see, e.g., Dacorogna & Marcellini [7, 8], Kirchheim [19], Müller & Šverák [24, 25, 26], Müller & Sychev [23], and Yan [28, 29]. We point out that in this connection the differential inclusion method has been recently used in De Lellis & Székelyhidi [21] to study the Euler equations. There are two well-known different approaches in solving the inclusion problem; however, both derive basically the same conclusion. The first method is the convex integration of Gromov [15], elaborated in [23, 24, 25, 26]. The other approach is the Baire's category method, exploited in [7, 8, 19, 28, 29]. We explore a simpler Baire's category method based on the density argument to study differential inclusion (1.10); our approach is quite different from that of Zhang [30] even for $n = 1$.

Let us compare our result with that of Ghisi & Gobbino [12]. Both papers deal with radial solutions for the Perona-Malik equation in dimensions $n \geq 2$. The paper [12] presents radial *classical* solutions over any annulus excluding the origin to avoid some technical difficulty due to the singularity of the corresponding one-dimensional equation at $s = 0$, while our result is to construct radial *weak* solutions on a ball including the singularity at $s = 0$ for the one-dimensional version. The major difference between the two works lies in the admissible classes of the initial data u_0 for solvability. In [12], the class of possible initial conditions for classical solvability is severely restricted due to the presence of backward (supercritical) region of a transcritical u_0. One has much freedom in choosing the initial values in the forward (subcritical) region of u_0, but then the initial values in the backward region are *determined* by the values of u_0 in the forward region. This phenomenon seems inevitable due to the inherent feature of the forward-backward radial problem. On the other hand, our result gives infinitely many radial weak solutions for all nonconstant smooth radial initial data u_0 (under certain natural compatibility conditions) whether it is transcritical or not. In fact, our result shows that, restricted to the smooth nonconstant radially symmetric initial data, no matter it is the specially selected initial condition in [12] or the initial condition which is all subcrtical (so the classical solution exists by the work of [17]), the problem (1.1) will always have infinitely many (Lipschitz) radial weak solutions.

The rest of this paper is organized as follows. In Section 2, we introduce more notations and gather some of the ingredients needed to prove Theorem 1.1. A Baire's category method is introduced in Subsection 2.1 and a classical result for uniformly parabolic Neumann problems is included in Subsection 2.2 as a building block that is to be used for a problem modified from problem (1.1). Section 3 contains the main setup of (1.1) as a differential inclusion and the main density result, Theorem 3.1, which plays a pivot role in constructing a weak solution via Baire's method. Section 4 is devoted to the proof of Theorem 1.1 based on Theorem 3.1. The construction of auxiliary functions needed in the proof of Theorem 3.1 is given in Section 5. The proof of Theorem 3.1 is finally given in Section 6.

2. Notation and preliminaries

We introduce some notations here. Let $N, n \in \mathbb{N}$. For any measurable set $X \subset \mathbb{R}^n$, $|X|$ denotes the Lebesgue measure of X. We denote by $\mathbb{R}^{N\times n}$ the space of $N \times n$ real matrices, and for each $A = (a_{ij}) \in \mathbb{R}^{N\times n}$, we let $|A|$ be the Hilbert-Schmidt norm of A, that is,

$$|A| := \left(\sum_{i=1}^{N} \sum_{j=1}^{n} a_{ij}^2 \right)^{1/2}.$$

We let $O(n)$ denote the space of $n \times n$ orthogonal real matrices. For each $A \in \mathbb{R}^{N\times n}$ and each $K \subset \mathbb{R}^{N\times n}$, the distance from A to the set K is defined by

$$\operatorname{dist}(A, K) := \inf_{B\in K} |A - B|.$$

For $1 \leq p \leq \infty$, let $W^{1,p}(\Omega; \mathbb{R}^N)$ denote the usual Sobolev space of functions $u \in L^p(\Omega; \mathbb{R}^N)$ whose first weak derivatives of each component exist and belong to $L^p(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is open. Also $W_0^{1,\infty}(\Omega; \mathbb{R}^N) := W^{1,\infty}(\Omega; \mathbb{R}^N) \cap W_0^{1,1}(\Omega; \mathbb{R}^N)$, where $W_0^{1,1}(\Omega; \mathbb{R}^N)$ is the closure of $C_0^\infty(\Omega; \mathbb{R}^N)$ in $W^{1,1}(\Omega; \mathbb{R}^N)$.

The following two lemmas are standard and used throughout this paper; see, e.g., [6, 8].

Lemma 2.1 (Vitali Covering Lemma)**.** *Let $\tilde{\Omega}$ and Ω be open sets in $\mathbb{R}^n$ with Ω bounded and $|\partial\Omega| = 0$. Then for each $\epsilon > 0$, there exist a sequence $\{x_j\}_{j\in\mathbb{N}}$ in $\mathbb{R}^n$ and a sequence $\{\epsilon_j\}_{j\in\mathbb{N}}$ of positive reals such that*

$$\begin{cases} x_j + \epsilon_j\Omega \subset \tilde{\Omega} \quad \textit{and} \quad \epsilon_j \leq \epsilon \quad \forall j \in \mathbb{N}, \\ (x_j + \epsilon_j\Omega) \cap (x_k + \epsilon_k\Omega) = \emptyset \quad \forall j, k \in \mathbb{N} \textit{ with } j \neq k, \\ |\Omega \setminus \cup_{j=1}^{\infty}(x_j + \epsilon_j\Omega)| = 0. \end{cases}$$

Lemma 2.2 (Gluing lemma)**.** *Let Ω be a bounded open set in $\mathbb{R}^n$, and let $\{\Omega_j\}_{j\in\mathbb{N}}$ be a sequence of disjoint open sets in Ω. Let $u \in W^{1,\infty}(\Omega;\mathbb{R}^N)$, and let $u_j \in u + W_0^{1,\infty}(\Omega_j;\mathbb{R}^N)$ for each $j \in \mathbb{N}$. If $\sup_{j\in\mathbb{N}} \|u_j\|_{W^{1,\infty}(\Omega_j;\mathbb{R}^N)} < \infty$ and $\tilde{u} := u\chi_{\Omega\setminus\cup_{j=1}^{\infty}\Omega_j} + \sum_{j=1}^{\infty} u_j\chi_{\Omega_j}$, then $\tilde{u} \in u + W_0^{1,\infty}(\Omega;\mathbb{R}^N)$.*

2.1. A Baire's category method

Definition 2.1 (Baire-one map)**.** *Let X and Y be metric spaces. Then $f : X \to Y$ is called a Baire-one map if it is pointwise limit of a sequence of continuous maps from X into Y.*

The proofs of the next two results can be found in [6, Chapter 10].

Theorem 2.1 (Baire's Category Theorem)**.** *Let X and Y be metric spaces with X complete. If $f : X \to Y$ is a Baire-one map, then $\mathcal{D}_f$ is of the first category, where $\mathcal{D}_f$ is the set of points in X at which f is discontinuous. Therefore, the set C_f of points in X at which f is continuous, that is, $C_f = X \setminus \mathcal{D}_f$, is dense in X.*

Proposition 2.1. *Let N and n be two positive integers. Let U be a bounded open set in $\mathbb{R}^n$, and let $X \subset W^{1,\infty}(U;\mathbb{R}^N)$ be equipped with the $L^\infty(U;\mathbb{R}^N)$-metric. Then the gradient operator*

$$\nabla : X \to L^p(U;\mathbb{R}^{N\times n})$$

is a Baire-one map for every $p \in [1,\infty)$.

Observe that if X in Proposition 2.1 is complete with respect to the L^∞-metric, it follows from Theorem 2.1 that the set C_∇ of points of continuity for the gradient operator ∇ is L^∞-dense in X. In our application we take $p = 1$ and X to be the L^∞-closure of the admissible class $\mathcal{P}^{n-1}_{\lambda,l_0}$ defined in Section 3 with $m = n-1$, so that C_∇ is L^∞-dense in X. This is a much shorter way to achieve the important principle that controlled L^∞ convergence implies $W^{1,1}$ convergence, exlpored in [23] by convex integration method. This explains that Baire's method is *somehow* equivalent to the convex integration.

2.2. Classical solution as building block

We need the following result to build the *nonempty* admissible class $\mathcal{P}^{n-1}_{\lambda,l_0}$ for the proof of Theorem 1.1.

Theorem 2.2. *Let $u_0 \in C^{3,\alpha}(\bar{\Omega})$ be a radially symmetric function with $u_0(x) = v_0(|x|)$ satisfying the compatibility condition* (1.3) *above. Let $a^* \in C^{2,\alpha}([0,\infty))$ be positive on* $[0,\infty)$*. Define $\sigma^*(p) := a^*(p^2)p$ for every $p \in \mathbb{R}$. Suppose that there exist two constants $C \geq c > 0$ such that*

$$c \leq (\sigma^*)'(p) \leq C \quad \forall p \geq 0. \tag{2.1}$$

Then the Neumann problem

$$\begin{cases} u^*_t = \operatorname{div}(a^*(|Du^*|^2)Du^*) & \text{in } \Omega_T \\ \frac{\partial u^*}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega \times (0,T) \\ u^*(x,0) = u_0(x) & \text{for } x \in \Omega \end{cases} \tag{2.2}$$

has a unique solution $u^* \in C^{3+\alpha,1+\alpha/2}(\bar{\Omega}_T)$. Moreover, u^* is radially symmetric in $\bar{\Omega}_T$, that is, $u^*(x,t) = v^*(|x|,t)$ for a function $v^*(s,t)$ on $[0,R]\times[0,T]$, and we have the gradient maximum principle:

$$\max_{\bar{\Omega}_T} |Du^*| = \max_{\bar{\Omega}} |Du_0| = \max_{[0,R]} |v_0'|.$$

Proof. By (2.1) and the positivity of a^*, the problem (2.2) is uniformly parabolic. Existence and uniqueness of classical solution to problem (2.2) under the compatibility condition (1.3) are standard for parabolic equations [20, 22]. We only include a proof for the radial symmetry and gradient maximum principle. In the case $n = 1$, the radial symmetry (i.e., $u^*(-x,t) = u^*(x,t)$) is easy and the gradient maximum principle is also standard; so let us assume $n \geq 2$. We first show that the solution u^* is radially symmetric in x on $\bar{\Omega}_T$. Suppose on the contrary that there exist two distinct points $x^0, y^0 \in \Omega$ with $|x^0| = |y^0|$ and a time $t^0 \in (0,T)$ such that $u^*(x^0,t^0) \neq u^*(y^0,t^0)$. We can choose a matrix $A \in O(n)$ such that $y^0 = Ax^0$, where x^0, y^0 are regarded as column vectors. Define

$$\tilde{u}^*(x,t) := u^*(Ax,t) \quad \forall (x,t) \in \bar{\Omega}_T.$$

Then it is straightforward to check that $\tilde{u}^* \in C^{3+\alpha,1+\alpha/2}(\bar{\Omega}_T)$ solves the problem (2.2). But

$$\tilde{u}^*(x^0,t^0) = u^*(Ax^0,t^0) = u^*(y^0,t^0) \neq u^*(x^0,t^0),$$

and this is a contradiction to the uniqueness of solution of (2.2). Thus u^* is radially symmetric in $\bar{\Omega}_T$. Note that $Du^*(0,t) = 0$ for all $t \in [0,T]$ by the radial symmetry and differentiability of u^* and that $Du^*(x,t) = 0$ for every $(x,t) \in \partial\Omega \times [0,T]$ by the Neumann boundary condition and the radial symmetry of u^*. Next, we establish the maximum principle

$$\max_{\bar{\Omega}_T} |Du^*| = \max_{\bar{\Omega}} |Du_0| = \max_{[0,R]} |v_0'|. \tag{2.3}$$

Let $u^*(x,t) = v^*(s,t)$, where $s = |x|$. Then $|v_s^*(s,t)| = |Du^*(x,t)|$ with $s = |x|$ and hence $v_s^*(0,t) = v_s^*(R,t) = 0$ for all $t \in [0,T]$. Similarly as in the introduction (or see (4.3) below), the function v^* solves the equation:

$$v_t^* = (\sigma^*(v_s^*))_s + \sigma^*(v_s^*)\frac{n-1}{s} \quad \text{in } (0,R)\times(0,T).$$

Let $w^* = v_s^*$. Then w^* solves the following equation in $(0,R)\times(0,T)$

$$\begin{cases} w_t^* = (\sigma^*)'(w^*)w_{ss}^* + (\sigma^*)''(w^*)(w_s^*)^2 + (\sigma^*)'(w^*)w_s^*\frac{n-1}{s} - \sigma^*(w^*)\frac{n-1}{s^2}, \\ w^*(0,t) = w^*(R,t) = 0 \quad \forall\, t \in [0,T], \\ w^*(s,0) = v_0'(s) \quad \forall\, s \in [0,R]. \end{cases} \tag{2.4}$$

It is then easy to show that

$$\max_{[0,R]\times[0,T]} |w^*| = \max_{[0,R]} |w^*(\cdot,0)|.$$

From this, (2.3) follows. (The compatibility condition (1.3) is easily seen needed from (2.4). The presence of the term $-\sigma^*(w^*)\frac{n-1}{s^2}$ in (2.4) makes the proof much easier.) □

3. Basic setup and the density theorem

In this section, we rephrase problem (1.1) into the framework of partial differential inclusion (1.10) with the set $\Sigma(s, v)$ replaced by a specific compact set $K^m_{\lambda,l_0}(s, v)$ with $m = n - 1$, and then present our main density result, Theorem 3.1, that is closely related to the reduction principle [23] or relaxation property [6]. To this end, we set up the relevant definitions and prove some lemmas building up on the definitions that are to be used in the proofs of Theorem 1.1 and Theorem 3.1. In doing so, we try to separate the arguments from these theorems to make our presentation as clear as possible.

3.1. A new function σ^* and several useful sets

In what follows, let $\sigma(p) = a(p^2)p$ be defined as above. It follows from (1.2) that for each $q \in (0, \sigma(1))$, there are exactly two $p^+_q,\ p^-_q \in \mathbb{R}$ such that

$$0 < p^-_q < 1 < p^+_q, \quad \sigma(p^\pm_q) = q. \quad \text{(See Figure 1.)}$$

For each $\lambda > 1$, let $\lambda^- := p^-_{\sigma(\lambda)}$; so, $0 < \lambda^- < 1 < \lambda$ and $\sigma(\lambda^-) = \sigma(\lambda)$.

We begin with the following technical lemma whose proof can be found in [30, Lemma 3.1]. (See Figure 2.)

Lemma 3.1. *Let $\lambda > 1$ and $\lambda^- < M < \lambda$. Then, there exists an odd function $\sigma^* \in C^{2,\alpha}(\mathbb{R})$ satisfying the following:*

(a) $\sigma^*(p) = \sigma(p)$ for $0 \le p \le \lambda^-$, $\sigma^*(p) < \min\{\sigma(p),\ \sigma(M)\}$ for $\lambda^- < p \le M$, and

(b) there exist two constants $C \ge c > 0$ such that

$$c \le (\sigma^*)'(p) \le C \ \ \textit{for every } p \ge 0.$$

We remark that the function σ^* depends on λ and M.

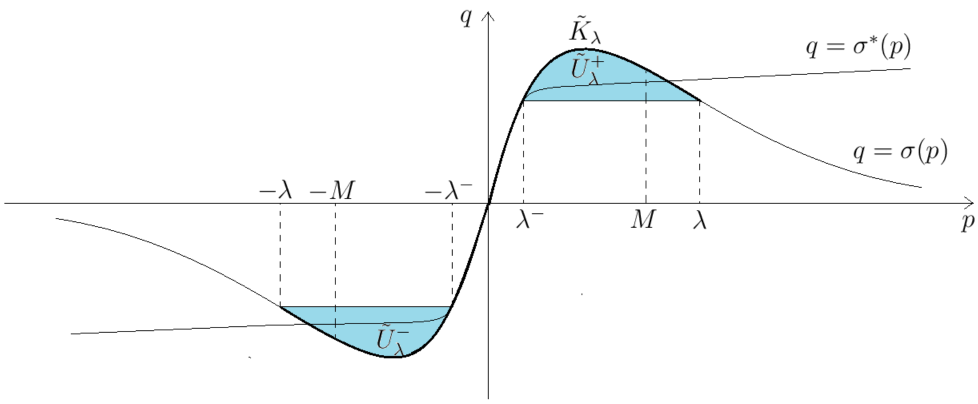

Figure 2: The graph of a new function $q = \sigma^*(p)$ from Lemma 3.1

For $\lambda > 1$, define the sets

$$\begin{array}{rcl} \tilde{K}_\lambda & := & \{(p, \sigma(p)) \in \mathbb{R}^2 : |p| \le \lambda\}, \\ \tilde{U}^+_\lambda & := & \{(p, q) \in \mathbb{R}^2 : \sigma(\lambda) < q < \sigma(1),\ p^-_q < p < p^+_q\}, \\ \tilde{U}^-_\lambda & := & \{(p, q) \in \mathbb{R}^2 : (-p, -q) \in \tilde{U}^+_\lambda\}. \end{array} \tag{3.1}$$

Let $m \ge 0$ be a fixed integer in the rest of this section. Here let us keep in mind that $m = n - 1$ in our application, where n is the space dimension in Theorem 1.1. For each $s > 0$ and $\lambda > 1$, define

$$\begin{array}{rcl} \tilde{K}^m_\lambda(s) & := & \{(p, s^m q) \in \mathbb{R}^2 : (p, q) \in \tilde{K}_\lambda\}, \\ \tilde{U}^m_\lambda(s) & := & \{(p, s^m q) \in \mathbb{R}^2 : (p, q) \in \tilde{U}^+_\lambda \cup \tilde{U}^-_\lambda\}, \\ I^m_\lambda(s, p) & := & \begin{cases} (s^m\sigma(\lambda), s^m\sigma(p)) \subset \mathbb{R} & \text{if } \lambda^- < p < \lambda \\ (s^m\sigma(p), -s^m\sigma(\lambda)) \subset \mathbb{R} & \text{if } -\lambda < p < -\lambda^-. \end{cases} \end{array} \tag{3.2}$$

Given any $l_0 > 0$, for $s > 0$ and $v \in \mathbb{R}$, define the sets in $\mathbb{R}^{2\times 2}$:

$$K^m_{\lambda,l_0}(s, v) := \left\{ \begin{pmatrix} p & l \\ s^m v & s^m q \end{pmatrix} \in \mathbb{R}^{2\times 2} : (p, q) \in \tilde{K}_\lambda,\ |l| \le l_0 \right\}, \tag{3.3}$$

$$U^m_{\lambda,l_0}(s, v) := \left\{ \begin{pmatrix} p & l \\ s^m v & s^m q \end{pmatrix} \in \mathbb{R}^{2\times 2} : (p, q) \in \tilde{U}^+_\lambda \cup \tilde{U}^-_\lambda,\ |l| < l_0 \right\}. \tag{3.4}$$

We also let $l_0 > 0$ be fixed throughout the rest of this section.

3.2. Properties of some distance functions

The following four lemmas are basically on the reformulations of some (inhomogeneous) distance functions into simpler expressions that we can easily manage for the proof of the density result, Theorem 3.1.

Lemma 3.2. *Let $s > 0$. Then for each $(p, q') \in \mathbb{R}^2$,*

$$(p, q') \in \tilde{U}^m_\lambda(s) \quad \text{if and only if} \quad p \in (-\lambda, -\lambda^-) \cup (\lambda^-, \lambda),\ \ q' \in I^m_\lambda(s, p).$$

For each $v' \in \mathbb{R}$, define

$$W_{v'} := \left\{ \begin{pmatrix} a & b \\ v' & d \end{pmatrix} \in \mathbb{R}^{2\times 2} : a, b, d \in \mathbb{R} \right\}.$$

If $K \subset W_{v'}$, let $\partial|_{W_{v'}} K$ denote the relative boundary of K in $W_{v'}$. Let $W := W_0$, and let P_W be the projection of $\mathbb{R}^{2\times 2}$ onto W, that is,

$$P_W\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^{2\times 2}.$$

For example, $K^m_{\lambda,l_0}(s, v),\ U^m_{\lambda,l_0}(s, v) \subset W_{s^m v}$, where $s > 0$ and $v \in \mathbb{R}$.

Lemma 3.3. *Let $s > 0$ and $v \in \mathbb{R}$, and let*

$$A = \begin{pmatrix} \tilde{p} & \tilde{l} \\ s^m v & \tilde{q}' \end{pmatrix} \in \mathbb{R}^{2\times 2}.$$

Then

$$\operatorname{dist}(A, K^m_{\lambda,l_0}(s,v) \cup \partial|_{W_{s^m v}} U^m_{\lambda,l_0}(s,v)) = \operatorname{dist}(P_W(A), K^m_{\lambda,l_0}(s,0) \cup \partial|_W U^m_{\lambda,l_0}(s,0)).$$

Lemma 3.4. *Let $F \subset \mathbb{R}_+ \times \mathbb{R}$ be a compact set, where $\mathbb{R}_+ := \{s \in \mathbb{R} : s > 0\}$. If $T : F \to W$ is a continuous mapping, then the mapping $d : F \to [0,\infty)$, defined by*

$$d(s,t) := \operatorname{dist}(T(s,t), K^m_{\lambda,l_0}(s,0) \cup \partial|_W U^m_{\lambda,l_0}(s,0)) \ \ \forall (s,t) \in F,$$

is also continuous.

Proof. Let $\epsilon > 0$. By the uniform continuity of T on F, there exists a $\delta > 0$ such that

$$|T(s_1,t_1) - T(s_2,t_2)| \le \frac{\epsilon}{2}$$

whenever $(s_1,t_1),(s_2,t_2) \in F$, $|(s_1,t_1)-(s_2,t_2)| \le \delta$. Fix any two $(s_1,t_1),(s_2,t_2) \in F$ with $|(s_1,t_1) - (s_2,t_2)| \le \delta$. Since $K^m_{\lambda,l_0}(s_1,0) \cup \partial|_W U^m_{\lambda,l_0}(s_1,0)$ is compact, we can choose a matrix $\begin{pmatrix} \tilde{p}_1 & \tilde{l}_1 \\ 0 & \tilde{q}'_1 \end{pmatrix}$ in this compact set so that

$$d(s_1,t_1) = \left| T(s_1,t_1) - \begin{pmatrix} \tilde{p}_1 & \tilde{l}_1 \\ 0 & \tilde{q}'_1 \end{pmatrix} \right|.$$

Put $\tilde{q}_1 := (s_1)^{-m}\tilde{q}'_1$. Then $(\tilde{p}_1,\tilde{q}_1) \in \tilde{K}_\lambda \cup \overline{(\tilde{U}^+_\lambda \cup \tilde{U}^-_\lambda)}$ if $\tilde{l}_1 \in \{l_0,-l_0\}$ or $(\tilde{p}_1,\tilde{q}_1) \in \tilde{K}_\lambda \cup (\partial\tilde{U}^+_\lambda \cup \partial\tilde{U}^-_\lambda)$ if $\tilde{l}_1 \in (-l_0,l_0)$. So we have

$$\begin{pmatrix} \tilde{p}_1 & \tilde{l}_1 \\ 0 & (s_2)^m\tilde{q}_1 \end{pmatrix} \in K^m_{\lambda,l_0}(s_2,0) \cup \partial|_W U^m_{\lambda,l_0}(s_2,0).$$

Note that

$$\begin{aligned} d(s_2,t_2) &\le \left| T(s_2,t_2) - \begin{pmatrix} \tilde{p}_1 & \tilde{l}_1 \\ 0 & (s_2)^m\tilde{q}_1 \end{pmatrix} \right| \\ &\le |T(s_2,t_2) - T(s_1,t_1)| + \left| T(s_1,t_1) - \begin{pmatrix} \tilde{p}_1 & \tilde{l}_1 \\ 0 & (s_1)^m\tilde{q}_1 \end{pmatrix} \right| \\ &\quad + \left| \begin{pmatrix} \tilde{p}_1 & \tilde{l}_1 \\ 0 & (s_1)^m\tilde{q}_1 \end{pmatrix} - \begin{pmatrix} \tilde{p}_1 & \tilde{l}_1 \\ 0 & (s_2)^m\tilde{q}_1 \end{pmatrix} \right|, \end{aligned}$$

and so

$$\begin{aligned} d(s_2,t_2) - d(s_1,t_1) &\le |T(s_2,t_2) - T(s_1,t_1)| + \left| \begin{pmatrix} \tilde{p}_1 & \tilde{l}_1 \\ 0 & (s_1)^m\tilde{q}_1 \end{pmatrix} - \begin{pmatrix} \tilde{p}_1 & \tilde{l}_1 \\ 0 & (s_2)^m\tilde{q}_1 \end{pmatrix} \right| \\ &\le \frac{\epsilon}{2} + \sigma(1)|(s_1)^m - (s_2)^m|. \end{aligned}$$

Let $I \subset \mathbb{R}_+$ be a compact interval with $\{s \in \mathbb{R}_+ : (s,t) \in F\} \subset I$. Then the mapping $s \mapsto s^m$ is uniformly continuous on I, so that there exists a $\delta' > 0$ such that

$$s_1, s_2 \in I,\ |s_1 - s_2| \le \delta' \Rightarrow |(s_1)^m - (s_2)^m| \le \frac{\epsilon}{2\sigma(1)}.$$

Thus if $(s_1,t_1),(s_2,t_2) \in F$ and $|(s_1,t_1)-(s_2,t_2)| \le \min\{\delta,\delta'\}$, then

$$d(s_2,t_2) - d(s_1,t_1) \le \epsilon.$$

Changing the roles of (s_1,t_1) and (s_2,t_2) and combining the results, we obtain the continuity of the mapping d on F. □

Lemma 3.5. *Let $s > 0$ and $v \in \mathbb{R}$, and let*

$$A := \begin{pmatrix} \tilde{p} & \tilde{l} \\ s^m v & \tilde{q}' \end{pmatrix} \in \mathbb{R}^{2\times 2}$$

be such that $|\tilde{l}| \le l_0$. Then

$$\mathrm{dist}(A, K^m_{\lambda,l_0}(s,v)) = \mathrm{dist}((\tilde{p},\tilde{q}'), \tilde{K}^m_\lambda(s)).$$

Proof. Choose any $(p,q) \in \tilde{K}_\lambda$. Then

$$\mathrm{dist}(A, K^m_{\lambda,l_0}(s,v)) \le \left| \begin{pmatrix} \tilde{p} & \tilde{l} \\ s^m v & \tilde{q}' \end{pmatrix} - \begin{pmatrix} p & \tilde{l} \\ s^m v & s^m q \end{pmatrix} \right| = |(\tilde{p},\tilde{q}') - (p, s^m q)|.$$

Taking an infimum on $(p,q) \in \tilde{K}_\lambda$, we have

$$\mathrm{dist}(A, K^m_{\lambda,l_0}(s,v)) \le \mathrm{dist}((\tilde{p},\tilde{q}'), \tilde{K}^m_\lambda(s)).$$

To show the reverse inequality, choose any $(p,q) \in \tilde{K}_\lambda$ and any $l \in \mathbb{R}$ with $|l| \le l_0$. Then

$$\mathrm{dist}((\tilde{p},\tilde{q}'), \tilde{K}^m_\lambda(s)) \le |(\tilde{p},\tilde{q}') - (p, s^m q)| = \left| \begin{pmatrix} \tilde{p} & \tilde{l} \\ s^m v & \tilde{q}' \end{pmatrix} - \begin{pmatrix} p & \tilde{l} \\ s^m v & s^m q \end{pmatrix} \right|$$

$$\le \left| \begin{pmatrix} \tilde{p} & \tilde{l} \\ s^m v & \tilde{q}' \end{pmatrix} - \begin{pmatrix} p & l \\ s^m v & s^m q \end{pmatrix} \right|,$$

so that taking an infimum on $(p,q,l) \in \tilde{K}_\lambda \times [-l_0, l_0]$, we have

$$\mathrm{dist}((\tilde{p},\tilde{q}'), \tilde{K}^m_\lambda(s)) \le \mathrm{dist}(A, K^m_{\lambda,l_0}(s,v)).$$

Thus the lemma is proved. □

3.3. Admissible class and the density theorem

Let $J := (0, R) \subset \mathbb{R}$, and $J_T := J \times (0, T) \subset \mathbb{R}^2$. Fix a $\delta_0 \in \mathbb{R}$ with $0 < \delta_0 < R/2$, and put $J_T^* := (\delta_0, R - \delta_0) \times (0, T) \subset J_T$.

Let $\Phi^* = (v^*, \varphi^*) \in W^{1,\infty}(J_T^*; \mathbb{R}^2)$ be a given piecewise C^1 function in J_T^*. We define the admissible class needed for later construction of the weak solutions as follows:

$$\mathcal{P}^m_{\lambda,l_0} := \left\{ \Phi \in \Phi^* + W_0^{1,\infty}(J_T^*; \mathbb{R}^2) : \begin{array}{l} \Phi = (v, \varphi) \text{ is piecewise } C^1 \text{ in } J_T^*, \\ \nabla\Phi(s,t) = \begin{pmatrix} v_s(s,t) & v_t(s,t) \\ \varphi_s(s,t) & \varphi_t(s,t) \end{pmatrix} \\ \in K^m_{\lambda,l_0}(s, v(s,t)) \cup U^m_{\lambda,l_0}(s, v(s,t)) \\ \text{for a.e. } (s,t) \in J_T^* \end{array} \right\}. \tag{3.5}$$

Note that this set may be empty; but in our application below, we will define a function Φ^* so that this class $\mathcal{P}^m_{\lambda,l_0}$ is nonempty.

We are now in a position to state the following main density result, whose proof will be postponed to Section 6.

Theorem 3.1 (Density Theorem)**.** *For each $\epsilon > 0$, the set*

$$\mathcal{P}^m_{\lambda,l_0,\epsilon} := \left\{ \Phi \in \mathcal{P}^m_{\lambda,l_0} : \int_{J_T^*} \operatorname{dist}(\nabla\Phi(s,t), K^m_{\lambda,l_0}(s, v(s,t)))dsdt \leq \epsilon |J_T^*| \right\}$$

is dense in $\mathcal{P}^m_{\lambda,l_0}$ with respect to the $L^\infty(J_T^*; \mathbb{R}^2)$-metric.

4. Proof of Theorem 1.1

In this section we aim to prove Theorem 1.1 based on the density theorem, Theorem 3.1. To this end, we assume a, σ and u_0 are functions given as above.

4.1. The modified parabolic problem

Let J, J_T be defined as in Subsection 3.3. Since u_0 is radial, let $u_0(x) = v_0(|x|)$ for a function $v_0 \in C^{3,\alpha}(\bar{J})$, and so

$$\max_{\bar{J}} |v_0'| = \max_{\bar{\Omega}} |Du_0| = M > 0. \tag{4.1}$$

Fix any $\epsilon > 0$. We define a number $\lambda > 1$ as follows: if $M \geq 1$, let $\lambda = M + \epsilon$; if $0 < M < 1$, let $\lambda \gg 1$ be such that $\sigma(\lambda) < \sigma(M)$. Then we always have $\lambda^- < M < \lambda$.

With this choice of M and λ, let σ^* be a function that can be determined by Lemma 3.1. Define $a^*(p) := \sigma^*(\sqrt{p})/\sqrt{p}$ for each $p > 0$. Then $a^*(p) = \sigma^*(\sqrt{p})/\sqrt{p} = \sigma(\sqrt{p})/\sqrt{p} = a(p)$ for every $p \in (0, (\lambda^-)^2]$. Since $a \in C^{2,\alpha}([0, \infty))$, we also have $a^* \in C^{2,\alpha}([0, \infty))$. Also the functions a^* and σ^* satisfy the hypotheses in Theorem 2.2. Therefore, for the given initial condition u_0, problem (2.2) has a unique radial solution $u^* \in C^{3+\alpha,1+\alpha/2}(\bar{\Omega}_T)$ with the maximum principle

$$\max_{\bar{\Omega}_T} |Du^*| = \max_{\bar{\Omega}} |Du_0| = M > 0. \tag{4.2}$$

Let $u^*(x,t) = v^*(|x|,t)$ for a function $v^* : \bar{J}_T \to \mathbb{R}$. Then $v^* \in C^{3+\alpha,1+\frac{\alpha}{2}}(\bar{J}_T)$. Let $(x,t) \in \{\Omega \setminus \{0\}\} \times (0,T)$. For each $i \in \{1,\ldots,n\}$,

$$\partial_i u^*(x,t) := \partial_{x_i} u^*(x,t) = v_s^*(|x|,t)\frac{x_i}{|x|}.$$

So $Du^*(x,t) = v_s^*(|x|,t)\frac{x}{|x|}$, and hence

$$a^*(|Du^*(x,t)|^2)Du^*(x,t) = a^*(v_s^*(|x|,t)^2)v_s^*(|x|,t)\frac{x}{|x|}.$$

Taking divergence on both sides, we obtain

$$\begin{aligned}\operatorname{div}(a^*(|Du^*(x,t)|^2)Du^*(x,t)) &= (a^*(v_s^*(s,t)^2)v_s^*(s,t))_s\big|_{s=|x|} \\ &\quad + a^*(v_s^*(|x|,t)^2)v_s^*(|x|,t)\frac{n-1}{|x|}.\end{aligned}$$

Since $u_t^*(x,t) = v_t^*(|x|,t)$, we thus have

$$v_t^*(s,t) = (a^*(v_s^*(s,t)^2)v_s^*(s,t))_s + a^*(v_s^*(s,t)^2)v_s^*(s,t)\frac{n-1}{s}. \quad (s = |x|) \tag{4.3}$$

In summary, $v^* \in C^{3+\alpha,1+\alpha/2}(\bar{J}_T)$ solves the following problem:

$$\begin{cases} (s^{n-1}v^*(s,t))_t = (s^{n-1}a^*(v_s^*(s,t)^2)v_s^*(s,t))_s & \text{for } (s,t) \in J_T \\ v_s^*(0,t) = v_s^*(R,t) = 0 & \text{for } t \in [0,T] \\ v^*(s,0) = v_0(s) & \text{for } s \in \bar{J}, \end{cases} \tag{4.4}$$

where

$$\max_{\bar{J}_T} |v_s^*| = \max_{\bar{J}} |v_0'| = M. \tag{4.5}$$

The uniform continuity of v_s^* on $\bar{J}_T$ and the second of (4.4) imply that there exists a $\delta_0 \in (0, R/2)$ such that

$$\max_{\{[0,\delta_0]\cup[R-\delta_0,R]\}\times[0,T]} |v_s^*| \le \lambda^-. \tag{4.6}$$

With this δ_0, let J_T^* be defined as in Subsection 3.3.

4.2. The starting function Φ^*

We define $\Phi^* := (v^*, \varphi^*)$, where $\varphi^* : \bar{J}_T \to \mathbb{R}$ is given by

$$\varphi^*(s,t) := \int_0^s w^{n-1}v^*(w,t)dw \quad \text{for every } (s,t) \in \bar{J}_T.$$

Then $\varphi^* \in C^{3+\alpha,1+\alpha/2}(\bar{J}_T)$, and

$$\begin{aligned}\varphi_s^*(s,t) &= s^{n-1}v^*(s,t), \\ \varphi_t^*(s,t) &= \int_0^s w^{n-1}v_t^*(w,t)dw \\ &= \int_0^s (w^{n-1}a^*(v_s^*(w,t)^2)v_s^*(w,t))_w dw \quad \text{(by (4.4))} \\ &= s^{n-1}a^*(v_s^*(s,t)^2)v_s^*(s,t)\end{aligned} \tag{4.7}$$

for every $(s,t) \in J_T$. So $\Phi^* = (v^*, \varphi^*) \in C^{3+\alpha,1+\alpha/2}(\bar{J}_T;\mathbb{R}^2)$, and

$$\nabla\Phi^*(s,t) = \begin{pmatrix} v^*_s(s,t) & v^*_t(s,t) \\ \varphi^*_s(s,t) & \varphi^*_t(s,t) \end{pmatrix} = \begin{pmatrix} v^*_s(s,t) & v^*_t(s,t) \\ s^{n-1}v^*(s,t) & s^{n-1}a^*(v^*_s(s,t)^2)v^*_s(s,t) \end{pmatrix}.$$

Put $l_0 := \max_{\bar{J}_T} |v^*_t| + 1 > 0$. Let $\tilde{K}_\lambda$ and $\tilde{U}^\pm_\lambda$ be defined as in (3.1). For each $(s,t) \in J_T$, since $|v^*_s(s,t)| \le M$, it follows from Lemma 3.1 that

$$(v^*_s(s,t), a^*(v^*_s(s,t)^2)v^*_s(s,t)) = (v^*_s(s,t), \sigma^*(v^*_s(s,t))) \in \tilde{K}_\lambda \cup \tilde{U}^+_\lambda \cup \tilde{U}^-_\lambda$$

and that

$$(v^*_s(s,t), a^*(v^*_s(s,t)^2)v^*_s(s,t)) = (v^*_s(s,t), \sigma(v^*_s(s,t))) \in \tilde{K}_\lambda \quad \text{(by (4.6))}$$

if $(s,t) \in J_T \setminus J^*_T$. Hence

$$\begin{cases} \nabla\Phi^*(s,t) \in K^{n-1}_{\lambda,l_0}(s, v^*(s,t)) \cup U^{n-1}_{\lambda,l_0}(s, v^*(s,t)) & \forall\, (s,t) \in J_T, \\ \nabla\Phi^*(s,t) \in K^{n-1}_{\lambda,l_0}(s, v^*(s,t)) & \forall\, (s,t) \in J_T \setminus J^*_T, \end{cases} \tag{4.8}$$

where the sets $K^{n-1}_{\lambda,l_0}(s,v)$ and $U^{n-1}_{\lambda,l_0}(s,v)$ are defined as in (3.3) and (3.4) with $m = n-1$.

We now define the admissible class $\mathcal{P}^{n-1}_{\lambda,l_0}$ by using this function Φ^* on J^*_T as in (3.5) with $m = n-1$. Then clearly,

$$\Phi^* \in \mathcal{P}^{n-1}_{\lambda,l_0} \neq \emptyset.$$

4.3. The Baire category method

Let X denote the closure of $\mathcal{P}^{n-1}_{\lambda,l_0}$ in the space $L^\infty(J^*_T;\mathbb{R}^2)$. Since the sets $K^{n-1}_{\lambda,l_0}(s,v)$ and $U^{n-1}_{\lambda,l_0}(s,v)$ are bounded, it is easily checked that

$$\mathcal{P}^{n-1}_{\lambda,l_0} \subset X \subset \Phi^* + W^{1,\infty}_0(J^*_T;\mathbb{R}^2).$$

Proposition 2.1 shows that the gradient operator $\nabla : X \to L^1(J^*_T;\mathbb{R}^{2\times 2})$ is a Baire-one map, and so the set C_∇ of points in X at which the map ∇ is continuous is dense in X by Theorem 2.1. So we have $C_\nabla \neq \emptyset$, since $X \neq \emptyset$. Later we show that C_∇ is actually an infinite set. But first we elaborate on how the density theorem (Theorem 3.1) guarantees that every function in C_∇ provides us with a solution to problem (1.1).

Let $\Phi = (v,\varphi) \in C_\nabla \subset X$. Let $k \in \mathbb{N}$. By the definition of X, we can choose a $\tilde{\Phi}_k \in \mathcal{P}^{n-1}_{\lambda,l_0}$ so that

$$\|\Phi - \tilde{\Phi}_k\|_{L^\infty} \le \frac{1}{k}.$$

By the density theorem, Theorem 3.1, we can choose a function $\Phi_k = (v_k, \varphi_k) \in \mathcal{P}^{n-1}_{\lambda,l_0,1/k}$ so that

$$\|\tilde{\Phi}_k - \Phi_k\|_{L^\infty} \le \frac{1}{k}.$$

Combining these two inequalities, we have

$$\|\Phi - \Phi_k\|_{L^\infty} \le \frac{2}{k} \to 0 \ \text{ as } k \to \infty.$$

Since the map ∇ is continuous at Φ, we thus have

$$\nabla\Phi_k \to \nabla\Phi \ \text{ in } L^1(J_T^*;\mathbb{R}^{2\times 2}) \ \text{ as } k\to\infty.$$

Upon passing to a subsequence (we do not relabel), we can assume that

$$\nabla\Phi_k(s,t) \to \nabla\Phi(s,t) \ \text{ in } \mathbb{R}^{2\times 2} \ \text{ as } k\to\infty, \ \text{ for a.e. } (s,t)\in J_T^*. \tag{4.9}$$

Since $\Phi_k \in \mathcal{P}^{n-1}_{\lambda,l_0,1/k}$, it follows from Lemma 3.5 that

$$\int_{J_T^*} \mathrm{dist}(((v_k)_s(s,t),(\varphi_k)_t(s,t)),\tilde{K}^{n-1}_\lambda(s))dsdt = \int_{J_T^*} \mathrm{dist}(\nabla\Phi_k(s,t),K^{n-1}_{\lambda,l_0}(s,v_k(s,t)))dsdt \le \frac{|J_T^*|}{k} \quad \forall\, k\in\mathbb{N}.$$

Applying Fatou's lemma to this inequality with (4.9), we obtain

$$\int_{J_T^*} \mathrm{dist}((v_s(s,t),\varphi_t(s,t)),\tilde{K}^{n-1}_\lambda(s))\, dsdt = 0.$$

Since $\tilde{K}^{n-1}_\lambda(s)$ is closed in $\mathbb{R}^2$ for each $s>0$, it follows that

$$(v_s(s,t),\varphi_t(s,t)) \in \tilde{K}^{n-1}_\lambda(s) \ \text{ for a.e. } (s,t)\in J_T^*. \tag{4.10}$$

Moreover, for each $k\in\mathbb{N}$, we have

$$|(v_k)_t(s,t)| \le l_0, \ \ (\varphi_k)_s(s,t) = s^{n-1}v_k(s,t) \ \text{ for a.e. } (s,t)\in J_T^*,$$

so that letting $k\to\infty$, it follows that

$$|v_t(s,t)| \le l_0, \ \varphi_s(s,t) = s^{n-1}v(s,t) \ \text{ for a.e. } (s,t)\in J_T^*. \tag{4.11}$$

Combining (4.10) and (4.11), we have

$$\nabla\Phi(s,t) \in K^{n-1}_{\lambda,l_0}(s,v(s,t)) \ \text{ for a.e. } (s,t)\in J_T^*.$$

Since $\Phi\in\Phi^* + W^{1,\infty}_0(J_T^*;\mathbb{R}^2)$, we can extend Φ from J_T^* to J_T by setting

$$\Phi := \Phi^* \ \text{ on } J_T\setminus J_T^*. \tag{4.12}$$

Then it follows that $\Phi\in\Phi^*+W^{1,\infty}_0(J_T;\mathbb{R}^2)$ and $\Phi\equiv\Phi^*$ on $\overline{J_T\setminus J_T^*}$, where we still write $\Phi=(v,\varphi)$ on $\bar{J}_T$. Observe now that by (4.8),

$$\nabla\Phi(s,t) \in K^{n-1}_{\lambda,l_0}(s,v(s,t)) \ \text{ for a.e. } (s,t)\in J_T. \tag{4.13}$$

Define

$$u(x,t) := v(|x|,t), \quad \psi(x,t) := \varphi(|x|,t) \quad \forall\,(x,t)\in\bar{\Omega}_T. \tag{4.14}$$

By (4.13) and (4.14), we have

$$Du(x,t) = v_s(s,t)\frac{x}{s}, \quad D\psi(x,t) = \varphi_s(s,t)\frac{x}{s} = |x|^{n-2}u(x,t)x, \quad s = |x| \neq 0. \tag{4.15}$$

Since $(v,\varphi) = (v^*,\varphi^*)$ on $\overline{J_T \setminus J_T^*}$, it is guaranteed from the definition of φ^*, (4.4), and (4.7) that for all $t \in [0,T]$,

$$\varphi(0,t) = 0, \quad \varphi(R,t) = \varphi(R,0) = \int_0^R w^{n-1}v_0(w)\,dw. \tag{4.16}$$

We now prove the following result.

Theorem 4.1. *The function u defined above solves problem (1.1) in the sense that, for every $\xi \in C^1(\bar{\Omega}_T)$,*

$$\int_\Omega (u(x,T)\xi(x,T) - u_0(x)\xi(x,0))\,dx = \int_{\Omega_T} (u\xi_t - a(|Du|^2)Du\cdot D\xi)\,dxdt. \tag{4.17}$$

Proof. It is sufficient to show that (4.17) holds for every $\xi \in C^\infty(\bar{\Omega}_T)$. Let $\xi \in C^\infty(\bar{\Omega}_T)$. By (4.15), $u = D\psi \cdot \frac{x}{|x|^n}$, and hence

$$\int_{\Omega_T} u\xi_t\,dxdt = \int_{\Omega_T} D\psi\cdot\frac{x}{|x|^n}\xi_t\,dxdt = \lim_{\epsilon\to 0^+}\int_{\Omega_T^\epsilon} D\psi\cdot\frac{x}{|x|^n}\xi_t\,dxdt,$$

where $\Omega_T^\epsilon = \Omega^\epsilon \times (0,T)$ with $\Omega^\epsilon = \{\epsilon < |x| < R\}$. For all sufficiently small $\epsilon > 0$, by the Divergence Theorem,

$$\begin{aligned}\int_{\Omega_T^\epsilon} D\psi\cdot\frac{x}{|x|^n}\xi_t\,dxdt &= \int_0^T\int_{\partial\Omega^\epsilon}\psi\xi_t\frac{x}{|x|^n}\cdot\mathbf{n}\,dS\,dt - \int_{\Omega_T^\epsilon}\psi\mathrm{div}\left(\frac{x}{|x|^n}\xi_t\right)dxdt\\
&= \frac{1}{R^{n-1}}\int_0^T\int_{|x|=R}\psi\xi_t\,dS\,dt - \frac{1}{\epsilon^{n-1}}\int_0^T\int_{|x|=\epsilon}\psi\xi_t\,dS\,dt - \int_{\Omega_T^\epsilon}\psi\mathrm{div}\left(\frac{x}{|x|^n}\xi_t\right)dxdt\\
&=: A - B_\epsilon - C_\epsilon,\end{aligned}$$

where $\mathbf{n}$ is outward unit normal on $\partial\Omega^\epsilon$. Since ψ is continuous on $\bar{\Omega}_T$ and $\psi(0,t) = \varphi(0,t) = 0$ for all $t \in [0,T]$, it is easily seen that

$$B_\epsilon \to 0 \quad \text{as } \epsilon \to 0^+.$$

By (4.16), $\psi(x,t) = C$ for all $(x,t) \in \partial\Omega\times[0,T]$, where $C = \int_0^R w^{n-1}v_0(w)\,dw$ is a constant; hence

$$A = \frac{1}{R^{n-1}}\int_0^T\int_{|x|=R}(\psi\xi)_t(x,t)\,dS\,dt = \frac{1}{R^{n-1}}\left[\int_{|x|=R}\psi\xi dS\right]_0^T.$$

For the term C_ϵ, using $\mathrm{div}\left(\frac{x}{|x|^n}\right) = 0$, we have, from integration by parts on t,

$$C_\epsilon = \int_{\Omega_T^\epsilon}\psi\frac{x}{|x|^n}\cdot D\xi_t\,dxdt = \left[\int_{\Omega^\epsilon}\frac{x}{|x|^n}\psi\cdot D\xi\,dx\right]_0^T - \int_{\Omega_T^\epsilon}\psi_t\frac{x}{|x|^n}\cdot D\xi\,dxdt$$

$$=: D_\epsilon - E_\epsilon.$$

Using the Divergence Theorem and $\operatorname{div}\left(\frac{x}{|x|^n}\right) = 0$ again, we have

$$\begin{aligned}
D_\epsilon &= \left[\int_{\Omega^\epsilon} \frac{x}{|x|^n}\psi \cdot D\xi\, dx\right]_0^T = \left[\int_{\partial\Omega^\epsilon} \psi\xi \frac{x}{|x|^n}\cdot \mathbf{n}\, dS\right]_0^T - \left[\int_{\Omega^\epsilon} D\psi \cdot \frac{x}{|x|^n}\xi\, dx\right]_0^T \\
&= \frac{1}{R^{n-1}}\left[\int_{|x|=R} \psi\xi dS\right]_0^T - \frac{1}{\epsilon^{n-1}}\left[\int_{|x|=\epsilon} \psi\xi dS\right]_0^T - \left[\int_{\Omega^\epsilon} u\xi\, dx\right]_0^T \\
&= A - \frac{1}{\epsilon^{n-1}}\left[\int_{|x|=\epsilon} \psi\xi dS\right]_0^T - \left[\int_{\Omega^\epsilon} u\xi\, dx\right]_0^T =: A - F_\epsilon - G_\epsilon,
\end{aligned}$$

where

$$\lim_{\epsilon\to 0^+} F_\epsilon = 0 \quad \text{since } \psi(0,t) = 0\ \forall t \in [0,T], \qquad \lim_{\epsilon\to 0^+} G_\epsilon = \left[\int_\Omega u\xi\, dx\right]_0^T.$$

Finally, using the equation $\psi_t \frac{x}{|x|^n} = a(|Du|^2)Du$ on Ω_T with $x \neq 0$, we have

$$E_\epsilon = \int_{\Omega_T^\epsilon} a(|Du|^2)Du \cdot D\xi\, dxdt \to \int_{\Omega_T} a(|Du|^2)Du \cdot D\xi\, dxdt \quad \text{as } \epsilon \to 0^+.$$

Therefore

$$\begin{aligned}
\int_{\Omega_T} u\xi_t\, dxdt &= \lim_{\epsilon\to 0^+}(A - B_\epsilon - C_\epsilon) = \lim_{\epsilon\to 0^+}(-B_\epsilon + F_\epsilon + G_\epsilon + E_\epsilon) \\
&= \left[\int_\Omega u\xi\, dx\right]_0^T + \int_{\Omega_T} a(|Du|^2)Du \cdot D\xi\, dxdt.
\end{aligned}$$

This is exactly (4.17), where $u(x,0) = u_0(x)$ in Ω as shown independently in (c) below. We remark that the fact that ψ is constant on $|x| = R$ plays an important role in the proof. This completes the proof. □

4.4. Completion of Proof of Theorem 1.1

Let us first verify that the radial function $u \in W^{1,\infty}(\Omega_T)$ defined above satisfies all of (a)-(f) in Theorem 1.1.

(a): This follows easily from (4.17).

(b): From (4.12), we have $v \equiv v^*$ on $\overline{J_T \setminus J_T^*}$. So by the definition of u,

$$u \equiv u^* \in C^{3+\alpha,1+\alpha/2}\left(\{\overline{B_{\delta_0}(0)} \cup \overline{(B_R(0) \setminus B_{R-\delta_0}(0))}\} \times [0,T]\right).$$

Observe that

$$\max_{\{\overline{B_{\delta_0}(0)}\cup\overline{(B_R(0)\setminus B_{R-\delta_0}(0))}\}\times[0,T]} |Du| = \max_{\{\overline{B_{\delta_0}(0)}\cup\overline{(B_R(0)\setminus B_{R-\delta_0}(0))}\}\times[0,T]} |Du^*| = \max_{\overline{J_T\setminus J_T^*}} |v_s^*| \le \lambda^-$$

by (4.6). Since $a \equiv a^*$ on $[0, (\lambda^-)^2]$ and u^* solves (2.2), it follows that u satisfies (b). At the end of this proof, we will check that C_∇ has infinitely many elements $\Phi = (v, \varphi)$. The first component v in every $\Phi \in C_\nabla$ is then extended to be the common v^* on $\overline{J_T \setminus J_T^*}$, so that each corresponding u satisfies (b) with the same $\delta_0 > 0$.

(c): By (4.4) and (4.12), we have

$$v(s, 0) = v^*(s, 0) = v_0(s) \ \text{ for every } s \in \bar{J}.$$

Thus from the definitions of u and v_0,

$$u(x, 0) = v(|x|, 0) = v_0(|x|) = u_0(x) \ \text{ for every } x \in \bar{\Omega}.$$

(d): This follows immediately from the observation in (b).

(e): Assume $M \geq 1$; then $\lambda = M + \epsilon$. Let $(s, t) \in J_T$ be any point such that

$$\nabla\Phi(s, t) \in K^{n-1}_{\lambda, l_0}(s, v(s, t)) \ \text{ and } \ v_s(s, t) \text{ exists in } \mathbb{R}.$$

Then for every $x \in \Omega$ with $|x| = s$, $Du(x, t)$ exists in $\mathbb{R}^n$,

$$|Du(x, t)| = |v_s(s, t)|$$

by the radial symmetry of u, and $|v_s(s, t)| \leq \lambda = M + \epsilon$ by (4.13). Note also that these hold for a.e. $(s, t) \in J_T$, so that

$$\|Du\|_{L^\infty(\Omega_T;\mathbb{R}^n)} = \|v_s\|_{L^\infty(J_T)} \leq M + \epsilon.$$

(f): This follows easily by taking $\xi \equiv 1$ in (4.17), which remains valid even when Ω_T and T are replaced by Ω_t and t with $0 < t \leq T$, respectively.

Finally, it remains to check that C_∇ is an infinite set. Suppose on the contrary that C_∇ is finite. Since C_∇ and $\mathcal{P}^{n-1}_{\lambda, l_0}$ are dense in X, we then have $C_\nabla = X = \mathcal{P}^{n-1}_{\lambda, l_0}$. So $\Phi^* \in \mathcal{P}^{n-1}_{\lambda, l_0} = C_\nabla$. By the above, Φ^* satisfies (4.13), that is,

$$\nabla\Phi^*(s, t) \in K^{n-1}_{\lambda, l_0}(s, v^*(s, t)) \quad \text{for a.e. } (s, t) \in J_T,$$

and so

$$(v^*_s(s, t), s^{n-1}\sigma^*(v^*_s(s, t))) = (v^*_s(s, t), \varphi^*_t(s, t)) \in \tilde{K}^{n-1}_\lambda(s) \quad \text{for a.e. } (s, t) \in J^*_T.$$

This is equivalent to saying that

$$(v^*_s(s, t), \sigma^*(v^*_s(s, t))) \in \tilde{K}_\lambda \quad \text{for a.e. } (s, t) \in J^*_T.$$

By the definition of the set $\tilde{K}_\lambda$, we have

$$\sigma^*(v^*_s(s, t)) = \sigma(v^*_s(s, t)) \quad \text{for a.e. } (s, t) \in J^*_T. \tag{4.18}$$

On the other hand, it follows from (4.5) and (4.6) with $\lambda^- < M$ that choosing a $\delta > 0$ so small that

$$\sigma^*(p) \neq \sigma(p) \quad \forall p \in [-M, -M + \delta] \cup [M - \delta, M],$$

we have

$$v_s^* \in [-M, -M + \delta] \cup [M - \delta, M]$$

on some set $W = W(\delta) \subset J_T^*$ of positive measure. Thus for each $(s, t) \in W$,

$$\sigma^*(v_s^*(s, t)) \neq \sigma(v_s^*(s, t)),$$

and this is a contradiction to (4.18). Therefore, C_∇ is an infinite set.

The theorem is now proved.

Remark 1. Assume $\max_{\bar{\Omega}} |Du_0| = M < 1$. For the moment, we select a different $\lambda > 1$ such that $\lambda^- = M$ and then select $M' \in (M, \lambda)$. With this choice of (M', λ) in place of (M, λ) in Lemma 3.1, we construct a function $\sigma^*(p)$. Define $a^*(p) := \sigma^*(\sqrt{p})/\sqrt{p}$ for each $p > 0$. Then $a^*(p) = a(p)$ for every $p \in (0, M^2]$ and the functions a^* and σ^* satisfy the hypotheses in Theorem 2.2. Therefore, for the given initial condition u_0, problem (2.2) has a unique radial solution $w^* \in C^{3+\alpha,1+\alpha/2}(\bar{\Omega}_T)$. Then w^* is also a classical solution to problem (1.1). However, Theorem 1.1 asserts that, even in this case, problem (1.1) still has infinitely many weak solutions. □

Remark 2. Let $M^+ \geq 1$ denote the unique number with $\sigma(M^+) = \sigma(M)$. (Note $M^+ = M$ when $M \geq 1$.) For any two $\lambda > \mu > M^+$, we have infinitely many weak solutions u of problem (1.1) such that $Du \in [0, \mu^-] \cup [\mu, \lambda]$ a.e. in Ω_T and that the two disjoint subsets of Ω_T at which $Du \in [0, \mu^-]$ and $Du \in [\mu, \lambda]$, respectively, are both of positive measure. To this end, one simply replaces the set $\tilde{K}_\lambda$ in (3.1) with $\tilde{K}_{\mu,\lambda} = \{(p, \sigma(p)) : |p| \in [0, \mu^-] \cup [\mu, \lambda]\}$ and changes the relevant sets in (3.1), (3.2), (3.3) and (3.4) accordingly. One also replaces $\sigma(M)$ in part (a) of Lemma 3.1 with $\sigma(\mu)$. Then one may repeat all the same arguments thereafter to obtain such a gradient result for weak solutions. So if μ is chosen large, then $0 < \mu^- \ll 1$ and the weak solutions u have a mixture of parts with $|Du| \geq \mu$ and with $|Du| \leq \mu^-$ at almost every $t \in [0, T]$. This also shows that there exists a sequence of weak solutions $\{u_k\}_{k=1}^\infty$ such that $\lim_{k\to\infty} \|Du_k\|_{L^\infty(\Omega_T)} = \infty$. □

5. Auxiliary functions

In this section, we construct some auxiliary functions that are needed to prove the density theorem, Theorem 3.1.

5.1. Construction lemma

We begin with the following useful result.

Lemma 5.1 (Construction Lemma)**.** *Let $a > 0$, $b > 0$, $L > 0$, $s_0 > 0$, and let $m \geq 0$ be an integer. Let $s_1,\ s_2 \in C^1(0, L)$ be two functions satisfying*

$$\begin{cases} 0 < s_1(t) < s_0 < s_2(t), \\ \frac{s_1(t)+s_2(t)}{2} = s_0, \end{cases} \qquad \forall\, t \in (0, L). \tag{5.1}$$

Let $D \subset \mathbb{R}^2$ be the bounded open set, defined by

$$D := \{(s,t) \in \mathbb{R}^2 : 0 < t < L, s_1(t) < s < s_2(t)\}.$$

For each $(s,t) \in D$, define

$$\begin{aligned} F(s,t) \quad := \quad & \int_{s_1(t)}^{\frac{as_1(t)+bs}{a+b}} \tau^m[-a(\tau - s_1(t))]d\tau + \int_{\frac{as_1(t)+bs}{a+b}}^{\frac{as_2(t)+bs}{a+b}} \tau^m b(\tau - s)d\tau \\ & + \int_{\frac{as_2(t)+bs}{a+b}}^{s_2(t)} \tau^m[-a(\tau - s_2(t))]d\tau. \end{aligned}$$

Then we have the following:

(a) $F \in C^1(D)$,

(b) there exists a unique function $\tilde{s} \in C^1(0,L)$ such that

$$s_1(t) < \tilde{s}(t) < s_2(t), \;\; F(\tilde{s}(t),t) = 0 \quad \forall\, t \in (0,L),$$

(c) $|\tilde{s}'(t)| \le \left[1 + \left(\frac{s_2(t)}{s_1(t)}\right)^m\right] |s_1'(t)|$ for all $t \in (0,L)$,

(d) if $s_1 \in C^1([0,L])$, $s_1(0) > 0$, and $s_1(L) > 0$, then $\tilde{s} \in C^1([0,L])$.

Proof. **(a):** Elementary computation shows that for each $(s,t) \in D$,

$$F(s,t) = c_m \left[\frac{(as_1(t) + bs)^{m+2}}{(a+b)^{m+1}} - as_1(t)^{m+2} - \frac{(as_2(t) + bs)^{m+2}}{(a+b)^{m+1}} + as_2(t)^{m+2} \right], \tag{5.2}$$

where $c_m = \frac{1}{m+1} - \frac{1}{m+2} > 0$. Since $s_1, s_2 \in C^1(0,L)$, it follows immediately from (5.2) that $F \in C^1(D)$.

(b): For each $t \in (0,L)$, using (5.2), it can be checked (mainly from the convexity of function s^{m+2} on $s > 0$) that $F(s_1(t),t) > 0$ and $F(s_2(t),t) < 0$. Moreover, on D,

$$\partial_s F(s,t) = \frac{(m+2)bc_m}{(a+b)^{m+1}} \left[(as_1(t) + bs)^{m+1} - (as_2(t) + bs)^{m+1} \right] < 0,$$

since $s_2(t) > s_1(t) > 0$. In particular, $\partial_s F(s,t) \neq 0$ for every $(s,t) \in D$. Therefore, by the Intermediate Value Theorem, for each $t \in (0,L)$, there exists a unique $\tilde{s}(t) \in (s_1(t), s_2(t))$ such that

$$F(\tilde{s}(t),t) = 0.$$

Furthermore, by the Implicit Function Theorem, it follows that $\tilde{s} \in C^1(0,L)$, and so (b) is proved.

(c): Clearly, by (5.2), $\tilde{s}(t)$ satisfies the equation

$$\begin{aligned} 0 = (as_1(t) + b\tilde{s}(t))^{m+2} - a(a+b)^{m+1} s_1(t)^{m+2} \\ -(as_2(t) + b\tilde{s}(t))^{m+2} + a(a+b)^{m+1} s_2(t)^{m+2} \end{aligned} \tag{5.3}$$

for each $t \in (0, L)$. Taking derivatives on both sides in (5.3) with respect to t, we obtain

$$\begin{aligned} \tilde{s}'(t) &= \frac{a}{b}\Big\{\frac{s_1'(t)[(as_1(t)+b\tilde{s}(t))^{m+1}-(as_1(t)+bs_1(t))^{m+1}]}{(as_2(t)+b\tilde{s}(t))^{m+1}-(as_1(t)+b\tilde{s}(t))^{m+1}} \\ &\quad + \frac{s_2'(t)[(as_2(t)+bs_2(t))^{m+1}-(as_2(t)+b\tilde{s}(t))^{m+1}]}{(as_2(t)+b\tilde{s}(t))^{m+1}-(as_1(t)+b\tilde{s}(t))^{m+1}}\Big\} \end{aligned}$$

for each $t \in (0, L)$. Applying the Mean Value Theorem, we have

$$\begin{aligned} \tilde{s}'(t) &= s_1'(t)\left(\frac{as_1(t)+b\bar{s}_1(t)}{a\bar{s}_3(t)+b\tilde{s}(t)}\right)^m \frac{\tilde{s}(t)-s_1(t)}{s_2(t)-s_1(t)} \\ &\quad + s_2'(t)\left(\frac{as_2(t)+b\bar{s}_2(t)}{a\bar{s}_3(t)+b\tilde{s}(t)}\right)^m \frac{s_2(t)-\tilde{s}(t)}{s_2(t)-s_1(t)} \end{aligned}$$

for some $\bar{s}_1(t),\ \bar{s}_2(t),\ \bar{s}_3(t) \in \mathbb{R}$ with

$$s_1(t) < \bar{s}_1(t) < \tilde{s}(t) < \bar{s}_2(t) < s_2(t), \quad s_1(t) < \bar{s}_3(t) < s_2(t).$$

So

$$|\tilde{s}'(t)| \le |s_1'(t)| + |s_1'(t)|\left(\frac{s_2(t)}{s_1(t)}\right)^m = \left[1+\left(\frac{s_2(t)}{s_1(t)}\right)^m\right]|s_1'(t)|,$$

since $s_2'(t) = -s_1'(t)$ by (5.1). Thus (c) is proved.

<u>**(d):**</u> Finally to prove (d), assume $s_1 \in C^1([0, L))$ with $s_1(0) > 0$, and we will show that $\tilde{s} \in C^1([0, L))$. (If $s_1 \in C^1((0, L])$ with $s_1(L) > 0$, we can prove that $\tilde{s} \in C^1((0, L])$ exactly in the same way.) Note that $0 < s_1(0) \le s_0$ by (5.1). If $0 < s_1(0) < s_0$, then we can extend s_1 and s_2 from $[0, L)$ to $(-\delta, L)$ for some $\delta > 0$ in a way that $s_1, s_2 \in C^1((-\delta, L))$ satisfy (5.1), and we can apply the previous argument to show $\tilde{s} \in C^1((-\delta, L))$. So let us assume $s_1(0) = s_0$; then, by (5.1), $s_2(0) = s_0$. From (5.3), we have

$$\lim_{t\to 0^+} \tilde{s}(t) = s_1(0) = s_2(0) = s_0.$$

We claim that

$$\lim_{t\to 0^+} \tilde{s}'(t) = 0, \tag{5.4}$$

and so $\tilde{s} \in C^1([0, L))$. To prove this claim, we rewrite (5.3) as

$$f(as_1(t)+b\tilde{s}(t), as_2(t)+b\tilde{s}(t)) = (a+b)^{m+1} f(s_1(t), s_2(t)) \quad \forall\, t \in (0, L), \tag{5.5}$$

where $f(s_1, s_2)$ is the polynomial in s_1 and s_2 determined through

$$s_1^{m+2} - s_2^{m+2} = (s_1 - s_2) f(s_1, s_2) \quad \forall\ (s_1, s_2) \in \mathbb{R}^2.$$

Note that $f(s_1, s_2)$ is symmetric in (s_1, s_2) and $\partial_1 f(s_1, s_2) > 0$ for all $s_1 > 0$ and $s_2 > 0$. To prove (5.4), note that, by (c), $\tilde{s}'(t)$ is bounded on $(0, L/2)$, and it suffices to show that if $\beta := \lim_{k\to\infty} \tilde{s}'(t_k)$ exists along a sequence $t_k \to 0^+$, then $\beta = 0$. Let $\alpha = s_1'(0^+)$; then $s_2'(0^+) = -\alpha$. Taking derivatives on both sides in (5.5) with respect to t, we have

$$\partial_1 f(as_2(t)+b\tilde{s}(t), as_1(t)+b\tilde{s}(t)) \cdot (as_2'(t)+b\tilde{s}'(t))$$

$$+\partial_2 f(as_2(t)+b\tilde{s}(t), as_1(t)+b\tilde{s}(t))\cdot(as_1'(t)+b\tilde{s}'(t))$$

$$=(a+b)^{m+1}[\partial_1 f(s_2(t),s_1(t))s_2'(t)+\partial_2 f(s_2(t),s_1(t))s_1'(t)]$$

for each $t \in (0, L)$. Letting $t = t_k \to 0^+$, we have

$$\partial_1 f(as_0+bs_0, as_0+bs_0)\cdot(-a\alpha+b\beta)+\partial_2 f(as_0+bs_0, as_0+bs_0)\cdot(a\alpha+b\beta)$$

$$=(a+b)^{m+1}[\partial_1 f(s_0,s_0)(-\alpha)+\partial_2 f(s_0,s_0)\alpha]=0,$$

by the symmetry of f. This yields that $2\beta b\partial_1 f(as_0+bs_0, as_0+bs_0)=0$, so that $\beta=0$, as desired. Hence (5.4) follows, and (d) is proved. □

5.2. Construction of auxiliary functions

We are now ready to construct auxiliary functions that will be used as local gradient modifiers in the proof of Theorem 3.1. Towards this goal, let $a>0$, $b>0$, $L>0$, $s_0^2>s_0^1>0$, $s_0:=\frac{s_0^2+s_0^1}{2}$, and let $m\geq 0$ be an integer. Define

$$s_1(t):=\frac{s_0^2-s_0^1}{2L}t+s_0^1 \quad\text{and}\quad s_2(t):=\frac{s_0^1-s_0^2}{2L}t+s_0^2$$

for each $t\in[0,L]$. (See Figure 3 with $t_0=0$.)

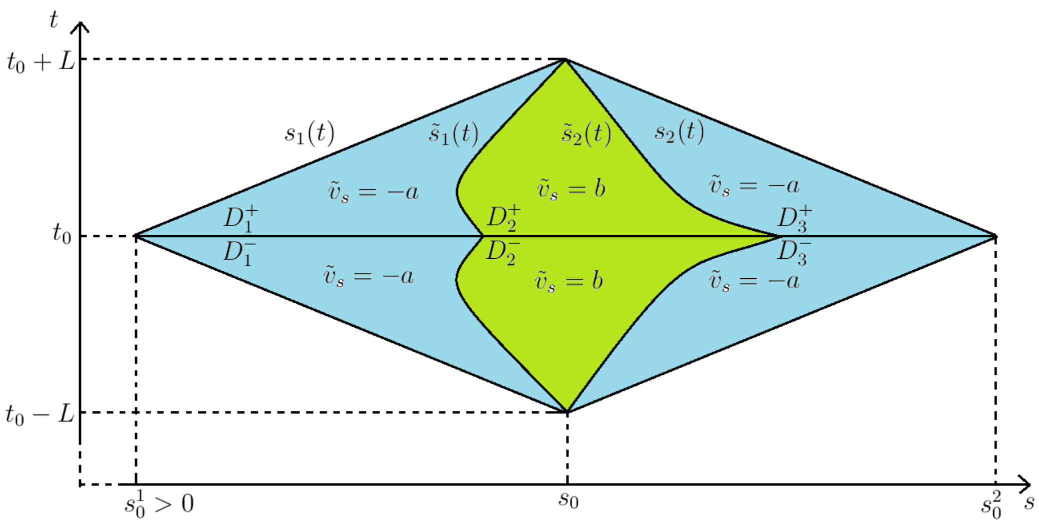

Figure 3: The s-derivatives of $\tilde{v}$ in Lemma 5.2 on the six regions separated by nonlinear piecewise C^1 curves $\tilde{s}_1(t)$ and $\tilde{s}_2(t)$

Let $D^+\subset\mathbb{R}^2$ be the bounded open set, given by

$$D^+:=\{(s,t)\in\mathbb{R}^2 : 0<t<L,\ s_1(t)<s<s_2(t)\}.$$

For each $(s,t) \in D^+$, define $F(s,t)$ as in Lemma 5.1, so that there exists a unique $\tilde{s} \in C^1([0,L])$ such that

$$\begin{cases} s_1(t) < \tilde{s}(t) < s_2(t),\ F(\tilde{s}(t),t) = 0 & \forall t \in [0,L), \\ |\tilde{s}'(t)| \leq \left[1 + \left(\frac{s_2(t)}{s_1(t)}\right)^m\right] |s_1'(t)| & \forall t \in [0,L]. \end{cases} \tag{5.6}$$

Let $\tilde{s}_1, \tilde{s}_2 : [0,L] \to \mathbb{R}$ be given by

$$\tilde{s}_i(t) := \frac{as_i(t) + b\tilde{s}(t)}{a+b} \quad \forall t \in [0,L],\ \forall i \in \{1,2\},$$

so that $\tilde{s}_1, \tilde{s}_2 \in C^1([0,L])$ and that by (5.6),

$$s_1(t) < \tilde{s}_1(t) < \tilde{s}(t) < \tilde{s}_2(t) < s_2(t) \quad \forall t \in [0,L).$$

Let $D_1^+, D_2^+, D_3^+ \subset \mathbb{R}^2$ be the bounded open sets, defined by

$$D_1^+ := \{(s,t) \in \mathbb{R}^2 : 0 < t < L,\ s_1(t) < s < \tilde{s}_1(t)\},$$

$$D_2^+ := \{(s,t) \in \mathbb{R}^2 : 0 < t < L,\ \tilde{s}_1(t) < s < \tilde{s}_2(t)\},$$

$$D_3^+ := \{(s,t) \in \mathbb{R}^2 : 0 < t < L,\ \tilde{s}_2(t) < s < s_2(t)\},$$

so that these are disjoint open subsets of D^+ with

$$\left| D^+ \setminus \cup_{i=1}^3 D_i^+ \right| = 0. \quad \text{(See Figure 3 with } t_0 = 0.)$$

Let $\tilde{v} : \bar{D}^+ \to \mathbb{R}$ be the function, defined by

$$\tilde{v}(s,t) := \begin{cases} -a(s - s_1(t)) & \forall (s,t) \in \bar{D}_1^+ \\ b(s - \tilde{s}(t)) & \forall (s,t) \in \bar{D}_2^+ \\ -a(s - s_2(t)) & \forall (s,t) \in \bar{D}_3^+. \end{cases}$$

It is easily checked that $\tilde{v} : \bar{D}^+ \to \mathbb{R}$ is well-defined and that $\tilde{v} \in W^{1,\infty}(D^+)$. It also follows from Lemma 5.1 that $\tilde{v} \in C^1(\bar{D}_i^+)$ for $i = 1,2,3$. If $0 \leq t \leq L$, then

$$\tilde{v}(s_1(t),t) = \tilde{v}(s_2(t),t) = 0.$$

Let $t \in [0,L]$. Then

$$\max_{s \in [s_1(t), s_2(t)]} |\tilde{v}(s,t)| = \max\{| - a(\tilde{s}_1(t) - s_1(t))|, | - a(\tilde{s}_2(t) - s_2(t))|\}$$

$$= \max\left\{\left|-a\left(\frac{as_1(t) + b\tilde{s}(t)}{a+b} - s_1(t)\right)\right|, \left|-a\left(\frac{as_2(t) + b\tilde{s}(t)}{a+b} - s_2(t)\right)\right|\right\}$$

$$= \max\left\{\frac{ab}{a+b}(\tilde{s}(t) - s_1(t)), \frac{ab}{a+b}(s_2(t) - \tilde{s}(t))\right\} \leq \frac{ab}{a+b}(s_0^2 - s_0^1).$$

Hence

$$\max_{\bar{D}^+} |\tilde{v}| \leq \frac{ab}{a+b}(s_0^2 - s_0^1) \leq \frac{a+b}{4}(s_0^2 - s_0^1).$$

Define

$$D^- := \{(s,t) \in \mathbb{R}^2 : (s,-t) \in D^+\},$$
$$D_i^- := \{(s,t) \in \mathbb{R}^2 : (s,-t) \in D_i^+\} \quad \forall i \in \{1,2,3\},$$
$$D := \operatorname{int}(\overline{D^+ \cup D^-}).$$

We do the even extensions for $s_1, \tilde{s}_1, \tilde{s}, \tilde{s}_2, s_2 : [-L, L] \to \mathbb{R}$ and for $\tilde{v} : \bar{D} \to \mathbb{R}$ along the t-axis, so that we have from the above observations that

$$\begin{cases} \tilde{v} \in W_0^{1,\infty}(D), \\ \tilde{v} \in C^1(\bar{D}_i^{\pm}) \ \ \forall i \in \{1,2,3\}, \\ \max_{\bar{D}} |\tilde{v}| \le \frac{a+b}{4}(s_0^2 - s_0^1). \end{cases} \tag{5.7}$$

It follows from (5.6) that for each $t \in [0, L]$,

$$\int_{s_1(t)}^{s_2(t)} \tau^m \tilde{v}(\tau, t) d\tau = F(\tilde{s}(t), t) = 0, \tag{5.8}$$

and this equality is valid for all $t \in [-L, L]$ by the definition $\tilde{v}$. Note also that

$$\nabla \tilde{v}(s,t) = \begin{cases} (-a, as_1'(t)) & \text{if } (s,t) \in D_1^+, \\ (b, -b\tilde{s}'(t)) & \text{if } (s,t) \in D_2^+, \\ (-a, as_2'(t)) & \text{if } (s,t) \in D_3^+, \\ (-a, -as_1'(-t)) & \text{if } (s,t) \in D_1^-, \\ (b, b\tilde{s}'(-t)) & \text{if } (s,t) \in D_2^-, \\ (-a, -as_2'(-t)) & \text{if } (s,t) \in D_3^-. \end{cases} \tag{5.9}$$

Also the second of (5.6) implies that

$$|\tilde{s}'(t)| \le \left[1 + \left(\frac{s_0^2}{s_0^1}\right)^m\right] \frac{s_0^2 - s_0^1}{2L} \quad \forall t \in [-L, L].$$

Combining this with (5.9), we have

$$|\partial_t \tilde{v}(s,t)| \le \max\{a,b\} \left[1 + \left(\frac{s_0^2}{s_0^1}\right)^m\right] \frac{s_0^2 - s_0^1}{2L} \quad \forall (s,t) \in \bigcup_{i=1}^{3} (D_i^+ \cup D_i^-). \tag{5.10}$$

Using the third of (5.7), we obtain

$$\max_{(s,t)\in\bar{D}} \left| \int_{s_1(t)}^{s} \tau^m \tilde{v}(\tau, t) d\tau \right| \le \frac{a+b}{4} (s_0^2)^m (s_0^2 - s_0^1)^2. \tag{5.11}$$

One can also easily check that

$$\frac{\partial}{\partial t}\left(\int_{s_1(t)}^{s} \tau^m \tilde{v}(\tau, t) d\tau \right) = \int_{s_1(t)}^{s} \tau^m \partial_t \tilde{v}(\tau, t) d\tau \quad \forall (s,t) \in \bigcup_{i=1}^{3} (D_i^+ \cup D_i^-). \tag{5.12}$$

Fix any $t_0 \in \mathbb{R}$. We now translate everything constructed above along the t-axis by t_0. So we define

$$\begin{array}{rcl} D(s_0^1, s_0^2, t_0, L) & := & \{(s,t) \in \mathbb{R}^2 : (s, t-t_0) \in D\}, \\ D_i^{\pm}(s_0^1, s_0^2, t_0, L) & := & \{(s,t) \in \mathbb{R}^2 : (s, t-t_0) \in D_i^{\pm}\} \quad \forall i \in \{1,2,3\}, \\ s_j(s_0^1, s_0^2, t_0, L; t) & := & s_j(t-t_0) \quad \forall t \in [t_0 - L, t_0 + L], \ \forall j \in \{1,2\}, \\ \tilde{v}(-a, b, s_0^1, s_0^2, t_0, L; s, t) & := & \tilde{v}(s, t-t_0) \quad \forall (s,t) \in \overline{D(s_0^1, s_0^2, t_0, L)}. \end{array} \tag{5.13}$$

Here is the right spot of mentioning a rather delicate feature of our construction. We should prohibit the auxiliary function $\tilde{v}$ in (5.7) from being translated in the s-axis as any s-translation will destroy the key properties to act as auxiliary functions for local gluing in the proof of the density theorem, Theorem 3.1. Accordingly, we construct $\tilde{v}$ on the positive s-axis from the start and allow translation in the t-axis only as in (5.13).

As a conclusion of this section, we suppress the letters $-a$, b, s_0^1, s_0^2, t_0, L in (5.13) for a notational simplicity and summarize the properties of $\tilde{v}$ inherited from (5.7), (5.8), (5.9), (5.10), (5.11), and (5.12) as follows. (See Figure 3.)

Lemma 5.2. *The function $\tilde{v} : \bar{D} \to \mathbb{R}$ constructed in (5.13) satisfies the following:*

(a) $\tilde{v} \in W_0^{1,\infty}(D)$,

(b) $\tilde{v} \in C^1(\bar{D}_i^{\pm}) \quad \forall i = 1,2,3$,

(c) $\partial_s \tilde{v}(s,t) = \begin{cases} -a & \forall (s,t) \in D_1^+ \cup D_1^- \cup D_3^+ \cup D_3^- \\ b & \forall (s,t) \in D_2^+ \cup D_2^-, \end{cases}$

(d) $|\partial_t \tilde{v}(s,t)| \leq \max\{a,b\} \left[1 + \left(\frac{s_0^2}{s_0^1}\right)^m\right] \frac{s_0^2 - s_0^1}{2L} \quad \forall (s,t) \in \bigcup_{i=1}^3 (D_i^+ \cup D_i^-)$,

(e) $\frac{\partial}{\partial t}\left(\int_{s_1(t)}^s \tau^m \tilde{v}(\tau,t)d\tau\right) = \int_{s_1(t)}^s \tau^m \partial_t \tilde{v}(\tau,t)d\tau \quad \forall (s,t) \in \bigcup_{i=1}^3 (D_i^+ \cup D_i^-)$,

(f) $\int_{s_1(t)}^{s_2(t)} \tau^m \tilde{v}(\tau,t)d\tau = 0 \quad \forall t \in [t_0 - L, t_0 + L]$,

(g) $\max_{\bar{D}} |\tilde{v}| \leq \frac{a+b}{4}(s_0^2 - s_0^1)$,

(h) $\max_{(s,t)\in\bar{D}} \left|\int_{s_1(t)}^s \tau^m \tilde{v}(\tau,t)d\tau\right| \leq \frac{a+b}{4}(s_0^2)^m(s_0^2 - s_0^1)^2$.

6. Proof of Theorem 3.1

In this long and final section, we present the proof of Theorem 3.1; that is, we prove the L^∞-density of $\mathcal{P}^m_{\lambda,l_0,\epsilon}$ in $\mathcal{P}^m_{\lambda,l_0}$ for each $\epsilon > 0$. To this end, assume $\Phi = (v,\varphi) \in \mathcal{P}^m_{\lambda,l_0}$, namely,

$$\begin{cases} \Phi \in \Phi^* + W_0^{1,\infty}(J_T^*; \mathbb{R}^2), \\ \Phi \text{ is piecewise } C^1 \text{ in } J_T^*, \text{ and} \\ \nabla\Phi(s,t) \in K^m_{\lambda,l_0}(s, v(s,t)) \cup U^m_{\lambda,l_0}(s, v(s,t)) \quad \text{for a.e. } (s,t) \in J_T^*. \end{cases} \tag{6.1}$$

Let $0 < \eta < 1$. Our goal is to prove that there exists a function $\Phi_\eta \in \mathcal{P}^m_{\lambda,l_0,\epsilon}$ such that $\|\Phi - \Phi_\eta\|_{L^\infty(J^*_T;\mathbb{R}^2)} \leq \eta$; that is, there exists a function $\Phi_\eta = (v_\eta, \varphi_\eta) \in \Phi^* + W^{1,\infty}_0(J^*_T;\mathbb{R}^2)$ satisfying that

$$
\begin{cases}
\Phi_\eta \text{ is piecewise } C^1 \text{ in } J^*_T, \\
\nabla\Phi_\eta(s,t) \in K^m_{\lambda,l_0}(s, v_\eta(s,t)) \cup U^m_{\lambda,l_0}(s, v_\eta(s,t)) \quad \text{for a.e. } (s,t) \in J^*_T, \\
\int_{J^*_T} \operatorname{dist}(\nabla\Phi_\eta(s,t), K^m_{\lambda,l_0}(s, v_\eta(s,t)))dsdt \leq \epsilon|J^*_T|, \\
\|\Phi - \Phi_\eta\|_{L^\infty(J^*_T;\mathbb{R}^2)} \leq \eta.
\end{cases}
\tag{6.2}
$$

We divide the proof into several parts.

6.1. Separation of domain J^*_T

By the second of (6.1), there is a sequence $\{G_i\}_{i\in\mathbb{N}}$ of disjoint open subsets of J^*_T such that

$$
\begin{cases}
|J^*_T \setminus \cup_{i=1}^\infty G_i| = 0, \\
\Phi \in C^1(\bar{G}_i;\mathbb{R}^2) \;\; \forall i \in \mathbb{N}.
\end{cases}
$$

Fix an index $i \in \mathbb{N}$ throughout this section. Since $\partial_s\varphi$ and v are continuous on $\bar{G}_i$, it follows from the third inclusion of (6.1) that

$$
\partial_s\varphi(s,t) = s^m v(s,t), \quad \text{i.e.,} \quad \nabla\Phi(s,t) \in W_{s^m v(s,t)} \quad \forall (s,t) \in \bar{G}_i.
$$

Applying Lemma 3.3, we have

$$
\begin{aligned}
d_i(s,t) \;&:=\; \operatorname{dist}(\nabla\Phi(s,t), K^m_{\lambda,l_0}(s, v(s,t)) \cup \partial|_{W_{s^m v(s,t)}} U^m_{\lambda,l_0}(s, v(s,t))) \\
&=\; \operatorname{dist}(P_W(\nabla\Phi(s,t)), K^m_{\lambda,l_0}(s,0) \cup \partial|_W U^m_{\lambda,l_0}(s,0))
\end{aligned}
$$

for every $(s,t) \in \bar{G}_i$, and it follows form Lemma 3.4 that the mapping $d_i : \bar{G}_i \to [0,\infty)$ is continuous. Let $0 < \delta < 1$, and put

$$
K_{i,\delta} := \{(s,t) \in G_i : d_i(s,t) \leq \delta\}.
$$

Define also

$$
\begin{aligned}
K^1_{i,\delta} &:= \{(s,t) \in K_{i,\delta} : \nabla\Phi(s,t) \notin K^m_{\lambda,l_0}(s, v(s,t)) \cup U^m_{\lambda,l_0}(s, v(s,t))\}, \\
K^2_{i,\delta} &:= \{(s,t) \in K_{i,\delta} : \nabla\Phi(s,t) \in K^m_{\lambda,l_0}(s, v(s,t))\}, \\
K^3_{i,\delta} &:= \{(s,t) \in K_{i,\delta} : \nabla\Phi(s,t) \in U^m_{\lambda,l_0}(s, v(s,t))\},
\end{aligned}
$$

so that $K_{i,\delta}$ is the disjoint union of $K^1_{i,\delta}$, $K^2_{i,\delta}$, and $K^3_{i,\delta}$. Note that $|K^1_{i,\delta}| = 0$ by the third of (6.1), and that

$$
\begin{gathered}
K^3_{i,\delta} \subseteq \{(s,t) \in G_i : \operatorname{dist}(\nabla\Phi(s,t), K^m_{\lambda,l_0}(s, v(s,t))) \leq \delta, \nabla\Phi(s,t) \in U^m_{\lambda,l_0}(s, v(s,t))\} \\
\cup\{(s,t) \in G_i : \operatorname{dist}(\nabla\Phi(s,t), \partial|_{W_{s^m v(s,t)}} U^m_{\lambda,l_0}(s, v(s,t))) \leq \delta, \nabla\Phi(s,t) \in U^m_{\lambda,l_0}(s, v(s,t))\} \\
=: K^{3,\alpha}_{i,\delta} \cup K^{3,\beta}_{i,\delta}.
\end{gathered}
$$

Hence

$$
\begin{gathered}
\int_{K_{i,\delta}} \operatorname{dist}(\nabla\Phi(s,t), K^m_{\lambda,l_0}(s, v(s,t))) \leq \int_{K^2_{i,\delta}} \operatorname{dist}(\nabla\Phi(s,t), K^m_{\lambda,l_0}(s, v(s,t))) \\
+ \int_{K^{3,\alpha}_{i,\delta}} \operatorname{dist}(\nabla\Phi(s,t), K^m_{\lambda,l_0}(s, v(s,t))) + \int_{K^{3,\beta}_{i,\delta}} \operatorname{dist}(\nabla\Phi(s,t), K^m_{\lambda,l_0}(s, v(s,t))) \\
= \int_{K^{3,\alpha}_{i,\delta}} \operatorname{dist}(\nabla\Phi(s,t), K^m_{\lambda,l_0}(s, v(s,t))) + \int_{K^{3,\beta}_{i,\delta}} \operatorname{dist}(\nabla\Phi(s,t), K^m_{\lambda,l_0}(s, v(s,t))) \\
\leq \delta|K^{3,\alpha}_{i,\delta}| + N_i|K^{3,\beta}_{i,\delta}| \leq \delta|J^*_T| + N_i|K^{3,\beta}_{i,\delta}|,
\end{gathered}
\tag{6.3}
$$

where $N_i := \max_{(s,t)\in\bar{G}_i} \operatorname{dist}(\nabla\Phi(s,t), K^m_{\lambda,l_0}(s,v(s,t)))$ is independent of δ. By the definition of $K^{3,\beta}_{i,\delta}$ $(0<\delta<1)$,

$$K^{3,\beta}_{i,\delta_1} \subset K^{3,\beta}_{i,\delta_2} \ \text{ whenever } \ 0<\delta_1<\delta_2<1.$$

Let us check that

$$\bigcap_{0<\delta<1} K^{3,\beta}_{i,\delta} = \emptyset. \tag{6.4}$$

Suppose on the contrary that there is a point $(s,t)\in\bigcap_{0<\delta<1} K^{3,\beta}_{i,\delta}$. Then

$$\operatorname{dist}(\nabla\Phi(s,t), \partial|_{W_{s^m v(s,t)}} U^m_{\lambda,l_0}(s,v(s,t))) = 0,$$

and so

$$\nabla\Phi(s,t) \in U^m_{\lambda,l_0}(s,v(s,t)) \cap \partial|_{W_{s^m v(s,t)}} U^m_{\lambda,l_0}(s,v(s,t)) \neq \emptyset.$$

This is a contradiction to the fact that $U^m_{\lambda,l_0}(s,v(s,t))$ is open in $W_{s^m v(s,t)}$, and so (6.4) holds. We thus have

$$\delta|J^*_T| + N_i|K^{3,\beta}_{i,\delta}| \to 0 \ \text{ as } \ \delta\to 0^+.$$

Note also that

$$|\{(s,t)\in G_i : d_i(s,t)=\delta\}| > 0$$

for at most countably many $\delta\in(0,1)$. So it is possible to choose a $\delta_i\in(0,\epsilon/2)$ so that

$$\begin{cases} \delta_i|J^*_T| + N_i|K^{3,\beta}_{i,\delta_i}| \le \frac{\epsilon}{2^{i+1}}|J^*_T|, \\ |\{(s,t)\in G_i : d_i(s,t)=\delta_i\}| = 0. \end{cases} \tag{6.5}$$

With this choice of δ_i, we define

$$\begin{aligned} \hat{K}_i &:= \{(s,t)\in G_i : d_i(s,t)<\delta_i\}, \\ \hat{H}_i &:= \{(s,t)\in G_i : d_i(s,t)=\delta_i\}, \\ \hat{G}_i &:= \{(s,t)\in G_i : d_i(s,t)>\delta_i\}, \end{aligned}$$

so that $K_{i,\delta_i} = \hat{K}_i\cup\hat{H}_i$, $|\hat{H}_i|=0$ by (6.5), and $\hat{K}_i$ and $\hat{G}_i$ are disjoint open subsets of G_i with $|G_i\setminus(\hat{K}_i\cup\hat{G}_i)|=0$ by the continuity of the mapping $d_i:\bar{G}_i\to[0,\infty)$. By (6.3) and (6.5), we have

$$\begin{aligned} &\int_{\hat{K}_i} \operatorname{dist}(\nabla\Phi(s,t), K^m_{\lambda,l_0}(s,v(s,t)))dsdt \\ &= \int_{K_{i,\delta_i}} \operatorname{dist}(\nabla\Phi(s,t), K^m_{\lambda,l_0}(s,v(s,t)))dsdt \le \frac{\epsilon}{2^{i+1}}|J^*_T|. \end{aligned} \tag{6.6}$$

Let us take a moment here to explain what we have done so far. We have separated the open set G_i into two disjoint open sets $\hat{K}_i$ and $\hat{G}_i$. On the set $\hat{K}_i$, the value of the integral in question is already "small" enough to the extent (6.6) as we wanted in the fulfillment of the third of (6.2). So no modification will be made to Φ on the set $\hat{K}_i$. But on the set $\hat{G}_i$, the (inhomogeneous) distance from the gradient of Φ to K^m_{λ,l_0} is relatively "large", and therefore a necessary modification will be made to Φ by gluing suitable functions constructed in Section 5, specifically in Lemma 5.2, so that the integral can be made "small" enough. This is what to be accomplished in the following subsections.

6.2. Properties of the gradient of Φ in $\hat{G}_i$

By the uniform continuity of $\nabla\Phi : \bar{G}_i \to \mathbb{R}^{2\times 2}$, there exists an $\eta_i = \eta_i(\rho, \delta_i) > 0$ such that

$$(s,t), (s',t') \in \bar{G}_i,\ |(s,t) - (s',t')| \le \eta_i \Rightarrow |\nabla\Phi(s,t) - \nabla\Phi(s',t')| \le \rho\delta_i, \tag{6.7}$$

where $\rho > 0$ is a constant with

$$\rho < \min\left\{\frac{1}{6}, \frac{1}{12R^m M_\sigma}\right\} \tag{6.8}$$

and

$$M_\sigma := \sup_{p_1,p_2\in[-2\lambda,2\lambda], p_1\neq p_2} \left|\frac{\sigma(p_1) - \sigma(p_2)}{p_1 - p_2}\right| < \infty.$$

Let us check that for each $(s,t) \in \hat{G}_i$,

$$\begin{cases} \nabla\Phi(s,t) \in U^m_{\lambda,l_0}(s, v(s,t)), \\ \operatorname{dist}((\partial_s v(s,t), \partial_t\varphi(s,t)), \partial\tilde{U}^m_\lambda(s)) > \delta_i, \\ |\partial_t v(s,t)| < l_0 - \delta_i. \end{cases} \tag{6.9}$$

To show this, choose any $(s,t) \in \hat{G}_i$. By the third of (6.1), we can take a sequence $\{(s_j,t_j)\}_{j\in\mathbb{N}}$ in $\hat{G}_i$ such that

$$\begin{cases} \nabla\Phi(s_j,t_j) \in K^m_{\lambda,l_0}(s_j, v(s_j,t_j)) \cup U^m_{\lambda,l_0}(s_j, v(s_j,t_j)) \ \ \forall j \in \mathbb{N}, \\ (s_j,t_j) \to (s,j) \quad \text{in } \mathbb{R}^2 \text{ as } j \to \infty. \end{cases}$$

So for each $j \in \mathbb{N}$, we have

$$\begin{pmatrix} \partial_s v(s_j,t_j) & \partial_t v(s_j,t_j) \\ \partial_s\varphi(s_j,t_j) & \partial_t\varphi(s_j,t_j) \end{pmatrix} = \begin{pmatrix} p_j & l_j \\ (s_j)^m v(s_j,t_j) & (s_j)^m q_j \end{pmatrix} \tag{6.10}$$

for some $(p_j,q_j) \in \tilde{K}_\lambda \cup \tilde{U}^+_\lambda \cup \tilde{U}^-_\lambda$ and some $l_j \in [-l_0, l_0]$. Passing to a subsequence (we do not relabel),

$$(p_j,q_j) \to (p,q) \ \text{ and } \ l_j \to l \quad \text{as } j \to \infty,$$

for some $(p,q) \in \overline{\tilde{K}_\lambda \cup \tilde{U}^+_\lambda \cup \tilde{U}^-_\lambda}$ and some $l \in [-l_0, l_0]$. Letting $j \to \infty$ on both sides of (6.10), we obtain

$$\begin{pmatrix} \partial_s v(s,t) & \partial_t v(s,t) \\ \partial_s\varphi(s,t) & \partial_t\varphi(s,t) \end{pmatrix} = \begin{pmatrix} p & l \\ s^m v(s,t) & s^m q \end{pmatrix}. \tag{6.11}$$

Since $(s,t) \in \hat{G}_i$, it follows from the definition of $\hat{G}_i$ that

$$\begin{aligned} \delta_i < d_i(s,t) &= \operatorname{dist}(\nabla\Phi(s,t), K^m_{\lambda,l_0}(s, v(s,t)) \cup \partial|_{W_{s^m v(s,t)}} U^m_{\lambda,l_0}(s, v(s,t))) \\ &= \operatorname{dist}(P_W(\nabla\Phi(s,t)), K^m_{\lambda,l_0}(s,0) \cup \partial|_W U^m_{\lambda,l_0}(s,0)). \end{aligned} \tag{6.12}$$

Note that $\overline{\tilde{K}_\lambda \cup \tilde{U}^+_\lambda \cup \tilde{U}^-_\lambda}$ is the disjoint union of $\tilde{K}_\lambda \cup \partial\tilde{U}^+_\lambda \cup \partial\tilde{U}^-_\lambda$ and $\tilde{U}^+_\lambda \cup \tilde{U}^-_\lambda$. So if $(p,q) \notin \tilde{U}^+_\lambda \cup \tilde{U}^-_\lambda$, then $P_W(\nabla\Phi(s,t)) \in K^m_{\lambda,l_0}(s,0) \cup \partial|_W U^m_{\lambda,l_0}(s,0)$ by (6.11), and so $d_i(s,t) = 0$. This is a contradiction to (6.12). Thus

$$(p,q) \in \tilde{U}^+_\lambda \cup \tilde{U}^-_\lambda. \tag{6.13}$$

Next, suppose that $\mathrm{dist}((p, s^m q), \partial\tilde{U}_\lambda^m(s)) \leq \delta_i$. Since $\partial\tilde{U}_\lambda^m(s)$ is compact, we can choose a point $(\tilde{p}, \tilde{q}) \in \partial\tilde{U}_\lambda^+ \cup \partial\tilde{U}_\lambda^-$ so that

$$|(p, s^m q) - (\tilde{p}, s^m\tilde{q})| = \mathrm{dist}((p, s^m q), \partial\tilde{U}_\lambda^m(s)) \leq \delta_i.$$

But

$$\begin{pmatrix} \tilde{p} & l \\ 0 & s^m\tilde{q} \end{pmatrix} \in \partial|_W U_{\lambda,l_0}^m(s, 0),$$

and so

$$\begin{aligned}
\delta_i &< \mathrm{dist}(P_W(\nabla\Phi(s,t)), K_{\lambda,l_0}^m(s,0) \cup \partial|_W U_{\lambda,l_0}^m(s,0)) \\
&\leq \mathrm{dist}(P_W(\nabla\Phi(s,t)), \partial|_W U_{\lambda,l_0}^m(s,0)) \\
&\leq \left| P_W(\nabla\Phi(s,t)) - \begin{pmatrix} \tilde{p} & l \\ 0 & s^m\tilde{q} \end{pmatrix} \right| = |(p, s^m q) - (\tilde{p}, s^m\tilde{q})|
\end{aligned}$$

by (6.11) and (6.12). This is a contradiction, and we thus have

$$\mathrm{dist}((p, s^m q), \partial\tilde{U}_\lambda^m(s)) > \delta_i. \tag{6.14}$$

Finally, suppose $|l| \geq l_0 - \delta_i$. Assume further that $l \geq l_0 - \delta_i$. Note

$$\begin{pmatrix} p & l_0 \\ 0 & s^m q \end{pmatrix} \in \partial|_W U_{\lambda,l_0}^m(s, 0),$$

and so

$$\begin{aligned}
\mathrm{dist}(P_W(\nabla\Phi(s,t)), K_{\lambda,l_0}^m(s,0) \cup \partial|_W U_{\lambda,l_0}^m(s,0)) &\leq \mathrm{dist}(P_W(\nabla\Phi(s,t)), \partial|_W U_{\lambda,l_0}^m(s,0)) \\
&\leq \left| P_W(\nabla\Phi(s,t)) - \begin{pmatrix} p & l_0 \\ 0 & s^m q \end{pmatrix} \right| \\
&= l_0 - l \leq \delta_i
\end{aligned}$$

by (6.11). This is a contradiction to (6.12), and thus $l < l_0 - \delta_i$. If $l \leq -(l_0 - \delta_i)$, then we also have a contradiction, so that we conclude that

$$|l| < l_0 - \delta_i. \tag{6.15}$$

Thus (6.9) follows from (6.11), (6.13), (6.14), and (6.15).

6.3. Local gradient modifiers in subdivisions of $\hat{G}_i$

By the Vitali Covering Lemma, we can take a sequence $\{Q_i^k\}_{k\in\mathbb{N}}$ of disjoint open squares in $\hat{G}_i$ whose sides are parallel to the axes such that

$$\left| \hat{G}_i \setminus \bigcup_{k=1}^{\infty} Q_i^k \right| = 0.$$

For each $k \in \mathbb{N}$, let $d_i^k > 0$ denote the side length of Q_i^k and (s_i^k, t_i^k) the center of Q_i^k. Dividing these squares further if necessary, we can have

$$d_i^k \leq \min\left\{ \frac{\eta_i}{\sqrt{2}}, \frac{4\eta}{\sqrt{2}(\lambda - \lambda^-)}, \sqrt{\frac{4\eta}{\sqrt{2}(\lambda - \lambda^-)R^m}}, \frac{\delta_i}{12 M_g \sigma(1)} \right\} \quad \forall k \in \mathbb{N}, \tag{6.16}$$

where $M_g := \max_{|s_1|,|s_2| \leq R} |g(s_1, s_2)|$ and g is the polynomial of two variables such that

$$(s_1)^m - (s_2)^m = (s_1 - s_2) g(s_1, s_2) \quad \forall s_1, s_2 \in \mathbb{R}. \text{ (Take } g \equiv 0 \text{ if } m = 0.)$$

We fix an index $k \in \mathbb{N}$ in the rest of the section. If $(s,t), (s',t') \in Q_i^k$, then $|(s,t) - (s',t')| < \sqrt{2} d_i^k \leq \eta_i$ by (6.16), and so

$$|\nabla\Phi(s,t) - \nabla\Phi(s',t')| \leq \rho\delta_i$$

by (6.7). In particular,

$$|\nabla\Phi(s,t) - \nabla\Phi(s_i^k, t_i^k)| \leq \rho\delta_i \quad \forall (s,t) \in Q_i^k. \tag{6.17}$$

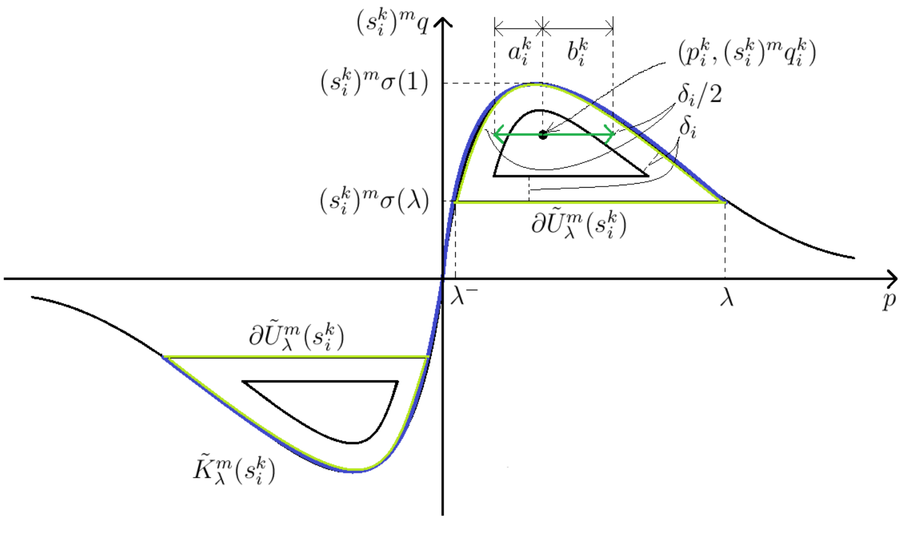

Figure 4: A necessary local s-derivative change of v in Q_i^k

Since $(s_i^k, t_i^k) \in Q_i^k \subset \hat{G}_i$, we have from (6.9) that

$$\begin{cases} \nabla\Phi(s_i^k, t_i^k) \in U_{\lambda,l_0}^m(s_i^k, v(s_i^k, t_i^k)), \\ \operatorname{dist}((\partial_s v(s_i^k, t_i^k), \partial_t \varphi(s_i^k, t_i^k)), \partial\tilde{U}_\lambda^m(s_i^k)) > \delta_i, \\ |\partial_t v(s_i^k, t_i^k)| < l_0 - \delta_i. \end{cases} \tag{6.18}$$

So

$$\begin{pmatrix} \partial_s v(s_i^k, t_i^k) & \partial_t v(s_i^k, t_i^k) \\ \partial_s \varphi(s_i^k, t_i^k) & \partial_t \varphi(s_i^k, t_i^k) \end{pmatrix} = \begin{pmatrix} p_i^k & l_i^k \\ (s_i^k)^m v(s_i^k, t_i^k) & (s_i^k)^m q_i^k \end{pmatrix} \tag{6.19}$$

for some $(p_i^k, q_i^k) \in \tilde{U}_\lambda^+ \cup \tilde{U}_\lambda^-$ and some $l_i^k \in \mathbb{R}$ with $|l_i^k| < l_0 - \delta_i$. Also

$$\mathrm{dist}((p_i^k, (s_i^k)^m q_i^k)), \partial \tilde{U}_\lambda^m(s_i^k)) > \delta_i. \tag{6.20}$$

So by the Intermediate Value Theorem, there exist two positive reals a_i^k and b_i^k such that

$$\begin{cases} \mathrm{dist}((p_i^k - a_i^k, (s_i^k)^m q_i^k), \tilde{K}_\lambda^m(s_i^k)) = \mathrm{dist}((p_i^k + b_i^k, (s_i^k)^m q_i^k), \tilde{K}_\lambda^m(s_i^k)) = \frac{\delta_i}{2}, \\ (p_i^k - a_i^k, (s_i^k)^m q_i^k), (p_i^k + b_i^k, (s_i^k)^m q_i^k) \in \tilde{U}_\lambda^m(s_i^k). \end{cases} \tag{6.21}$$

(See Figure 4.) Observe

$$a_i^k + b_i^k < \lambda - \lambda^-.$$

Let $\xi_i^k > 0$ be a constant with

$$\xi_i^k \leq \min\left\{ \frac{\delta_i}{2(\lambda - \lambda^-)\left[1 + \left(\frac{R-\delta_0}{\delta_0}\right)^m\right]}, \frac{\delta_i}{6(R - 2\delta_0)(R - \delta_0)^m(\lambda - \lambda^-)\left[1 + \left(\frac{R-\delta_0}{\delta_0}\right)^m\right]} \right\}. \tag{6.22}$$

Define the diamond-shaped $\tilde{D}_i^k$ in $\mathbb{R}^2$ as

$$\tilde{D}_i^k := \mathrm{int}\left(\mathrm{co}\{(0,1), (0,-1), (\xi_i^k, 0), (-\xi_i^k, 0)\}\right).$$

By the Vitali Covering Lemma, there exist a sequence $\{(s_{i,j}^k, t_{i,j}^k)\}_{j\in\mathbb{N}}$ in Q_i^k and a sequence $\{\epsilon_{i,j}^k\}_{j\in\mathbb{N}}$ of positive reals such that $\{(s_{i,j}^k, t_{i,j}^k) + \epsilon_{i,j}^k \tilde{D}_i^k\}_{j\in\mathbb{N}}$ is a sequence of disjoint open subsets of Q_i^k whose union has measure $|Q_i^k|$. Following the notations in (5.13), we have

$$(s_{i,j}^k, t_{i,j}^k) + \epsilon_{i,j}^k \tilde{D}_i^k = D(s_{i,j}^k - \epsilon_{i,j}^k \xi_i^k, s_{i,j}^k + \epsilon_{i,j}^k \xi_i^k, t_{i,j}^k, \epsilon_{i,j}^k) =: D_{i,j}^k \quad \forall j \in \mathbb{N}.$$

Let $j \in \mathbb{N}$. We also define according to the notations in (5.13) that

$$\begin{array}{rcl} (D_{i,j}^k)_r^\pm & := & D_r^\pm(s_{i,j}^k - \epsilon_{i,j}^k \xi_i^k, s_{i,j}^k + \epsilon_{i,j}^k \xi_i^k, t_{i,j}^k, \epsilon_{i,j}^k) \quad \forall r \in \{1,2,3\}, \\ (s_{i,j}^k)_r(t) & := & s_r(s_{i,j}^k - \epsilon_{i,j}^k \xi_i^k, s_{i,j}^k + \epsilon_{i,j}^k \xi_i^k, t_{i,j}^k, \epsilon_{i,j}^k; t) \quad \forall t \in [t_{i,j}^k - \epsilon_{i,j}^k, t_{i,j}^k + \epsilon_{i,j}^k], \forall r \in \{1,2\}, \\ \tilde{v}_{i,j}^k(s,t) & := & \tilde{v}(-a_i^k, b_i^k, s_{i,j}^k - \epsilon_{i,j}^k \xi_i^k, s_{i,j}^k + \epsilon_{i,j}^k \xi_i^k, t_{i,j}^k, \epsilon_{i,j}^k; s, t) \quad \forall (s,t) \in \overline{D_{i,j}^k}. \end{array}$$

Then Lemma 5.2 can be restated as follows in a bit more specific form:

(a) $\tilde{v}_{i,j}^k \in W_0^{1,\infty}(D_{i,j}^k)$,

(b) $\tilde{v}_{i,j}^k \in C^1\left(\overline{(D_{i,j}^k)_r^\pm}\right) \quad \forall r \in \{1,2,3\}$,

(c) $\partial_s \tilde{v}_{i,j}^k(s,t) = \begin{cases} -a_i^k & \forall (s,t) \in (D_{i,j}^k)_1^+ \cup (D_{i,j}^k)_1^- \cup (D_{i,j}^k)_3^+ \cup (D_{i,j}^k)_3^- \\ b_i^k & \forall (s,t) \in (D_{i,j}^k)_2^+ \cup (D_{i,j}^k)_2^-, \end{cases}$

(d)

$$\begin{array}{rcl} |\partial_t \tilde{v}_{i,j}^k(s,t)| & \leq & \max\{a_i^k, b_i^k\}\left[1 + \left(\dfrac{s_{i,j}^k + \epsilon_{i,j}^k \xi_i^k}{s_{i,j}^k - \epsilon_{i,j}^k \xi_i^k}\right)^m\right] \dfrac{2\epsilon_{i,j}^k \xi_i^k}{2\epsilon_{i,j}^k} \\ & \leq & (\lambda - \lambda^-)\left[1 + \left(\dfrac{R - \delta_0}{\delta_0}\right)^m\right] \xi_i^k \\ & \leq & \dfrac{\delta_i}{2} \quad \forall (s,t) \in \displaystyle\bigcup_{r=1}^3 [(D_{i,j}^k)_r^+ \cup (D_{i,j}^k)_r^-], \quad \text{(by (6.22))} \end{array}$$

(e) $\frac{\partial}{\partial t}\left(\int_{(s^k_{i,j})_1(t)}^{s} \tau^m \tilde{v}^k_{i,j}(\tau,t)d\tau\right) = \int_{(s^k_{i,j})_1(t)}^{s} \tau^m \partial_t \tilde{v}^k_{i,j}(\tau,t)d\tau \quad \forall (s,t) \in \bigcup_{r=1}^{3}[(D^k_{i,j})^+_r \cup (D^k_{i,j})^-_r]$,

(f) $\int_{(s^k_{i,j})_1(t)}^{(s^k_{i,j})_2(t)} \tau^m \tilde{v}^k_{i,j}(\tau,t)d\tau = 0 \quad \forall t \in [t^k_{i,j} - \epsilon^k_{i,j}, t^k_{i,j} + \epsilon^k_{i,j}]$,

(g) $\max_{\overline{D^k_{i,j}}} |\tilde{v}^k_{i,j}| \le \frac{a^k+b^k_i}{4} 2\epsilon^k_{i,j}\xi^k_i \le \frac{\lambda-\lambda^-}{4} d^k_i \le \frac{\eta}{\sqrt{2}}$, (by (6.16))

(h)

$$
\begin{aligned}
\max_{(s,t)\in\overline{D^k_{i,j}}} \left| \int_{(s^k_{i,j})_1(t)}^{s} \tau^m \tilde{v}^k_{i,j}(\tau,t)d\tau \right| &\le \frac{a^k + b^k_i}{4}(s^k_{i,j} + \epsilon^k_{i,j}\xi^k_i)^m (2\epsilon^k_{i,j}\xi^k_i)^2 \\
&\le \frac{\lambda - \lambda^-}{4} R^m (d^k_i)^2 \le \frac{\eta}{\sqrt{2}}. \quad \text{(by (6.16))}
\end{aligned}
$$

6.4. New function Φ_η from old Φ

We now define

$$\tilde{v} := \sum_{i,j,k\in\mathbb{N}} \tilde{v}^k_{i,j} \chi_{D^k_{i,j}} \quad \text{in } J^*_T.$$

Note that $\forall i, j, k \in \mathbb{N}$,

$$
\begin{aligned}
\|\tilde{v}^k_{i,j}\|_{W^{1,\infty}(D^k_{i,j})} &= \|\tilde{v}^k_{i,j}\|_{L^\infty(D^k_{i,j})} + \|\partial_s \tilde{v}^k_{i,j}\|_{L^\infty(D^k_{i,j})} + \|\partial_t \tilde{v}^k_{i,j}\|_{L^\infty(D^k_{i,j})} \\
&\le \frac{\eta}{\sqrt{2}} + \max\{a^k_i, b^k_j\} + \frac{\delta_i}{2} \quad \text{(by (c), (d), and (g))} \\
&\le \frac{\eta}{\sqrt{2}} + (\lambda - \lambda^-) + \frac{\epsilon}{4}, \quad \text{(by (6.5))}
\end{aligned}
$$

that is, $\sup_{i,j,k\in\mathbb{N}} \|\tilde{v}^k_{i,j}\|_{W^{1,\infty}(D^k_{i,j})} \le \frac{\eta}{\sqrt{2}} + (\lambda - \lambda^-) + \frac{\epsilon}{4} < \infty$. Applying the Gluing Lemma, it follows from this inequality, (a), and (b) that

$$
\begin{cases} \tilde{v} \in W^{1,\infty}_0(J^*_T), \\ \tilde{v} \text{ is piecewise } C^1 \text{ in } J^*_T. \end{cases} \tag{6.23}
$$

Define

$$\tilde{\varphi}(s,t) := \int_{\delta_0}^{s} \tau^m \tilde{v}(\tau,t)d\tau \quad \forall (s,t) \in \bar{J}^*_T.$$

It is then clear that $\tilde{\varphi} \in W^{1,\infty}(J^*_T)$. Also, by (f) and the definitions of $\tilde{\varphi}$ and $\tilde{v}$,

$$\tilde{\varphi}(s,t) = 0 \quad \forall (s,t) \in \bar{J}^*_T \setminus \bigcup_{i,j,k\in\mathbb{N}} D^k_{i,j}, \tag{6.24}$$

and hence $\tilde{\varphi} \equiv 0$ on ∂J^*_T. Thus $\tilde{\varphi} \in W^{1,\infty}_0(J^*_T)$.

Let $r \in \{1,2,3\}$ and $(s,t) \in (D^k_{i,j})^\pm_r$. By (e) and (f),

$$\partial_t \tilde{\varphi}(s,t) = \int_{(s^k_{i,j})_1(t)}^{s} \tau^m \partial_t \tilde{v}^k_{i,j}(\tau,t)d\tau. \tag{6.25}$$

So it is easily deduced from (b) that

$$\partial_t \tilde{\varphi} \in C^0\left(\overline{(D^k_{i,j})^\pm_r}\right).$$

By the definition of $\tilde{\varphi}$,

$$\partial_s \tilde{\varphi}(s,t) = s^m \tilde{v}(s,t) \quad \forall (s,t) \in J^*_T. \tag{6.26}$$

Since $\tilde{v} \in C^0(\bar{J}^*_T)$, we have $\partial_s \tilde{\varphi} \in C^0(\bar{J}^*_T)$. In particular,

$$\partial_s \tilde{\varphi} \in C^0\left(\overline{(D^k_{i,j})^\pm_r}\right),$$

so that

$$\tilde{\varphi} \in C^1\left(\overline{(D^k_{i,j})^\pm_r}\right). \tag{6.27}$$

Thus

$$\begin{cases} \tilde{\varphi} \in W^{1,\infty}_0(J^*_T), \\ \tilde{\varphi} \text{ is piecewise } C^1 \text{ in } J^*_T. \end{cases} \tag{6.28}$$

Finally, we define

$$v_\eta := v + \tilde{v}, \;\; \varphi_\eta := \varphi + \tilde{\varphi}, \;\text{ and }\; \Phi_\eta := (v_\eta, \varphi_\eta) \quad \text{in } J^*_T.$$

6.5. Completion of Proof of Theorem 3.1

To finish the proof of the density theorem, Theorem 3.1, we will show that the function Φ_η defined above belongs to $\Phi^* + W^{1,\infty}_0(J^*_T;\mathbb{R}^2)$ and satisfies all of (6.2).

First, it follows from (6.1), (6.23), and (6.28) that

$$\begin{cases} \Phi_\eta \in \Phi^* + W^{1,\infty}_0(J^*_T;\mathbb{R}^2), \\ \Phi_\eta \text{ is piecewise } C^1 \text{ in } J^*_T. \end{cases}$$

It remains to verify the rest of (6.2).

The fourth of (6.2): Note

$$\|\Phi - \Phi_\eta\|_{L^\infty(J^*_T;\mathbb{R}^2)} = \sup_{i,j,k\in\mathbb{N}} \|(\tilde{v},\tilde{\varphi})\|_{L^\infty(D^k_{i,j};\mathbb{R}^2)} \leq \sup_{i,j,k\in\mathbb{N}} \left(\|\tilde{v}\|^2_{L^\infty(D^k_{i,j})} + \|\tilde{\varphi}\|^2_{L^\infty(D^k_{i,j})}\right)^{1/2},$$

since $\tilde{v} = \tilde{\varphi} = 0$ on $\bar{J}^*_T \setminus \bigcup_{i,j,k\in\mathbb{N}} D^k_{i,j}$. But for every $(i,j,k) \in \mathbb{N}^3$,

$$\|\tilde{v}\|_{L^\infty(D^k_{i,j})} = \|\tilde{v}^k_{i,j}\|_{L^\infty(D^k_{i,j})} \leq \frac{\eta}{\sqrt{2}} \quad \text{(by (g))}$$

and

$$\|\tilde{\varphi}\|_{L^\infty(D^k_{i,j})} \leq \frac{\eta}{\sqrt{2}}. \quad \text{(by (f) and (h))}$$

Thus

$$\|\Phi - \Phi_\eta\|_{L^\infty(J^*_T;\mathbb{R}^2)} \leq \eta.$$

The second of (6.2): By the third of (6.1) and (6.26),

$$\begin{aligned} \partial_s\varphi_\eta(s,t) &= \partial_s(\varphi(s,t)+\tilde\varphi(s,t)) \\ &= s^m v(s,t) + s^m\tilde v(s,t) \\ &= s^m v_\eta(s,t) \end{aligned} \tag{6.29}$$

for a.e. $(s,t) \in J_T^*$. Since $\Phi_\eta \equiv \Phi$ on $J_T^* \setminus \bigcup_{i,j,k\in\mathbb{N}} D_{i,j}^k$, it follows from the third of (6.1) that

$$\nabla\Phi_\eta(s,t) = \nabla\Phi(s,t) \in K_{\lambda,l_0}^m(s,v(s,t)) \cup U_{\lambda,l_0}^m(s,v(s,t)) = K_{\lambda,l_0}^m(s,v_\eta(s,t)) \cup U_{\lambda,l_0}^m(s,v_\eta(s,t))$$

for a.e. $(s,t) \in J_T^* \setminus \bigcup_{i,j,k\in\mathbb{N}} D_{i,j}^k$. Let $i,j,k \in \mathbb{N}$. To finish the proof of this part, it now suffices to show that

$$\nabla\Phi_\eta(s,t) \in U_{\lambda,l_0}^m(s,v_\eta(s,t)) \quad \text{for a.e. } (s,t) \in D_{i,j}^k. \tag{6.30}$$

To this end, we will show that for a.e. $(s,t) \in D_{i,j}^k$, we have

$$|\partial_t v_\eta(s,t)| < l_0 \tag{6.31}$$

and

$$\begin{cases} \partial_s v_\eta(s,t) \in (-\lambda,-\lambda^-) \cup (\lambda^-,\lambda), \\ \partial_t\varphi_\eta(s,t) \in I_\lambda^m(s,\partial_s v_\eta(s,t)). \end{cases} \tag{6.32}$$

Then combining (6.29), (6.31), and (6.32) and appealing to Lemma 3.2, we obtain (6.30).

Since $D_{i,j}^k \subset Q_i^k$, it follows from (6.17) that for each $(s,t) \in D_{i,j}^k$,

$$|\partial_t v(s,t) - \partial_t v(s_i^k,t_i^k)| \le |\nabla\Phi(s,t) - \nabla\Phi(s_i^k,t_i^k)| \le \rho\delta_i.$$

But $|\partial_t v(s_i^k,t_i^k)| < l_0 - \delta_i$ by the third of (6.18). Observe also that for a.e. $(s,t) \in D_{i,j}^k$,

$$|\partial_t\tilde v(s,t)| = |\partial_t\tilde v_{i,j}^k(s,t)| \le \frac{\delta_i}{2}. \quad \text{(by (d))}$$

Thus for a.e. $(s,t) \in D_{i,j}^k$,

$$\begin{aligned} |\partial_t v_\eta(s,t)| &= |\partial_t v(s,t) + \partial_t\tilde v(s,t)| \\ &\le |\partial_t v(s,t) - \partial_t v(s_i^k,t_i^k)| + |\partial_t v(s_i^k,t_i^k)| + |\partial_t\tilde v(s,t)| \\ &< \rho\delta_i + l_0 - \delta_i + \frac{\delta_i}{2} < l_0 - \frac{\delta_i}{3} < l_0, \quad \text{(by (6.8))} \end{aligned}$$

and hence (6.31) holds.

As above for each $(s,t) \in D_{i,j}^k$,

$$\begin{aligned} \rho\delta_i &\ge |\nabla\Phi(s,t) - \nabla\Phi(s_i^k,t_i^k)| \\ &\ge |(\partial_s v(s,t),\partial_t\varphi(s,t)) - (\partial_s v(s_i^k,t_i^k),\partial_t\varphi(s_i^k,t_i^k))| \\ &= |(\partial_s v(s,t),\partial_t\varphi(s,t)) - (p_i^k,(s_i^k)^m q_i^k)| \quad \text{(by (6.19))} \\ &\ge \max\{|\partial_s v(s,t) - p_i^k|, |\partial_t\varphi(s,t) - (s_i^k)^m q_i^k|\}, \end{aligned} \tag{6.33}$$

where $(p_i^k, q_i^k) \in \tilde{U}_\lambda^+ \cup \tilde{U}_\lambda^-$. Let us assume that $(p_i^k, q_i^k) \in \tilde{U}_\lambda^+$. (The other case that $(p_i^k, q_i^k) \in \tilde{U}_\lambda^-$ can be shown in the same way.) We have to show that for a.e. $(s,t) \in D_{i,j}^k$,

$$\begin{cases} \lambda^- < \partial_s v_\eta(s,t) = \partial_s v(s,t) + \partial_s \tilde{v}(s,t) < \lambda, \\ \partial_t \varphi_\eta(s,t) \in I_\lambda^m(s, \partial_s v_\eta(s,t)), \\ \text{or equivalently } s^m\sigma(\lambda) < \partial_t\varphi(s,t) + \partial_t\tilde{\varphi}(s,t) < s^m\sigma(\partial_s v(s,t) + \partial_s\tilde{v}(s,t)). \end{cases} \tag{6.34}$$

Case 1: Assume $(s,t) \in (D_{i,j}^k)_1^+ \cup (D_{i,j}^k)_1^- \cup (D_{i,j}^k)_3^+ \cup (D_{i,j}^k)_3^-$.
In this case, we have

$$\partial_s \tilde{v}_{i,j}^k(s,t) = -a_i^k$$

by (c). Let $0 < p_i^{k,-} < 1 < p_i^{k,+}$ be such that

$$\sigma(p_i^{k,\pm}) = q_i^k,$$

so that $(s_i^k)^m\sigma(p_i^{k,\pm}) = (s_i^k)^m q_i^k$. Then by (6.21),

$$p_i^{k,-} + \frac{\delta_i}{3} < p_i^k - a_i^k < p_i^k < p_i^k + b_i^k < p_i^{k,+} - \frac{\delta_i}{3}.$$

(See Figure 4.) Also by (6.8) and (6.33),

$$-\frac{\delta_i}{6} < \partial_s v(s,t) - p_i^k < \frac{\delta_i}{6}.$$

Thus

$$\begin{aligned} \lambda^- < p_i^{k,-} &< p_i^k - \frac{\delta_i}{3} - a_i^k < \partial_s v(s,t) + \frac{\delta_i}{6} - \frac{\delta_i}{3} - a_i^k \\ &< \partial_s v(s,t) - a_i^k = \partial_s v(s,t) + \partial_s \tilde{v}_{i,j}^k(s,t) = \partial_s v_\eta(s,t) \\ &< \partial_s v(s,t) - \frac{\delta_i}{6} + b_i^k + \frac{\delta_i}{3} < p_i^k + b_i^k + \frac{\delta_i}{3} < p_i^{k,+} < \lambda, \end{aligned}$$

that is,

$$\lambda^- < \partial_s v_\eta(s,t) < \lambda. \tag{6.35}$$

Next, note from (6.22), (6.25), and (d) that

$$|\partial_t\tilde{\varphi}(s,t)| \le (R - 2\delta_0)(R-\delta_0)^m(\lambda - \lambda^-)\left[1 + \left(\frac{R-\delta_0}{\delta_0}\right)^m\right]\xi_i^k \le \frac{\delta_i}{6}. \tag{6.36}$$

By (6.21),

$$(s_i^k)^m q_i^k \le (s_i^k)^m \sigma(p_i^k - a_i^k) - \frac{\delta_i}{2}. \tag{6.37}$$

(See Figure 4.) But

$$\begin{gathered} |(s_i^k)^m\sigma(p_i^k - a_i^k) - s^m\sigma(\partial_s v(s,t) - a_i^k)| \\ = |((s_i^k)^m - s^m)\sigma(p_i^k - a_i^k) + s^m(\sigma(p_i^k - a_i^k) - \sigma(\partial_s v(s,t) - a_i^k))| \\ \le |s_i^k - s||g(s_i^k, s)|\sigma(1) + R^m M_\sigma |p_i^k - \partial_s v(s,t)| \\ \le M_g\sigma(1)d_i^k + R^m M_\sigma \rho\delta_i \quad \text{(by (6.33))} \\ < \tfrac{\delta_i}{6}. \quad \text{(by (6.8) and (6.16))} \end{gathered}$$

Combining this with (6.37), we get

$$(s_i^k)^m q_i^k < s^m\sigma(\partial_s v(s,t) - a_i^k) - \frac{\delta_i}{3}. \tag{6.38}$$

So

$$\begin{aligned}\partial_t\varphi(s,t) + \partial_t\tilde{\varphi}(s,t) &\le (s_i^k)^m q_i^k + \rho\delta_i + |\partial_t\tilde{\varphi}(s,t)| \quad \text{(by (6.33))}\\ &< s^m\sigma(\partial_s v(s,t) - a_i^k) - \frac{\delta_i}{3} + \frac{\delta_i}{6} + \frac{\delta_i}{6} \quad \text{(by (6.8), (6.36), and (6.38))}\\ &= s^m\sigma(\partial_s v(s,t) + \partial_s\tilde{v}(s,t)). \end{aligned} \tag{6.39}$$

Note (See Figure 4.)

$$(s_i^k)^m q_i^k > (s_i^k)^m\sigma(\lambda) + \delta_i \quad \text{(by (6.20))} \tag{6.40}$$

and

$$|s^m\sigma(\lambda) - (s_i^k)^m\sigma(\lambda)| \le M_g\sigma(1)d_i^k \le \frac{\delta_i}{12}. \quad \text{(by (6.16))} \tag{6.41}$$

So

$$\begin{aligned}\partial_t\varphi(s,t) + \partial_t\tilde{\varphi}(s,t) &\ge (s_i^k)^m q_i^k - \rho\delta_i - |\partial_t\tilde{\varphi}(s,t)| \quad \text{(by (6.33))}\\ &\ge (s_i^k)^m\sigma(\lambda) + \delta_i - \frac{\delta_i}{6} - \frac{\delta_i}{6} \quad \text{(by (6.8), (6.36), and (6.40))}\\ &\ge s^m\sigma(\lambda) - \frac{\delta_i}{12} + \delta_i - \frac{\delta_i}{6} - \frac{\delta_i}{6} \quad \text{(by (6.41))}\\ &> s^m\sigma(\lambda). \end{aligned} \tag{6.42}$$

Combining (6.35), (6.39), and (6.42), we have (6.34) whenever $(s,t) \in (D_{i,j}^k)_1^+ \cup (D_{i,j}^k)_1^- \cup (D_{i,j}^k)_3^+ \cup (D_{i,j}^k)_3^-$.

Case 2: (6.34) also holds whenever $(s,t) \in (D_{i,j}^k)_2^+ \cup (D_{i,j}^k)_2^-$. To show this, we just follow the lines of Case 1 with minor modifications whenever it is necessary. We skip the details.
We conclude from Cases 1 and 2 that (6.34) holds for a.e. $(s,t) \in D_{i,j}^k$.

The third of (6.2): Observe

$$\int_{J_T^*} \mathrm{dist}(\nabla\Phi_\eta(s,t), K_{\lambda,l_0}^m(s, v_\eta(s,t)))dsdt = \sum_{i=1}^\infty \int_{\hat{K}_i} \mathrm{dist}(\nabla\Phi(s,t), K_{\lambda,l_0}^m(s, v(s,t)))dsdt$$

$$+ \sum_{i,j,k\in\mathbb{N}} \int_{D_{i,j}^k} \mathrm{dist}(\nabla\Phi(s,t) + \nabla\tilde{\Phi}(s,t), K_{\lambda,l_0}^m(s, v(s,t) + \tilde{v}(s,t)))dsdt =: A + B,$$

where $\tilde{\Phi} := (\tilde{v}, \tilde{\varphi})$. By (6.6), we have $A \le \frac{\epsilon}{2}|J_T^*|$. Let $i,j,k \in \mathbb{N}$, and let $(s,t) \in \bigcup_{r=1}^3[(D_{i,j}^k)_r^+ \cup (D_{i,j}^k)_r^-]$ be any point at which (6.30) holds. Then by Lemma 3.5,

$$\mathrm{dist}(\nabla\Phi(s,t) + \nabla\tilde{\Phi}(s,t), K_{\lambda,l_0}^m(s, v(s,t) + \tilde{v}(s,t)))$$

$$= \mathrm{dist}((\partial_s v(s,t) + \partial_s\tilde{v}(s,t), \partial_t\varphi(s,t) + \partial_t\tilde{\varphi}(s,t)), \tilde{K}_\lambda^m(s)).$$

We assume further that $(s,t) \in (D^k_{i,j})^+_1 \cup (D^k_{i,j})^-_1 \cup (D^k_{i,j})^+_3 \cup (D^k_{i,j})^+_3$, so that $\partial_s \tilde{v}(s,t) = -a^k_i$. Choose any $(p,q) \in \tilde{K}_\lambda$. Then

$$
\begin{aligned}
&|(\partial_s v(s,t) - a^k_i, \partial_t \varphi(s,t) + \partial_t \tilde{\varphi}(s,t)) - (p, s^m q)| \\
&= |(\partial_s v(s,t), \partial_t \varphi(s,t)) - (\partial_s v(s^k_i, t^k_i), \partial_t \varphi(s^k_i, t^k_i)) + (\partial_s v(s^k_i, t^k_i), \partial_t \varphi(s^k_i, t^k_i)) \\
&\quad + (-a^k_i, 0) + (0, \partial_t \tilde{\varphi}(s,t)) - (p, s^m q) - (p, (s^k_i)^m q) + (p, (s^k_i)^m q)| \\
&\leq \rho\delta_i + |(p^k_i - a^k_i, (s^k_i)^m q^k_i) - (p, (s^k_i)^m q)| \\
&\quad + |\partial_t \tilde{\varphi}(s,t)| + |q||s^m - (s^k_i)^m| \\
&\leq \frac{\delta_i}{6} + \frac{\delta_i}{6} + \frac{\delta_i}{12} + |(p^k_i - a^k_i, (s^k_i)^m q^k_i) - (p, (s^k_i)^m q)|
\end{aligned}
$$

as in the verification for the second of (6.2). Taking an infimum on $(p,q) \in \tilde{K}_\lambda$ for the far-left and -right terms of the inequalities, we have

$$
\begin{aligned}
&\operatorname{dist}((\partial_s v(s,t) + \partial_s \tilde{v}(s,t), \partial_t \varphi(s,t) + \partial_t \tilde{\varphi}(s,t)), \tilde{K}^m_\lambda(s)) \\
&\leq \frac{5\delta_i}{12} + \operatorname{dist}((p^k_i - a^k_i, (s^k_i)^m q^k_i), \tilde{K}^m_\lambda(s^k_i)) = \frac{5\delta_i}{12} + \frac{\delta_i}{2} < \delta_i < \frac{\epsilon}{2}
\end{aligned}
$$

by (6.5) and (6.21). We can get the same result when $(s,t) \in (D^k_{i,j})^+_2 \cup (D^k_{i,j})^-_2$, but we omit the details. We now have

$$
\operatorname{dist}((\partial_s v(s,t) + \partial_s \tilde{v}(s,t), \partial_t \varphi(s,t) + \partial_t \tilde{\varphi}(s,t)), \tilde{K}^m_\lambda(s)) \leq \frac{\epsilon}{2} \quad \text{for a.e. } (s,t) \in D^k_{i,j}.
$$

So we obtain $B \leq \sum_{i,j,k\in\mathbb{N}} \frac{\epsilon}{2} |D^k_{i,j}| \leq \frac{\epsilon}{2} |J^*_T|$. Thus $A + B \leq \epsilon |J^*_T|$.

The theorem is finally proved.

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References

[1] J.M. Ball and R.D. James. Fine phase mixtures as minimizers of energy. *Arch. Rational Mech. Anal.*, 100:13–52, 1987.

[2] J.M. Ball and R.D. James. Proposed experimental tests of a theory of fine microstructure and the two-well problem. *Phil. Trans. Roy. Soc. London A*, 338:389–450, 1992.

[3] G. Bellettini and G. Fusco. The Γ-limit and the related gradient flow for singular perturbation functionals of Perona-Malik type. *Trans. Amer. Math. Soc.*, 360:4929–4987, 2008.

[4] Y. Chen and K. Zhang. Young measure solutions of the two-dimensional Perona-Malik equation in image processing. *Commun. Pure Appl. Anal.*, 5:615–635, 2006.

[5] M. Chipot and D. Kinderlehrer. Equillibrium configurations of crystals. *Arch. Rational Mech. Anal.*, 103:237–277, 1988.

[6] B. Dacorogna. *Direct methods in the caculus of variations*. Springer, second edition, 2008.

[7] B. Dacorogna and P. Marcellini. General existence theorems for Hamilton-Jacobi equations in the scalar and vectorial cases. *Acta Mathematica*, 178:1–37, 1997.

[8] B. Dacorogna and P. Marcellini. *Implicit partial differential equations. Progress in Nonlinear Differential Equations and their Applications*, volume 37. Birkhäuser, 1999.

[9] S. Esedoglu. An analysis of the Perona-Malik scheme. *Comm. Pure Appl. Math.*, 54:1442–1487, 2001.

[10] S. Esedoglu and J.B. Greer. Upper bounds on the coarsening rate of discrete, ill-posed nonlinear diffusion equation. *Comm. Pure Appl. Math.*, 62:57–81, 2009.

[11] M. Ghisi and M. Gobbino. A class of local classical solutions for the one-dimensional Perona-Malik equation. *Trans. Amer. Math. Soc.*, 361:6429–6446, 2009.

[12] M. Ghisi and M. Gobbino. An example of global classical solution for the Perona-Malik equation. *Comm. Partial Differential Equations*, 36:1318–1352, 2011.

[13] M. Ghisi and M. Gobbino. On the evolution of subcritical regions for the Perona-Malik equation. *Interfaces Free Bound.*, 13:105–125, 2011.

[14] M. Gobbino. Entire solutions of the one-dimensional Perona-Malik equation. *Comm. Partial Differential Equations*, 32:719–743, 2007.

[15] M. Gromov. Convex integration of differential relations. *Izw. Akad. Nauk S.S.S.R.*, 37:329–343, 1973.

[16] K. Höllig. Existence of infinitely many solutions for a forward backward heat equation. *Trans. Amer. Math. Soc.*, 278:299–316, 1983.

[17] B. Kawohl and N. Kutev. Maximum and comparison priciple for one-dimensional anisotrotic diffusion. *Math. Ann.*, 311:107–123, 1998.

[18] S. Kichenassamy. The Perona-Malik Paradox. *SIAM J. Appl. Math.*, 57:1328–1342, 1997.

[19] B. Kirchheim. Deformations with finitely many gradients and stability of quasiconvex hulls. *C. R. Acad. Sci. Paris Ser. I Math*, 332:289–294, 2001.

[20] O.A. Ladyženskaja, V.A. Solonnikov, and N.N. Ural'ceva. *Linear and quasilinear equations of parabolic type, Translations of Mathematical Monographs*, volume 23. American Mathematical Society, Providence, R.I.

[21] C. De Lellis and L. Székelyhidi Jr. The Euler equations as a differential inclusion. *Ann. of Math.*, 170:1417–1436, 2009.

[22] G.M. Lieberman. *Second order parabolic differential equations*. World Scientific, 1996.

[23] S. Müller and M. Sychev. Optimal existence theorems for nonhomogeneous differential inclusions. *J. Funct. Anal.*, 181:447–475, 2001.

[24] S. Müller and V. Šverák. Attainment results for the two-well problem by convex integration. In *Geometric analysis and the calculus of variations*, pages 239–251. International Press, Cambridge, MA, 1996.

[25] S. Müller and V. Šverák. Convex integration with constraints and applications to phase transitions and partial differential equations. *J. European Math. Sco.*, 1:393–422, 1999.

[26] S. Müller and V. Šverák. Convex integration for Lipschitz mappings and counterexamples to regularity. *Ann. of Math.*, 157:715–742, 2003.

[27] P. Perona and J. Malik. Scale space and edge detection using anistropic diffusion. *IEEE Trans. Pattern Anal. Mach. Intell*, 12:629–639, 1990.

[28] B. Yan. A Baire's category method for the Dirichlet problems of quasiregular mappings. *Trans. Amer. Math. Soc.*, 355:4755–4765, 2003.

[29] B. Yan. On $W^{1,p}$ solvability for special vectorial Hamilton-Jacobi systems. *Bull. Sci. Math.*, 127:467–483, 2003.

[30] K. Zhang. Existence of infinitely many solutions for the one-dimensional Perona-Malik model. *Calc. Var. Partial Differential Equations*, 26:171–199, 2006.

[31] K. Zhang. On existence of weak solutions for one-dimenstional forward-backward diffusion equations. *J. Differential Equations*, 220:322–353, 2006.