Power series link invariants and the Thurston norm

Efstratia Kalfagianni

E–mail: *ekal@math.rutgers.edu*

Abstract. We generalize a result of Scharlemann and Thompson ([14]), to obtain a relation between the Thurston norms of links related by "skein moves", in irreducible homology 3-spheres. Then we apply this result to the study of "skein trees" of links and we formulate an obstruction to the convergence of the HOMFLY power series link invariants constructed in [10].

Mathematics Subject Classification: 57M25, 57M40.

Keywords: Homology spheres, link invariants, skein moves, Thurston norm.

1. Introduction

In [10] we used the techniques developed in [9] and [12], to show the existence and uniqueness of link power series invariants obeying the 2-variable Jones polynomial ([3, 7, 8]) skein relation, in a large class of rational homology 3- spheres. Subsequently, Jones-type power series for links in rational homology 3-spheres were also constructed by Le, Murakami and Ohtsuki ([11]). We should point out here that the methods of [9, 10, 12] and [11] are different. The construction of the invariants in [9, 10, 12] is direct and it is based on the use of intrinsic 3-manifold topology. In [11] the authors use combinatorial techniques and obtain the power series link invariants as an application of their construction of a universal "quantum" 3-manifold invariant.

For links in S^3 these power series are known to converge to Laurent polynomials (two variable Jones polynomial). However, it is not known whether

The author was supported by NSF grant DMS-9626140.

they can be normalized so that they converge to Laurent polynomials for links in other 3-manifolds.

The purpose of this paper is to formulate an obstruction to the convergence of the power series in terms of the Thurston norm. This obstruction is a class of links (*infinite links*) that can't be "simplified" by crossing changes. See Definition 3.3.

Suppose that M is an orientable 3-manifold. Let $\pi = \pi_1(M)$ and let $\hat{\pi}$ denote the set of non-trivial conjugancy classes in π. Notice, that $\hat{\pi}$ can be identified with the set of free non-trivial homotopy classes of oriented loops in M. An n-component *link* is a collection of n unordered oriented circles, tamely and disjointly embedded in M. Hence, a link corresponds to an unordered n-tuple of conjugancy classes in π. In every homotopy class we will fix, once and for all, a link CL and call it a *"trivial link"*. If CL has k components which are homotopically trivial, our choice will be such that $CL = CL^* \coprod U^k$, where U^k is the standard unlink with k components in a small ball neighborhood disjoint from CL^*. We will denote by $\mathcal{CL}^*$ the set of all *trivial links* with none of their components homotopically trivial. It is obvious that $\mathcal{CL}^*$ is in 1-1 correspondence with unordered n-tuples of elements in $\hat{\pi}$, and that every link L is homotopic to a certain $CL^* \coprod U^k$ for some $CL^* \in \mathcal{CL}^*$, possibly empty.

Let $\mathcal{L}$ be the set of isotopy classes of links in M and let $R = \mathbb{C}[v^{\pm 1}, z^{\pm 1}]$ be the ring of Laurent polynomials in v and z.

Now set $z = t^{\frac{1}{2}} - t^{-\frac{1}{2}}$ and let $\mathcal{I}$ be the ideal of $R[t]$ generated by $v - v^{-1}$ and t. Let $\hat{R}$ be the pro-$\mathcal{I}$ completion of $R[t]$, i.e. the inverse limit of

$$\cdots \to R[t]/\mathcal{I}^n \to R[t]/\mathcal{I}^{n-1} \to \cdots.$$

Definition A. *We will say that a $\mathbb{Z}$- homology 3-sphere M, is admissible iff the following is true: There exists a unique map $J_M : \mathcal{L} \to \hat{R}$ satisfying the HOMFLY skein relation*

$$v^{-1} J_M(L_+) - v J_M(L_-) = z J_M(L_o)$$

and with given values $J_M(U)$ and $J_M(CL^*)$ for every $CL^* \in \mathcal{CL}^*$.

Here the three links L_+, L_- and L_o appearing in the skein relation above differ only in a small ball neighborhood in M where they intersect at a positive crossing, a negative crossing, and a smoothing of a crossing, respectively. See Figure 1.

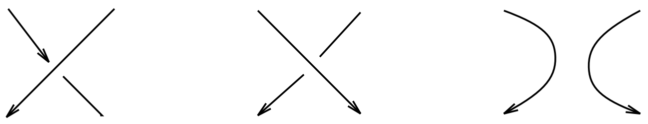

FIGURE 1. The links L_+, L_- and L_o

By the constructions in [3, 7, 8], S^3 is *admissible*. Also by [10, Theorem, 4.2.1] every irreducible homology 3-sphere that is either atoroidal or a Seifert fibered space is *admissible*. In [10] we asked the question of whether we can choose the set of links $\mathcal{CL}^*$ and the values $J_M(U)$ and $J_M(CL^*)$ appropriately, so that J_M is a Laurent polynomial in $z^{\pm 1}$ and $v^{\pm 1}$.

To illustrate the difficulty of the problem, let us recall that every link $L \subset S^3$ can be reduced to unlinks by a series of finitely many *skein moves*; that is, by replacing L_+ (resp. L_-) with the pair of links L_- (resp. L_+) and L_o. An immediate consequence of this fact is that if the power series of Definition A are normalized to take polynomial values for the unlinks, then they are Laurent polynomials for all links in S^3.

Now let M be a homology 3-sphere. As a first step in understanding the convergence behavior of the HOMFLY power series in M we ask the following question:

Question 1. Is there a choice of *trivial links* $\mathcal{CL}^*$ such that, every link $L \subset M$ can be reduced to disjoint unions of unlinks and elements in $\mathcal{CL}^*$, by a series of finitely many *skein moves*?

The main difficulty in answering the question above, lies in finding a complexity function for links, that behaves well under the skein moves.

In [14] Scharlemann and Thompson used machinery of Gabai ([5, 6]) to relate the Euler characteristics of the three links L_+, L_- and L_o in S^3. Here, we generalize the main result of [14] for links in irreducible homology spheres (see Theorem 2.4). Then, we use it to show that an obstruction to a positive answer to Question 1 is the set of *infinite links* in the manifold M. A link $L \subset M$ is called an *infinite link* iff i) no component of L lies in a 3-ball in M; and ii) no crossing change increases the Euler characteristic. More precisely we have:

Theorem B. *Let M be an irreducible $\mathbb{Z}$-homology sphere which is admissible, and suppose that $J_M(IL)$ is a Laurent polynomial in $v^{\pm 1}$ and $z^{\pm 1}$, for every infinite link, IL, in M. Moreover, assume that the values of J_M on the trivial links are chosen to be Laurent polynomials, in $v^{\pm 1}$ and $z^{\pm 1}$. Then, $J_M(L)$ is a Laurent polynomial for all $L \in \mathcal{L}$.*

2. A relation between the genus of L_+, L_- and L_o

In this section we use machinery of Gabai ([5, 6]), as used by Scharlemann and Thompson in [14], to relate the Euler characteristics of the three links L_+, L_- and L_o, in an irreducible $\mathbb{Z}$-homology sphere.

2a. Preliminaries

For an oriented surface S, define the *complexity* $\chi^-(S)$, of S to be the negative of the Euler characteristic of the union of the non-simply connected components of S.

Definition 2.1.([15]) *Let N be a compact, oriented 3-manifold with boundary ∂N, and let Σ be a (possibly empty) surface in ∂N. To a homology class $\alpha \in H_2(N, \Sigma)$, assign $x(\alpha)$ to be the minimum complexity of all oriented properly embedded surfaces that represent α. The function $x : H_2(N, \Sigma) \longrightarrow \mathbb{N}$, is called the Thurston norm.* An oriented surface $(S, \partial S) \subset (N, \partial N)$ is called taut if it is incompressible and $\chi^-(S) = x[S, \partial S]$ in $H_2(N, n(\partial S))$. Here $n(\partial S)$ is a regular neighborhood of ∂S in ∂N.*

Let M be a $\mathbb{Z}$-homology sphere, and let L be a link in M. Then L bounds an embedded, oriented, surface (a Seifert surface) in M. The maximal Euler characteristic, over all Seifert surfaces of a link L, is called the

* Actually, in general it is a pseudonorm

Euler characteristic of L, $\chi(L)$. Let $n(L)$ denote a regular neighborhood of L and let $N = M \setminus n(L)$. A Seifert surface S of L gives rise to a surface $(S,\ \partial S) \subset (N,\ \partial N)$.

Lemma 2.2. *S is taut if and only if $\chi(S) = \chi(L)$.*

Proof. The conclusion follows from the fact that $\chi^{-}(S) = d - \chi(S)$, where d is the number of the disc components of S. See Lemma 1.2 of [14]. □

We say that two closed, connected, oriented surfaces $C_1,\ C_2 \subset M$ are I-cobordant, if there is a compact oriented 3-manifold $V \subset M$ such that $\partial V = C_1 \cup C_2$ and the maps

$$j_i : H_1(C_i) \longrightarrow H_1(V)$$

induced by inclusion, are injective ($i = 1,\ 2$).

Definition 2.3. ([6, Definition 1.6]) *Let N be a compact oriented 3-manifold and let $(S,\ \partial S) \subset (N,\ \partial N)$ be a properly embedded oriented surface, such that each component of ∂N is a torus. Let P be a component of ∂N, such that $P \cap S = \emptyset$. We say that N is S_P-atoroidal if boundary parallel tori are the only incompressible tori that are I-cobordant to P by a cobordism that lies in $N \setminus S$.*

2b. Thurston norm and skein moves

Recall that for a link $L \subset M$, we denoted by $n(L)$ a regular neighborhood of L. Our purpose in this paragraph is to prove the following theorem:

Theorem 2.4. *Assume that M is an irreducible $\mathbb{Z}$-homology sphere, and let L_+, L_- and L_o be links in M, which differ at a crossing as in the HOMFLY skein relation. Moreover, assume that one of $M \setminus n(L_+)$ and $M \setminus n(L_-)$ is irreducible. Then two of $\chi(L_+)$, $\chi(L_-)$ and $\chi(L_o) - 1$ are equal and not larger than the third.*

The proof, is similar to the proof of Theorem 1.4 of [14]. However, and for the convenience of the reader, we will present the details of our arguments.

Assume that $M \setminus n(L_+)$ is irreducible, let D be a disc that intersects $L = L_+$ in exactly two points of opposite orientation (see Figure 2), and let $K = \partial D$.

Since by [14] the conclusion is true for S^3 we may, and will, assume that $M \neq S^3$. Since $M \setminus n(L_+)$ is irreducible, so is $M \setminus n(L_+ \cup D)$. Let

$$N = M \setminus n(L_+ \cup K)$$

and let N_+, N_- and N_o be the manifolds obtained from N by filling in a solid torus along $\partial(n(K))$ with slope ∞, -1 and 0, respectively.

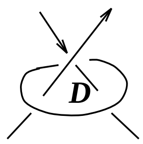

FIGURE 2

We have that $N_+ = M \setminus n(L_+)$ and $N_- = M \setminus n(L_-)$. The relation between N_o and $M \setminus n(L_o)$ is given in Lemma 2.6 below.

Lemma 2.5. *Let S be a Seifert surface for L_+ which has maximal Euler characteristic over all Seifert surfaces in N. At least one of N_- and N_o is irreducible. Moreover, S still has maximal Euler characteristic in at least two of N_+, N_- and N_o.*

Proof. First suppose that the manifold N is S_P*-atoroidal*, where $P = \partial(n(K))$. Then by Corollary 2.4 of [6] there is at most one essential simple closed curve α on P so that S is not *taut* in the manifold $N(\alpha)$, obtained from N by filling in P along α. Moreover, if α is not this exceptional curve then $N(\alpha)$ is irreducible. Now the conclusion in this case follows by Lemma 2.2.

Now suppose that there exists a non-boundary parallel, incompressible torus $T \subset N$ which is I-cobordant to $\partial(n(K)) = P$ by a cobordism V, that lies in $N \setminus S$.

Claim 1: T remains incompressible in $N_\pm$ and N_- is irreducible.

Proof of Claim 1: If T remains incompressible in M then the conclusion is clearly true. Now assume that T is compressible in M. Since M is irreducible and $M \neq S^3$, T must bound a solid torus in M, on exactly one side. Then, since T in not parallel to one of the boundary tori, the following must be true: There must be a solid torus $U \subset M$, with $T = \partial U$, such that i) a sublink of L_+ containing the crossing in question lies U; ii) T is incompressible in $\bar{N} = U \setminus L_+$; and iii) the linking number of U and K is ± 1.

Since, by assumption, $N_+ = M \setminus n(L_+)$ is irreducible, U cannot lie in a 3-ball in M. Thus, and since the crossing change from L_+ to L_- is done in the interior of U, $N_\pm \setminus U$ and $M \setminus U$ are irreducible. Now since $M \neq S^3$, $T = \partial U$ must remain incompressible in $M \setminus U$. Thus, it remains incompressible in both N_+ and N_-.

To see that N_- is irreducible, notice that $U \setminus (U \cap L_+)$ is irreducible. Then, N_- is irreducible since it is obtained by gluing $U \setminus (U \cap L_+)$ and $N_- \setminus U$, along the incompressible torus T. This finishes the proof of the claim.

To continue with the proof of the Lemma, let $N(\alpha)$ (resp. $V(\alpha)$) be the manifold obtained from N (resp. V) by filling in P with slope α. We have:

Claim 2: If T is incompressible in $N(\alpha)$ then S remains taut in this manifold.

Proof of Claim 2: Let Σ be a Seifert surface for L_+ in $N(\alpha)$. Since the linking number of K and L_+ is zero, $\Sigma \cap T$ is homologically trivial in T. Thus, we can replace the components of $\Sigma \cap V(\alpha)$ by annuli, attached along the components of

$$\partial(\Sigma \cap U) \cap T$$

near T and away from P. Then, we can isotope to obtain a Seifert surface $\Sigma^{'}$ in N. Since no component of $\Sigma \cap V(\alpha)$ is a disc (because T is incompressible) we have

$$\chi(\Sigma) \leq \chi(\Sigma^{'}).$$

But we also have

$$\chi(\Sigma^{'}) \leq \chi(S)$$

by the definition of S. Thus S has maximal Euler characteristic in $N(\alpha)$, For more details see also the proof of Corollary 2.4 in [6]. This finishes the proof of Claim 2 and the Lemma. $\square$

Now, we consider the connection between N_o and $M \setminus n(L_o)$. We isotope S so that it intersects the disc D in an arc δ, whose endpoints lie on L_+. Let S_o be the surface obtained from S by deleting a neighborhood of δ. Clearly S_o is a Seifert surface of L_o. One can see that $M \setminus n(S_o)$ is obtained from $M \setminus n(S)$ by attaching a 2-handle to $n(S)$ along the curve

$$\lambda = \partial(n(S)) \cap D$$

Here $n(S)$ (resp. $n(S_o)$) denotes a regular neighborhood of S (resp S_o).

Lemma 2.6. *If N_o is irreducible and S is taut in N_o, then $M \setminus n(L_o)$ is irreducible and S_o is a taut Seifert surface for L_o, in M.*

Proof. We see that λ and K cobound an annulus. Since N_o is obtained from N by attaching a meridional disc to K, λ bounds a disc in N_o. Denote this disc by D^*. Observe that attaching a 2-handle to $M \setminus n(S)$ along λ, is the same as deleting from $N_o \setminus n(S)$ a neighborhood of D^*. Let $n(\partial S)$ denote a neighborhood of ∂S in the boundary of N_o. Now if N_o is irreducible and S is taut in N_o, the manifold $(N_o,\ n(\partial S))$ is *sutured* in the sense of [5]. Moreover, the induced *sutured* manifold structure on $N_o \setminus n(S)$ is *taut*. That is $N_o \setminus n(S)$ is irreducible and S is a taut surface. The boundary of the disc D^* intersects the sutures of $N_o \setminus n(S)$ at exactly two points. By Lemma 3.12 of [5], deleting such a disc preserves tautness. Thus,

$$M \setminus n(S_o) = (N_o \setminus n(S)) \setminus D^*$$

is irreducible and S_o is a taut surface. Thus, $M \setminus n(L_o)$ must be irreducible. But then, by Lemma 3.6 in [5] S_o must remain taut in $M \setminus n(L_o)$. $\square$

The completion of the proof of Theorem 2.4. By combining Lemmas 2.5 and 2.6 together with the observation that a Seifert surface for L_- in N_- corresponds to Seifert surface for L_- in M we obtain Theorem 2.4. $\square$

Theorem 2.4 was also proven by U. Kaiser ([16]).

3. Applications to link skein trees

3a. Preliminaries and some special cases

Let M be an irreducible $\mathbb{Z}$-homology sphere. Let $L : \coprod S^1 \longrightarrow M$ be an n component link, and let $\mathcal{M}^L$ denote the space of maps

$$\coprod S^1 \longrightarrow M$$

which are homotopic to L. The spaces $\mathcal{M}^L$ corresponding to links with n components are in one to one correspondence with the unordered n-tuples of conjugancy classes in $\pi = \pi_1(M)$. In every such space we will fix, once and for all, a link CL and call it a *"trivial link"*. If CL has k components which are homotopically trivial, our choice will be such that $CL = CL^* \coprod U^k$, where U^k is the standard unlink with k components in a small ball neighborhood disjoint from CL^*. Moreover, for the purposes of this paper, we will choose each CL^* to have minimal complexity χ^-, over all links in $\mathcal{M}^{CL^*}$.

Let $\mathcal{L}$ (resp. $\mathcal{CL}$) denote the set of all isotopy classes of links (resp. of our chosen trivial links), in M . Let us begin by recalling our construction of HOMFLY power series from [10].

Theorem 3.1. ([10, Theorem 4.2.1]) *Let M be an irreducible Z-homology sphere which is either atoroidal or a Seifert fibered space. Let $\mathcal{L}$ and $\mathcal{CL}$ be as above. For every $n \in \mathbb{N}$, there exists a unique sequence of complex valued link invariants $v_n^0,\ v_n^1, \dots\ v_n^m, \dots$, with given values on the links in $\mathcal{CL}$, such that if we define a formal power series*

$$J_{\{M,n\}}(L) = \sum_{m=0}^{\infty} v_n^m(L) z^m$$

for $L \in \mathcal{L}$ then

$$t^{\frac{n+1}{2}} J_{\{M,n\}}(L_+) - t^{-\frac{n+1}{2}} J_{\{M,n\}}(L_-) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) J_{\{M,n\}}(L_o) \qquad (3.1)$$

where $t = e^z = 1 + z + \frac{z^2}{2} + \ldots$. *Moreover, we have*

$$J_{\{M,n\}}(L \coprod U) = u_n(t)\ J\{M,n\}(L) \tag{3.2}$$

where U *denotes the trivial knot in* M, $L \coprod U$ *is obtained from* L *by adding* U, *in a ball neighborhood disjoint from* L, *and*

$$u_n(t) = \frac{t^{\frac{n+1}{2}} - t^{-\frac{n+1}{2}}}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}$$

Now set $z = t^{\frac{1}{2}} - t^{-\frac{1}{2}}$ and let $\mathcal{I}$ be the ideal of $R[t]$ generated by $v - v^{-1}$ and t. Let $\hat{R}$ be the pro-$\mathcal{I}$ completion of $R[t]$, i.e. the inverse limit of

$$\cdots \to R[t]/\mathcal{I}^n \to R[t]/\mathcal{I}^{n-1} \to \cdots.$$

It is not hard to see that the existence of the power series $\{J_{\{M,n\}}\}_{n\in\mathbb{N}}$, is equivalent to the existence of a map $J_M : \mathcal{L} \to \hat{R}$, as in Definition A of the introduction.

Thus, Theorem 3.1 implies that a homology 3-sphere which is either atoroidal or a Seifert fibered space is *admissible*. In this section we will study the question of whether we can renormalize the power series of Theorem 3.1 so that they become polynomials in $t^{\pm 1}$ and $(t^{\frac{1}{2}} - t^{-\frac{1}{2}})^{\pm 1}$. Let us first look at a special case. Let M be a homology 3-sphere and let $J_L = J_{\{M,-1\}}(L)$. We define

$$\nabla_L(t) \overset{def}{=} J_L(t^2)$$

By (3.1) we obtain that $\nabla_L(t)$ satisfies the following skein relation

$$\nabla_{L_+}(t) - \nabla_{L_-}(t) = (t - t^{-1})\nabla_{L_o}(t) \tag{3.3}$$

By (3.3) and the results in [2] we see that, possibly after normalization, $\nabla_L(t)$ is the *Conway potential function*, which is known to be a Laurent rational function in t. More precisely, it is known that

$$\nabla_L(t) = \frac{\Delta_L(t^2)}{t - t^{-1}} \tag{3.4}$$

if L is a knot, and

$$\nabla_L(t) = \Delta_L(t^2, \dots, t^2) \tag{3.5}$$

if L is a link. Here Δ_L denotes the Alexander polynomial of L, and the number of variable entries in the right hand side of (3.5) is the same as the number of components of L.

To simplify our notation, we will write J_n instead of $J_{\{M,n\}}$.

Lemma 3.2. *Suppose that L lies in a ball $B^3 \subset M$. Then $J_n(L)$ is a Laurent polynomial in t.*

Proof. Instead of working with embeddings in $B^3 \subset M$ we can work with link projections on a plane inside B^3. For an embedding $L \subset M$ let $a(L)$ denote the minimum, over all projections of L, number of crossings. Moreover, let $b(L)$ denote the number of crossing changes needed to go from L to an embedding of the *unlink*. Define a complexity function

$$c(L) = (a(L),\ b(L))$$

We order the complexities lexicographically, and proceed by induction. By the fact that $J_n(unknot) = 1$, and (3.2) we see that J_n is a Laurent polynomial for all *unlinks*. If $c(L) = (0,\ 0)$, then L is the *unlink*. Otherwise, use (3.1) to write $J_n(L)$ as a linear combination of the invariants on links with strictly smaller complexity. $\square$

3b. Obstructions to polynomial invariants

Let M be an irreducible homology 3-sphere and let $L \subset M$ be a link. The maximal sublink $L^{'} \subset L$ that can be separated from the rest of L by a 3-ball $B \subset M$ will be called *the inessential sublink* of L. The complement

$$L^* = L \setminus L^{'}$$

will be called *the essential sublink* of L. Define $m(L)$ by

$$m(L) = \begin{cases} \chi^-(L^*) = -\chi(L^*), & \text{if } L^* \text{ is non-empty} \\ 0 \quad , & \text{if } L^* \text{ is empty,} \end{cases}$$

where $\chi(L^*)$ is the Euler characteristic of L^*.

Definition 3.3. *We will say that a link $L \subset M$ is* infinite *if and only if: (i) No component of L lies in a 3-ball in M and (ii) If $L_\pm$ is a link obtained from L by a single crossing change, then $m(L_\pm) \geq m(L)$. Moreover, we will say that the* infinite *link L is of type (I) iff there is a link $L_\pm$ as above, which is non-isotopic to L, and such that $m(L_\pm) = m(L)$. Otherwise we will say that L is of type (II).*

Definition 3.4. *Let $L \subset M$ be a link and let $L_\pm$ be the result of a single crossing change on L. We will say that the crossing change is* good *if and only if $m(L_\pm) \leq m(L)$.*

Remark. By the proof of Theorem 2.4, if $M \setminus L_+$ is irreducible and one of $M \setminus L_-$ and $M \setminus L_o$ is reducible, then the crossing change from L_+ to L_- is *good*.

We will say that two infinite links of type (I), L and $L^{'}$, are equivalent iff there exists a finite sequense of links

$$L = L_1 \to L_2 \to \ldots \to L_{n-1} \to L_n = L^{'}$$

such that a) each L_i is an infinite link of type (I), b) L_{i+1} is obtained from L_i by a single crossing change, c) L_{i+1} is not isotopic to L_i and d) $m(L_{i+1}) = m(L_i)$ for $i = 1, \ldots, n$.

A *skein move* on a link L_+ (resp. L_-), is the replacement of L_+ (resp. L_-) with the pair of links L_- (resp. L_+) and L_o. To any sequence of *skein moves* we can associate a binary tree, with a vertex for each link and edges between a link and the pair of links obtained by a skein move. Such a tree T, will be called *a skein tree.* If T is finite then the end corresponding to the link to begin with is called the *root* and the other ends are called the *leaves.*

Let $\mathcal{IL} = \mathcal{IL}(M)$ denote the set of isotopy classes of all infinite links of type (II) in M, together with a set of isotopy classes of representatives of equivalence classes of infinite links of type (I).

Lemma 3.5. *Let M be an irreducible $\mathbb{Z}$-homology sphere and let $\mathcal{CL}$ denote the set of our chosen trivial links. Every link $L \subset M$, has a finite skein tree T with the following properties:*

i) Every leaf L_T of T, is a union of a link in $\mathcal{CL}$ and a link in $\mathcal{IL}$.

ii) All the crossing changes used in the skein moves corresponding to T are good.

Proof. Let L be a link in M and let L^* (resp $L^{'}$) denote its essential (resp. inessential) sublink.

Let CL (resp. CL^*) denote the chosen *trivial link* in the homotopy class of L (resp. L^*). Clearly,

$$CL = CL^* \coprod U^l$$

where U^l is an unlink.

Define $s = s(L)$ to be the number of components of L^*, and define $k = k(L)$ to be the minimum number of *good* crossing changes required to change L^* to CL^* or to a union of a (possibly-empty) link in $\mathcal{CL}$ and a link in $\mathcal{IL}$. In particular, if L^* is already in $\mathcal{IL}$ or L^* is empty then $k(L) = 0$.

Now, define the complexity function $d(L)$ by

$$d(L) := (m(L),\ s(L),\ k(L)),$$

where $m(L)$ is defined as before Definition 3.3, and order the complexities lexicographically. If L^* is the empty link the conclusion is clearly true. So, assume that L^* is not empty. If the link L^* admits no (non-essential) *good* crossings, then it is an infinite link of type (II) already, and there is nothing to prove. So we suppose that L^* admits *good* crossings. Without loss of generality we can assume that $L = L_+$.

Claim: We have that $d(L_-) < d(L)$ and $d(L_o) < d(L)$.

Proof of the claim: By the definition of a good crossing we have that $m(L_-) \leq m(L)$ and since $s(L_-) \leq s(L_+)$ and $k(L_-) < k(L_+)$ we obtain that $d(L_-) < d(L)$. Now let us show that $d(L_o) < d(L)$.

Since M is irreducible and no component of L^*_+ lies in a 3-ball we have that $M \setminus n(L^*_+)$ is irreducible. By Theorem 2.4 we have that $\chi^-(L^*_+)$ is equal to one of $\chi^-(L^*_-)$ and $1 + \chi^-(L^*_o)$ and no smaller than the other. Moreover, at most one of $M \setminus n(L^*_-)$ and $M \setminus n(L^*_o)$ can be reducible. We will show that in all cases arising from Theorem 2.4, we have that $m(L_o) < m(L_+)$.

Case 1. Suppose that $M \setminus n({L^*}_o)$ is reducible. Then $M \setminus n(L^*_-)$ must be irreducible, and we have

$$m(L_-) = \chi^-(L^*_-) = \chi^-(L^*_+) = m(L_+)$$

and $\chi(L^*_o) \geq \chi(L^*_+) + 1$. Since $M \setminus L^*_o$ is not irreducible, there must be a 2-sphere S^2 in M, separating a sublink $L_1 \subset L^*_o$, from the rest of ${L^*}_o$, and such that

$$\bar{L}^*_o := L^*_o \setminus L_1$$

is the *essential sublink of* of L_o. Let S be a Seifert surface of maximal Euler characteristic for L^*_+ and let S_o be the Seifert surface for L^*_o, obtained from S as in the proof of Theorem 2.4. Then,

$$\chi(L^*_o) \geq \chi(S_o) = \chi(L^*_+) + 1$$

By capping off the components of $S^2 \cap S_o$ (if there are any that cannot be removed by isotopy) with discs away from S^2, we obtain a Seifert surface S_1 for $\bar{L}^*_o$ such that $\chi(S_1) \geq \chi(S_o)$. Then we have

$$m(L_o) = \chi^-(\bar{L}^*_o) \leq -\chi(S_1) \leq \chi^-(S_o) = \chi^-(L^*_+) - 1 = m(L_+) - 1$$

Thus, $m(L_o) < m(L_+)$ as desired.

Case 2. Suppose that $M \setminus n({L^*}_o)$ is irreducible. Thus, the *essential sublink* of L_o is L^*_o itself. Again we have that $\chi(L^*_o) \geq \chi(L^*_+) + 1$, and thus

$$m(L_o) = \chi^-(L^*_o) \leq \chi^-(L^*_+) - 1 = m(L_+) - 1$$

Finally, we have $m(L_o) < m(L_+)$ as desired. This finishes the proof of the claim and that of Lemma 3.5. $\square$

Theorem 3.6. *Let M be an irreducible $\mathbb{Z}$-homology sphere for which Theorem 3.1 is true, and assume that the values of the power series J_n on the trivial links, are chosen to be polynomials in $t^{\pm 1}$ and $(t^{\frac{1}{2}} - t^{-\frac{1}{2}})^{\pm 1}$. If $J_n(IL)$ is a polynomial in $t^{\pm 1}$ and $(t^{\frac{1}{2}} - t^{-\frac{1}{2}})^{\pm 1}$ for every link $IL \in \mathcal{IL}$, then so is $J_n(L)$ for every $L \in \mathcal{L}$.*

Proof. It follows from Lemma 3.5. $\square$

Now Theorem B of the introduction is an immediate consequence of Theorem 3.6 above.

3c. Generators of skein modules

Recall that $\mathcal{L}$ is the set of isotopy classes of links in M and let $R = \mathbb{C}[v^{\pm 1}, z^{\pm 1}]$ be the ring of Laurent polynomials in v and z. With the notation as in the previous paragraph, the HOMFLY skein module $\mathcal{S}_3(M)$ is defined to be the R-module spanned by $\mathcal{L}$, and subject to the skein relations

$$v^{-1}L_+ - vL_- = zL_o \tag{3.6}$$

and

$$L \coprod U^k = u^{k-1}\, L \tag{3.7}$$

where U^k denotes the k-component unlink in M, $L \coprod U^k$ is obtained from L by adding U^k in a ball neighborhood disjoint from L, and

$$u = u(v,z) = \frac{v^{-1} - v}{z}$$

Let $S(R\hat{\pi})$ be the symmetric tensor algebra of the free R-module $R\hat{\pi}$ generated by $\hat{\pi}$. J. Przytycki conjectured the following:

Conjecture. *If M is compact and contains no closed non-separating surfaces, then*

$$\mathcal{S}_3(M) \cong S(R\hat{\pi})$$

as R-modules.

In the special case of $M = X \times [0,1]$ where X is a compact surface, it was proved by Przytycki (see [4] and references therein) that $\mathcal{S}_3(M) \cong S(R\hat{\pi})$ as R-algebras. Many other partial verifications of the conjecture are known.

Proposition 3.7. *a) Let M be an irreducible homology sphere, and let $\tau = \hat{\pi} \cup \mathcal{IL}$. Moreover, let $S(R\tau)$ be the symmetric tensor algebra of the free R-module $R\tau$, generated by τ. Then we have*

$$\mathcal{S}_3(M) \cong S(R\tau)$$

as R-modules.

b) If M is a homotopy 3-sphere, then $\mathcal{S}_3(M) \cong S(R(\mathcal{IL}))$. In particular,

$$\mathcal{S}_3(M) \cong \mathcal{S}_3(S^3) \cong R$$

if and only if $\mathcal{IL}$ is empty.

Proof. The proof of a) and that of the "if" part of b) follows from Lemma 3.5. Now let us prove the "only if" part of b). Assume that

$$\mathcal{S}_3(M) \cong \mathcal{S}_3(S^3) \cong R \tag{3.8}$$

and suppose that $\mathcal{IL}$ is not empty. Let $IL \in \mathcal{IL}$. Then it follows from (3.6), (3.7) and (3.8) that IL must have a finite *skein tree* T, each leaf of which is an unlink. By the proof of Theorem 2.4 we have the following: If for some $L = L_+ \subset M$ two of $M \setminus L_+$, $M \setminus L_-$ and $M \setminus L_o$ are reducible then so is the third one. (This was also shown by Bullock in [1].) By applying this to every node of T we obtain that $M \setminus IL$ is reducible. But this is a contradiction since by Definition 3.3 no component of IL lies in a 3-ball in M. $\square$

Remark. Notice that by Bing's Theorem (see for example [13]), if the set of *infinite links* in a homotopy 3-sphere M is empty then $M \cong S^3$.

Acknowledgment. The author thanks Xiao-Song Lin for many helpful discussions on this work and Abigail Thompson for helpful correspondence about Theorem 2.4. She also thanks Doug Bullock for a conversation during the AMS meeting at the University of Maryland.

References

1. D. Bullock: *Skein related links in 3-manifolds*; Topology Appl. **60** (1994), no. 3, 235–248.
2. S. Boyer and D. Lines: *A Conway potential function for Rational Homology Spheres*; Proc. Edinburgh Math. Soc. (2) **35** (1992) no 1, 53–69.
3. P. Freyd, J. Hoste, W.B.R. Lickorish, K. Millet, A. Ocneanu and D. Yetter: *A new polynomial invariant of knots and links*, Bulletin AMS **12**(1985),239-246.

4. J. Hoste and J. Przytycki, *A survey of skein modules of 3-manifolds*, in Knots 90 (Osaka, 1990), de Gruyter, 1992.
5. D. Gabai: *Foliations and the topology of 3-manifolds,* J. Diff. Geom. vol **18** (1983), 461-478.
6. D. Gabai: *Foliations and the topology of 3-manifolds II,* J. Diff. Geom. vol **26** (1987), 445-503.
7. V.F.R. Jones: *A polynomial invariant for knots via von Neumann algebras*, Bull. AMS **12** (1985), 103–111.
8. ___________: *Hecke algebra representations of braid groups and link polynomials* Annals of Math. **126** (1987), 335-338.
9. E. Kalfagianni: *Finite type invariants for knots in 3-manifolds* , Topology vol. **37**, no 3 (1998), 673-607.
10. E. Kalfagianni and X.-S. Lin: *The HOMFLY polynomial for links in Rational homology spheres*, Topology, in press.
11. T. Q. T. Le, J. Murakami and T. Ohtsuki: *On a universal invariant of 3-manifolds*, Topology vol. **37**, no 3 (1998), 539-574.
12. X.-S. Lin: *Finite type link invariants of 3-manifolds*,Topology, vol. **33**, no 1 (1994), 45-71.
13. D. Rolfsen: *Knots and Links*, MLS 7, Publish or perish (1976).
14. M. Scharlemann and A. Thompson: *Link genus and the Conway moves*, Comen. Math Helveciti, **64** (1989), 527-535.
15. W. Thurston: *A norm for the homology of 3-manifolds*, Mem AMS , vol. **59**, no 339, (1986), 99-130.
16 . U. Kaiser: *Link theory in manifolds*, Lecture Notes in Mathematics, 1669. Springer-Verlag, Berlin, 1997.

Department of Mathematics, Rutgers University, Piscataway, NJ 08854-8019

Address after September 1, 1998:

Department of Mathematics, Michigan State University, East Lansing, MI 48824