SKEIN MODULES AND CHARACTER VARIETIES OF SEIFERT MANIFOLDS

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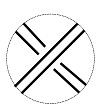
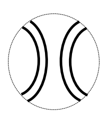
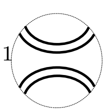
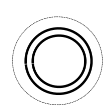
Abstract. We show that the Kauffman bracket skein module of a closed Seifert fibered 3-manifold M is finitely generated over $\mathbb{Z}[A^{\pm 1}]$ if and only if M is irreducible and non-Haken. We analyze in detail the character varieties $\mathcal{X}(M)$ of such manifolds and show that under mild conditions they are reduced. We compute the Kauffman bracket skein modules for these 3-manifolds (over $\mathbb{Q}(A)$) and show that their dimensions coincide with $|\mathcal{X}(M)|$.

1. Introduction

?⟨sec:intro⟩?

Throughout the paper, M will denote an oriented, closed 3-manifold. Let $\mathcal{S}(M,R)$ be the Kauffman bracket skein module of M with coefficients in a commutative ring R, with a distinguished invertible element $A \in R$. The module $\mathcal{S}(M,R)$ is the quotient of the free R-module on framed unoriented links in M, including the empty link $\emptyset$, by the relations:

K1:  $= A$  $+A^{-1}$  K2: $L \sqcup$  $= (-A^2 - A^{-2})L$

In this paper we will consider the coefficient rings $R = \mathbb{Z}[A^{\pm 1}], \mathbb{Q}[A^{\pm 1}]$, and $\mathbb{Q}(A)$.

Recall that a $\mathbb{Q}[A^{\pm 1}]$-module S is tame if it is a direct sum of cyclic $\mathbb{Q}[A^{\pm 1}]$-modules and, for at least one odd N, S does not contain $\mathbb{Q}[A^{\pm 1}]/(\phi_{2N})$ as a submodule, where ϕ_{2N} is the $2N$-th cyclotomic polynomial. In particular, every finitely generated $\mathbb{Q}[A^{\pm 1}]$-module is tame. We propose the following:

⟨ourconjecture⟩ **Conjecture 1.1.** *For any closed 3-manifold M the following are equivalent:*

(a) M contains no 2-sided essential surface (i.e. M is irreducible, non-Haken).
(b) The skein module $\mathcal{S}(M, \mathbb{Z}[A^{\pm 1}])$ is finitely generated.
(c) The skein module $\mathcal{S}(M, \mathbb{Q}[A^{\pm 1}])$ is tame.

All the essential surfaces we discuss in this paper will be 2-sided.

Conjecture 1.1 is inspired and closely related to a conjecture of Przytycki (Conjecture (E) of [Kir97, Problem 1.92]) asserting that $S(M, \mathbb{Q}[A^{\pm 1}])$ is free for manifolds M containing no essential surfaces. Since $\mathcal{S}(M) := S(M, \mathbb{Q}(A))$ is finitely generated over $\mathbb{Q}(A)$ by [GJS23], Przytycki's conjecture implies that $S(M,\mathbb{Q}[A^{\pm 1}]))$ is finitely generated for closed 3-manifold without essential surfaces.

Let

$$\mathcal{X}(M)=\mathrm{Hom}(\pi_1(M),\mathrm{SL}_2(\mathbb{C}))/\!\!/\mathrm{SL}_2(\mathbb{C}),$$

denote the $\mathrm{SL}_2(\mathbb{C})$-character variety of M, considered as a scheme, as defined for example in [LM85, BH95]. The coordinate ring $\mathbb{C}[\mathcal{X}(M)]$ is the algebra of global sections of the structure sheaf of $\mathcal{X}(M)$. The above variety may be non-reduced, that is $\mathbb{C}[\mathcal{X}(M)]$ may have a nontrivial nil-radical [KM17]. We denote by $X(M)$ the algebraic set underlying $\mathcal{X}(M)$ and by $|X(M)|$ the cardinality of $X(M)$.By the definitions, $\mathbb{C}[X(M)]=\mathbb{C}[\mathcal{X}(M)]/\sqrt{0}$, where $\sqrt{0}$ is the nil-radical of $\mathbb{C}[\mathcal{X}(M)]$.

Our results in [DKS23] imply if $X(M)$ is infinite, then $\mathcal{S}(M,\mathbb{Q}[A^{\pm 1}])$ is non-tame. On the other hand, by the Culler-Shalen theory [CS83], M contains an essential surface. Hence, Conjecture 1.1 holds for 3-manifolds with infinite $X(M)$.

One of our main results is:

⟨main-i⟩ **Theorem 1.2.** *Conjecture 1.1 holds for all Seifert fibered 3-manifolds.*

An infinite family of non-irreducible 3-manifolds with finite $X(M)$, is formed by the connected sum $M=\mathbb{RP}^3\#L(2p,1)$, for $p>0$. For $M=\mathbb{RP}^3\#\mathbb{RP}^3$ the skein module $\mathcal{S}(M,\mathbb{Q}[A^{\pm 1}])$ was computed in [Mro11] and was shown to be non-tame. More recently, Belletti and Detcherry [BD24] proved that $\mathcal{S}(M,\mathbb{Z}[A^{\pm 1}])$ is not finitely generated for any $M:=\mathbb{RP}^3\#L(2p,1)$. Hence Conjecture 1.1 also holds for this infinite family of non-Seifert manifolds as well.

In the course of proving Theorem 1.2, we also established the following statement, which we have not been able to find in the literature in this generality:

thm:HakenSFSchar **Theorem 1.3.** *If M is Seifert manifold, then* $\dim X(M)>0$ *if and only if M is Haken or $M=S^2\times S^1$.*

More generally, any 3-manifold M with infinite $X(M)$ contains an essential surface, by [CS83], but the opposite implication is known to be false in general, as there are Haken or non-irreducible 3-manifolds with finite $X(M)$. Besides the connected sums $\mathbb{RP}^3\#L(n,1)$ considered above, examples of Haken manifolds M with finite $X(M)$ were constructed for example in [Mot88].

Combining Theorem 1.2 with an earlier result of the authors [DKS23, Theorem 1.1], leads to the following corollary proved in Section 4:

⟨reduced⟩ **Corollary 1.4.** *For any non-Haken Seifert fibered manifold M, we have*

$$|X(M)|\leq \dim_{\mathbb{Q}(A)}\mathcal{S}(M)\leq \dim_{\mathbb{C}}\mathbb{C}[\mathcal{X}(M)].$$

In particular, if $\mathcal{X}(M)$ is reduced then $\dim_{\mathbb{Q}(A)}\mathcal{S}(M)=|X(M)|$.

Corollary 1.4 motivates the question of when $\mathcal{X}(M)$ is reduced. Non-Haken, irreducible Seifert manifolds fiber over S^2 with at most three exceptional fibers and non-zero Euler number $e(M)$. In Section 5 we study $\mathcal{X}(M)$ of such manifolds in detail, and we compute the number of their irreducible representations $|X^{irr}(M)|$ (Proposition 5.8). We also show that, under mild conditions on the multiplicities of the exceptional fibers, $\mathcal{X}(M)$ is reduced.

We call the integers p_1, p_2, p_3 are weakly coprime if one of them is coprime with the other two.

⟨thm:reducedness⟩ **Theorem 1.5.** *Let M be a Seifert fibered manifold that fibers over S^2 with at most three exceptional fibers of multiplicities $p_1, p_2, p_3 > 0$ and $e(M) \neq 0$. If either $H_1(M, \mathbb{Z})$ is 2-torsion or if p_1, p_2, p_3 are weakly coprime, then $\mathcal{X}(M)$ is reduced and*

$$|X^{irr}(M)| = p_1^+ p_2^+ p_3^+ + p_1^- p_2^- p_3^-,$$

where $p_i^+ = \lceil \frac{p_i}{2} \rceil - 1$ and $p_i^- = \lfloor \frac{p_i}{2} \rfloor$.

If p_1, p_2, p_3 are weakly coprime and at most one is even, our count of irreducible characters in Theorem 1.5 becomes

$$|X^{irr}(M)| = \frac{(p_1 - 1)(p_2 - 1)(p_3 - 1)}{4}.$$

When p_1, p_2, p_3 are pairwise coprime, and furthermore, q_1, q_2, q_3 are chosen so that M is a homology sphere, the Seifert manifold M is the Brieskorn homology sphere $\Sigma(p_1, p_2, p_3)$ and the above formula for $|X^{irr}(M)|$ was previously discovered in [BC06].

Since any rational homology sphere M has $\frac{1}{2}(|H_1(M, \mathbb{Z})| + |H_1(M, \mathbb{Z}/2\mathbb{Z})|)$ abelian $SL(2, \mathbb{C})$-characters (Lemma 5.2), we conclude:

Corollary 1.6. *Under the assumptions of Theorem 1.5,*

$$\dim_{\mathbb{Q}(A)} \mathcal{S}(M) = p_1^+ p_2^+ p_3^+ + p_1^- p_2^- p_3^- + \frac{1}{2}(|H_1(M, \mathbb{Z})| + |H_1(M, \mathbb{Z}/2\mathbb{Z})|).$$

For the case of weakly coprime p_1, p_2, p_3 we also find explicit bases of $\mathbb{C}[\mathcal{X}(M)]$ and of $\mathcal{S}(M, \mathbb{Q}(A))$ (Theorem 5.15).

It is conjectured [GJS23, Section 6.3] that $\dim_{\mathbb{Q}(A)} S(M, \mathbb{Q}(A))$ is equal to the dimension of the zero degree part of the Abouzaid-Manolescu homology $HP^{\bullet}_{\#}(M)$ [AM20]. By [AM20, Theorem 1.4], if $\mathcal{X}(M)$ is finite and reduced, the latter dimension is $|X(M)|$. So Theorem 1.5 and Corollary 1.4 provide new infinite families of 3-manifolds for which the conjecture holds.

1.1. **Outline of contents.**

The paper is organized as follows: In Section 2 we discuss relations between different, related, properties of skein modules. In Section 3 we prove Theorem 1.3. Section 4 is devoted to the proof of Theorem 1.2. First, using our earlier work [DKS23, Theorem 1.1] and Theorem 1.3, we reduce the proof of Theorem 1.2 to showing that the skein module $\mathcal{S}(M, \mathbb{Z}[A^{\pm 1}])$ is finitely generated for any Seifert fibered space over S^2 with at most three exceptional fibers and with a non-zero Euler number. We prove the latter statement by skein-theoretic techniques. In Section 5, we compute $|X(M)|$ for all non-Haken Seifert fibered 3-manifolds and, in particular, we prove Theorem 1.5. In the case where p_1, p_2, p_3 are weakly coprime we also compute a basis of $\mathcal{S}(M)$.

1.2. **Acknowledgement.** During the course of this work the first author was partially supported by the projects "NAQI-34T" (ANR-23-ERCS-0008) and by the project "CLICQ" of the Région Bourgogne Franche Comté. The second author was partially supported by the NSF grants DMS-2004155 and DMS-2304033. The third author was partially supported by the Simons Foundation grant 957582.

2. Relations between different properties of skein modules

⟨s.skeinmodules⟩

In this section we clarify the relation of commonly used properties of 3-manifolds skein modules $\mathcal{S}(M,R)$ for $R=\mathbb{Q}[A^{\pm 1}], \mathbb{Z}[A^{\pm 1}]$, and $\mathbb{Q}(A)$. The properties and the corresponding rings are as follows:

(1) $\mathcal{S}(M,R)$ is free over $R=\mathbb{Z}[A^{\pm 1}]$
(2) $\mathcal{S}(M,R)$ is finitely generated over $R=\mathbb{Z}[A^{\pm 1}]$.
(3) $\mathcal{S}(M,R)$ is tame (over $R=\mathbb{Q}[A^{\pm 1}]$).
(4) $\mathcal{S}(M,R)$ is torsion free over $R=\mathbb{Z}[A^{\pm 1}]$.
(5) $\mathcal{S}(M)$ has no $(A+1)$- nor $(A-1)$-torsion over $R=\mathbb{Z}[A^{\pm 1}]$.
(6) $X(M)$ is finite.
(7) $X(M)$ is finite and $\mathcal{S}(M,R)$ is generated over $R=\mathbb{Z}[A^{\pm 1}]$ by $|X(M)|$ elements.

The next proposition describes the relations between above properties.

prop:implications⟩?

Proposition 2.1. *Let M be a closed 3-manifold. We have the following implications between the properties (1)-(7) above:*

$$\begin{array}{ccccccccc} & & (2) & \Longrightarrow & (3) & & \\ & & \Uparrow & & & & \\ (7) & \Longrightarrow & (1) & \Rightarrow (4) \Rightarrow & (5) & \Longrightarrow & (6) \end{array}$$

Moreover, if $\mathcal{X}(M)$ is reduced, then we also have $(3) \Rightarrow (5)$.

Proof. The implications $(1) \Rightarrow (4) \Rightarrow (5)$ and $(2) \Rightarrow (3)$ are immediate from the definitions.

Suppose that $\mathcal{S}(M,\mathbb{Z}[A^{\pm 1}])$ is free. Then it must have finite rank since by the finiteness theorem [GJS23] the dimension $\dim_{\mathbb{Q}(A)} \mathcal{S}(M,\mathbb{Q}(A))$ is finite. Hence we have the implication $(1) \Rightarrow (2)$.

The implication $(5) \Rightarrow (6)$ is also a consequence of the Finiteness theorem, though less immediate. A proof is given in [BD24].

The implication $(3) \Rightarrow (6)$ follows from the main theorem of [DKS23]. Indeed, if $\mathcal{S}(M)$ is tame, then $\dim_{Q(A)}(\mathcal{S}(M,\mathbb{Q}(A))) \geq |X(M)|$, therefore $X(M)$ is finite.

For reduced $\mathcal{X}(M)$, the implication (3) $\Rightarrow$ (5) is the last part of [DKS23, Theorem 3.1].

Finally, if $\mathcal{S}(M,\mathbb{Z}[A^{\pm 1}])$ is generated by $|X(M)|$ elements, then it is isomorphic to $\mathbb{Z}[A^{\pm1}]^{|X(M)|}/I$ where I is some $\mathbb{Z}[A^{\pm1}]$-submodule of $\mathbb{Z}[A^{\pm1}]^{|X(M)|}$. If I is not the trivial submodule, then we would have that $S(M,\mathbb{Q}(A)) = \left(\mathbb{Z}[A^{\pm1}]^{|X(M)|}/I\right) \underset{\mathbb{Z}[A^{\pm1}]}{\otimes} \mathbb{Q}(A)$ would have dimension $< |X(M)|$ over $\mathbb{Q}(A)$, which contradicts [DKS23, Theorem 3.1]. Hence, we must have $I = 0$, and (7) $\Rightarrow$ (1). $\square$

3. Haken Seifert fibered manifolds

⟨sec:Haken⟩ Recall that all 3-manifolds M and essential surfaces in them are assumed closed and oriented in this paper. A closed, irreducible 3-manifold is Haken if it contains an embedded incompressible surface. The only non-irreducible closed Seifert fibered 3-manifolds are $S^2 \times S^1$ and $\mathbb{RP}^3 \# \mathbb{RP}^3$ [Jac80]. In this section, we will prove Theorem 1.3.

We denote by $M(B; \frac{q_1}{p_1} \dots \frac{q_n}{p_n})$ the Seifert fibered 3-manifold with the fiber space being the 2-orbifold B with exceptional fiber invariants $(q_1, p_1) \dots (q_n, p_n)$ in the notation of [JN83].

⟨Zetner⟩ **Proposition 3.1.** *Suppose that a closed 3-manifold M admits a Seifert fibration over S^2 with at least four exceptional fibers or over $\mathbb{RP}^2$ with at least two exceptional fibers. Then M is Haken and $X(M)$ is infinite.*

We prove Proposition 3.1 by constructing infinitely many non-conjugate $SU(2)$-representations of $\pi_1(M)$. We will say that a matrix $M \in SU(2)$ has angle $\theta \in [0,\pi]$ if it is conjugated to the diagonal matrix with entries $e^{i\theta}$ and $e^{-i\theta}$.

We will separate the proof of the proposition in two cases, according to whether the base is S^2 or $\mathbb{RP}^2$.

We recall that by [JN83, Theorem 6.1], the fundamental group of a Seifert manifold of the form $M = M(S^2; \frac{q_1}{p_1}, \dots, \frac{q_n}{p_n})$ has a presentation

(1) eq:presentation_pi1 $$\pi_1(M) = \langle c_1, \dots, c_n, h \mid [h, c_i] = c_i^{p_i} h^{q_i} = 1 \ \forall i,\ c_1 \dots c_n = 1 \rangle.$$

To treat this case, we will use the following lemma:

⟨lemma:SZ⟩ **Lemma 3.2.** [SZ22, Lemma 2.4] *There exists a representation*

$$\rho : \langle c_1, c_2, c_3 | c_1 c_2 c_3 \rangle \longrightarrow SU(2)$$

that maps each c_i to an element of $SU(2)$ of angle θ_i if and only if

(2) eq:condition $$|\theta_1 - \theta_2| \le \theta_3 \le \min(\theta_1 + \theta_2, 2\pi - \theta_1 - \theta_2).$$

Note that since the conjugacy class of an element of $SU(2)$ is determined by its angle, the lemma can be alternatively stated as follows: given $\theta_1, \theta_2, \theta_3$ that satisfy Equation 2, and $A \in SU(2)$ of angle θ_1, one can find $B \in SU(2)$ of angle θ_2 so that AB has angle θ_3.

We can now prove the following, which will prove the first case of Proposition 3.1:

⟨lemma:S2base⟩ **Lemma 3.3.** *Let $M = M(S^2;\ \frac{q_1}{p_1}, \dots, \frac{q_n}{p_n})$ with $2 \leq p_1 \leq \dots \leq p_n$, and let $k \geq 6$. Then for any angles $\varphi_2, \dots, \varphi_{n-2} \in [\frac{5\pi}{12}, \frac{\pi}{2}]$, there is a representation $\rho : \pi_1(M) \to SU(2)$, such that $\rho(h) = -I$ and for each $2 \leq i \leq n-2$, the element $\rho(c_1c_2 \dots c_i)$ has angle φ_i.*

Proof. Note that the condition $\rho(h) = -I$ implies that the commutation relations in the presentation 1 are automatically satisfied. Moreover, for each i, $\rho(c_i)$ must satisfy $\rho(c_i)^{p_i} = (-1)^{q_i}I$. As remarked in the proof of [SZ22, Proposition 2.8], for $p_i \geq 2$ and (p_i, q_i) coprime, $\rho(c_i)$ satisfing the latter condition can always be chosen of angle $\frac{\pi}{4} \leq \theta_i \leq \frac{2\pi}{3}$.

We will pick the values of $\rho(c_1), \dots, \rho(c_n)$ inductively so that the angles of the $\rho(c_1c_2 \dots c_i)$ are always φ_i. First, we choose $\rho(c_1)$ and $\rho(c_2)$ so that $\rho(c_1c_2)$ has angle φ_2. This is possible since, as we can assume $\frac{\pi}{4} \leq \theta_1, \theta_2 \leq \frac{2\pi}{3}$, the left-hand side of the condition in Lemma 3.2 is at most $\frac{5\pi}{12}$, and the right hand side at least $\frac{\pi}{2}$, and $\varphi_2 \in [\frac{5\pi}{12}, \frac{\pi}{2}]$.

For the inductive step, assuming we have chosen $\rho(c_1), \dots, \rho(c_i)$, let us pick $\rho(c_{i+1})$ so that $\rho(c_1 \dots c_{i+1})$ has angle φ_{i+1}. Note that the left hand-side of 2 is less than $\frac{5\pi}{12}$ again, and the right hand side is more than $\frac{\pi}{2}$, so we can pick $\rho(c_{i+1})$ accordingly, by Lemma 3.2, since $\varphi_{i+1} \in [\frac{5\pi}{12}, \frac{\pi}{2}]$.

Finally, when $\rho(c_1), \dots, \rho(c_{n-2})$ have been chosen, we can pick $\rho(c_{n-1}), \rho(c_n)$ by the same reasoning.

The representation ρ now satisfies all relations in the presentation 1, and furthermore, maps each $c_1 \dots c_i$ for $2 \leq i \leq n-2$ to an element of angle φ_i. $\square$

We note that Lemma 3.3 actually implies that the dimension of $X(M)$ is at least $n-3$ for M a Seifert manifold over S^2 with $n \geq 4$ exceptional fibers. We will however not need this fact, but just that $X(M)$ is infinite. The proof of Proposition 3.1 now reduces to the other case, manifolds which fiber over $\mathbb{RP}^2$ with at least 2 exceptional fibers:

Proof of Proposition 3.1. Thanks to Lemma 3.3, we only need to treat the case of M which fibers over $\mathbb{RP}^2$ with at least 2 exceptional fibers, i.e. $M = M(\mathbb{RP}^2;\ \frac{q_1}{p_1}, \dots, \frac{q_n}{p_n})$ with $n \geq 2$ and $2 \leq p_1 \leq \dots \leq p_n$. By [JN83, Theorem 6.1],

$$\pi_1(M) = \langle c_1, \dots, c_n, h, a \ \mid \ aha^{-1} = h^{-1}, [h, c_i] = c_i^{p_i} h_{q_i} = 1\ \forall i,\ c_1 \dots c_n a^2 = 1 \rangle$$

Recall that $SU(2)$ can be thought as the unit sphere of the quaternions, with elements

$$aI + b\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + c\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

where $a^2+b^2+c^2+d^2 = 1$ and I is the identity matrix. Unitary quaternions with coordinate $a = 0$ along the identity matrix are called purely imaginary. Following [SZ22, Proposition 3.4], we define a representation $\rho : \pi_1(M) \longrightarrow SU(2)$ by setting $\rho(h) = -I$ and

$$\rho(c_k) = \cos\left(\frac{q_k}{p_k}\pi\right) I + \sin\left(\frac{q_k}{p_k}\pi\right) v_k,$$

for $1 \leq k \leq n$ and any purely imaginary unit quaternions $v_1, \dots, v_n$. Then taking $\rho(a)$ to be a square root of $\rho(c_1 \dots c_n)^{-1}$, one gets a representation of $\pi_1(M)$. (Note that square roots always exist in $SU(2)$.)

Now, we notice that

$$\mathrm{Tr}(\rho(c_1c_2)) = 2\cos\left(\frac{q_1}{p_1}\pi\right)\cos\left(\frac{q_2}{p_2}\pi\right) + \sin\left(\frac{q_1}{p_1}\pi\right)\sin\left(\frac{q_2}{p_2}\pi\right)\mathrm{Tr}(v_1v_2).$$

Since $\frac{q_1}{p_1}$ and $\frac{q_2}{p_2}$ are not integers, $\sin\left(\frac{q_1}{p_1}\pi\right)\sin\left(\frac{q_2}{p_2}\pi\right) \neq 0$. Consequently, for different choices of v_1, v_2, the trace $\mathrm{Tr}(\rho(c_1c_2))$ can take any value in the interval $[\alpha - \beta, \alpha + \beta]$, for

$$\alpha := 2\cos\left(\frac{q_1}{p_1}\pi\right)\cos\left(\frac{q_2}{p_2}\pi\right) \text{ and } \beta := 2\left|\sin\left(\frac{q_1}{p_1}\pi\right)\sin\left(\frac{q_2}{p_2}\pi\right)\right|.$$

Since $\mathrm{Tr}(\rho(c_1c_2))$ can take infinitely many values for $SU(2)$-representations ρ of $\pi_1(M)$, its character variety is infinite. □

Next we prove Theorem 1.3 whose statement we recall for the convenience of the reader:

Theorem 1.3. If M is Seifert manifold, then $\dim X(M) > 0$ if and only if M is Haken or $M = S^2 \times S^1$.

Proof of Theorem 1.3. By the work of Culler and Shallen [CS83], if $X(M)$ is infinite, then M contains an essential surfaces and, hence, it is either Haken or reducible. Since $S^2 \times S^1$ and $\mathbb{RP}^3\#\mathbb{RP}^3$ are only reducible Seifert manifolds [Jac80, Lemma VI.7] and $X(\mathbb{RP}^3\#\mathbb{RP}^3)$ is finite, the implication $\Rightarrow$ follows.

Proof of implication $\Leftarrow$: Since $X(S^2 \times S^1)$ is infinite, we can assume that M is Haken. Let (M, π, B) be a Seifert fibered space structure on M where B is the orbifold of the fibration and $\pi : M \longrightarrow B$ is the canonical projection. Recall that B is a closed surface that may be orientable or non-orientable. Since π induces a surjection of $\pi_1(M) \to \pi_1(B)$, if $B \neq S^2, \mathbb{RP}^2$, then $H_1(M)$ projects onto $\mathbb{Z}$, implying an infinite $X(M)$ in this case.

Therefore, it is enough to assume that B is either S^2 or $\mathbb{RP}^2$. In the first case, by Proposition 3.1, it is enough to assume that M has at most three exceptional fibers. Such a manifold is Haken precisely when the fibration has exactly three exceptional fibers and $H_1(M)$ is infinite, see [Jac80, VI.15. Theorem]. Thus $X(M)$ is infinite in this case.

If $B = \mathbb{RP}^2$, then again by Proposition 3.1, it is enough to assume that the fibration has at most one exceptional fiber. Such a 3-manifold M is one of the following [Jac80, page 97].

- $M = \mathbb{RP}^3\#\mathbb{RP}^3$
- M is a lens space
- M is a prism manifold, that is a manifold which fibers over S^2 with exactly three exceptional fibers of multiplicities $p_1 = 2, p_2 = 2$, and $p_3 > 1$.

In all these cases either M is non-Haken or the result follows from the previous case. □

4. Non-Haken Seifert fibered manifolds

⟨sec:non-Haken⟩ The purpose of this section is to prove Theorem 1.2, which we recall here:

Theorem 1.2. For any closed Seifert fibered 3-manifold the following are equivalent:

(a) M contains no 2-sided essential surface (i.e. M is irreducible, non-Haken).
(b) The skein module $\mathcal{S}(M, \mathbb{Z}[A^{\pm 1}])$ is finitely generated.
(c) The skein module $\mathcal{S}(M, \mathbb{Q}[A^{\pm 1}])$ is tame.

Proof. (a) $\Rightarrow$ (b): If M contains no 2-sided essential surface then it fibers over S^2 with at most three exceptional fibers; that is $M = M(S^2; \frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3})$. Furthermore, it has finite $H_1(M)$, which happens precisely when the Euler number,

$$e(M) := \frac{q_1}{p_1} + \frac{q_2}{p_2} + \frac{q_3}{p_3}$$

is non-zero, [EJ73].

Now the statement follows from Theorem 4.1 below.

The implication (b) $\Rightarrow$ (c) by definition of tameness.

(c) $\Rightarrow$ (a): By [DKS23, Theorem 1.1], we have $|X(M)| \leq \dim_{\mathbb{Q}(A)} \mathcal{S}(M)$ and since by [GJS23], the dimension $\dim_{\mathbb{Q}(A)} \mathcal{S}(M)$ is finite, we conclude that $X(M)$ is finite. Now by Theorem 1.3, M has to be non-Haken, implying it is either reducible or it contains an incompressible surface. $\square$

⟨SFSfingen⟩ **Theorem 4.1.** *For any Seifert 3-manifold M that fibers over S^2 with at most three exceptional fibers and with $e(M) \neq 0$, the skein module $\mathcal{S}(M, \mathbb{Z}[A^{\pm 1}])$ is finitely generated.*

The proof of Theorem 4.1 will occupy the remaining of the section.

4.1. The torus skein algebra and the Frohman-Gelca basis.

Let T be a torus with a choice of a basis of $H_1(T)$ consisting of simple closed curves c, h oriented to have a single positive intersection. Let $\mathcal{S}(T, \mathbb{Z}[A^{\pm 1}])$ denote the skein algebra of the torus T with the product given by stacking of skeins. For coprime integers p, q, let $(p, q)_T$ denote the simple closed curve on T of slope p/q; that is a simple closed curve on T representing $pc + qh$ in $H_1(T)$. More generally, for $p, q \in \mathbb{Z}$, non both zero, we set

$$(p, q)_T = T_d((p/d, q/d)) \in \mathcal{S}(T, \mathbb{Z}[A^{\pm 1}]),$$

where $d = gcd(p, q)$ and $T_d(X)$ is the d-th Chebyshev polynomial of the first kind.

We use $(0, 0)_T$ to be the empty multicurve, denoted by $\emptyset$.

Since multicurves on T without contractible components form a basis of $\mathcal{S}(T, \mathbb{Z}[A^{\pm 1}])$, the skeins $(p, q)_T$ for $\pm(p, q) \in \mathbb{Z}^2/_{\{\pm 1\}}$ form a basis of $\mathcal{S}(T, \mathbb{Z}[A^{\pm 1}])$.

We have the product-to-sum formula by [FG00]:

$$(3) \quad \boxed{\texttt{e.prod}} \qquad (p, q)_T \cdot (r, s)_T = A^{ps-qr}(p + r, q + s)_T + A^{qr-ps}(p - r, q - s)_T.$$

4.2. **Realizing Seifert spaces by Dehn fillings on $S_{0,3}\times S^1$.** Let $S_{0,3}$ denote the 3-holed S^2, with the boundary components, c_1, c_2, c_3, with their orientation induced by that of S^2. Fix a base point $b_i \in c_i$, for $i = 1, 2, 3$. Let $N = S_{0,3} \times S^1$ and let $T_i = \{c_i\} \times S^1$ and $h_i = \{b_i\} \times S^1$.

Let

$$\varepsilon_1 := q_1/p_1, \quad \varepsilon_2 := q_2/p_2, \quad \varepsilon_3 := q_3/p_3,$$

and let $M(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ denote the Dehn filling of N along the boundary slopes, $\varepsilon_1, \varepsilon_2, \varepsilon_3$. That is

$$M = M(S^2;\ \varepsilon_1, \varepsilon_2, \varepsilon_3) := N \cup (V_1 \cup V_2 \cup V_3),$$

where V_i is a solid torus attached to T_i by identifying the meridian of ∂V_i with $p_i c_i + q_i h_i$ in $H_1(T_i)$. The S^1-bundle structure of N extents to the Seifert fibration of $M(S^2;\ \varepsilon_1, \varepsilon_2, \varepsilon_3)$ over S^2 with at most three exceptional fibers. The core of V_i is an exceptional fiber iff $p_i > 1$, and p_i is the multiplicity of the exceptional fiber in that case.

From now on, let us denote $M(S^2;\ \varepsilon_1, \varepsilon_2, \varepsilon_3)$ by $M(\ \varepsilon_1, \varepsilon_2, \varepsilon_3)$ for simplicity.

Without loss of generality we will assume that $p_1, p_2, p_3 \geq 1$ since any closed 3-dimensional Seifert fibered space with fiber space S^2 with at most three exceptional fibers (as in Theorem 4.1) is obtained from N in this way. We will also assume that the Euler number $e(M)$ is positive, since the orientation reversal of M negates its Euler number and does not affect the statement of Theorem 4.1.

Note that

$$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = e(M),$$

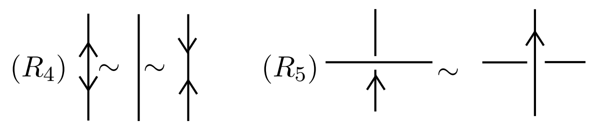
and shifting $\varepsilon_1, \varepsilon_2, \varepsilon_3$ by integers which add up to zero does not change the homeomorphism type of $M(\varepsilon_1, \varepsilon_2, \varepsilon_3)$, cf. [JN83, Theorem 1.5]. Therefore, without loss of generality, we assume that

(4) ?e.ve23? $$\varepsilon_2, \varepsilon_3 \leq 0.$$

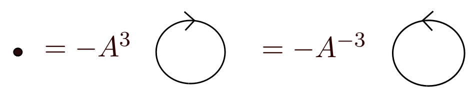
Note that since $e(M) > 0$, this implies that $\varepsilon_1 > 0$, and since all p_i are positive, we obtain

(5) e.q123 $$q_1 > 0 \text{ and } q_2, q_2 \leq 0.$$

4.3. **The skein module $\mathcal{S}(S_{0,3} \times S^1, \mathbb{Z}[A^{\pm 1}])$.** Set $N := S_{0,3} \times S^1$. The skein module $S(N, \mathbb{Z}[A^{\pm 1}])$ was studied in [MD09], where it was shown that it is a free $\mathbb{Z}[A^{\pm 1}]$-module. To prove this, [MD09] developed a diagrammatic presentation of framed links in products $S \times S^1$, over surfaces S. Links are isotoped to be either disjoint from $S \times \{1\}$ or to intersect $S \times \{1\}$ transversally and are studied via their projections (diagrams) on $S \times \{1\}$. In this setting, link diagrams without arrows represent links in a tubular neighborhood of $S \times \{1\}$ in $S \times S^1$, and an arrow marking on an arc C of a diagram indicates that the arc makes a full loop in $S \times S^1$ in the direction of S^1. Changing the direction of an arrow on a diagram amounts to changing the direction in which the link runs along the S^1 direction. They show that to study framed links via their diagrams, besides the three usual Reidemeister moves on "unarrowed" parts of diagrams, one needs the following two moves:



To simplify notation we will represent S^1-fibers by "dots". Note that in the Mroczkowski-Dabkowski notation such dots are equal to



where the two diagrams on the right are arrowed trivial curves on S.

In this paper, we will actually not use the result of [MD09] that $S(N, \mathbb{Z}[A^{\pm 1}])$ is a free over $\mathbb{Z}[A^{\pm 1}]$. Instead, we will be interested in understanding its $S(\partial N, \mathbb{Z}[A^{\pm 1}])$-module structure. Recall that the skein module $\mathcal{S}(N, \mathbb{Z}[A^{\pm 1}])$ has a natural left module structure over the skein algebra $\mathcal{S}(\partial N, \mathbb{Z}[A^{\pm 1}])$ of the boundary, induced by the homeomorphism $N \coprod_{\partial N} \partial N \times [0,1] \simeq N$.

Since the boundary components of $S_{0,3}$ are ordered, the above skein algebra is canonically isomorphic with $\mathcal{S}(T, \mathbb{Z}[A^{\pm 1}]) \otimes \mathcal{S}(T, \mathbb{Z}[A^{\pm 1}]) \otimes \mathcal{S}(T, \mathbb{Z}[A^{\pm 1}])$, which has a basis

$$\{(k_1, l_1)_T \otimes (k_2, l_2)_T \otimes (k_3, l_3)_T : \quad (k_1, l_1), (k_2, l_2), (k_3, l_3) \in \mathbb{Z}^2/\{\pm 1\}\}.$$

We will denote a basis element corresponding to $v = (k_1, l_1, k_2, l_2, k_3, l_3) \in \mathbb{Z}^6$ by $L_v := L_{k_1,l_1,k_2,l_2,k_3,l_3}$. Given v we will denote the skein $L_v \cdot \emptyset \in \mathcal{S}(N, \mathbb{Z}[A^{\pm 1}])$ by $\overline{L}_v = \overline{L}_{k_1,l_1,k_2,l_2,k_3,l_3}$, for simplicity.

Since all links in N can be isotoped into a tubular neighborhood of ∂N, $\mathcal{S}(N, \mathbb{Z}[A^{\pm 1}])$ is generated by the set

$$\{\overline{L}_v \mid v := (k_1, l_1, k_2, l_2, k_3, l_3) \in \mathbb{Z}^6\}.$$

Given a submodule $V \subset \mathbb{Z}^n$ and $w \in \mathbb{Z}^n$, we call the set $w + V$ an affine subspace of $\mathbb{Z}^n$ directed by V.

Lemma 4.2. *Suppose that $\mathcal{S}(N, \mathbb{Z}[A^{\pm 1}]))$ is generated by the elements $\overline{L}_v$, for which v belongs to a finite collection of affine subspaces of $\mathbb{Z}^6$ directed by*

$$V_{p_1,q_1,p_2,q_2,p_3,q_3} = Span_{\mathbb{Z}}((p_1, q_1, 0, 0, 0, 0), (0, 0, p_2, q_2, 0, 0), (0, 0, 0, 0, p_3, q_3)).$$

Then $\mathcal{S}(M(\varepsilon_1, \varepsilon_2, \varepsilon_3), \mathbb{Z}[A^{\pm 1}])$ is finitely generated.

Proof. Since the curve of slope $\varepsilon_i = p_i/q_i$ in T_i bounds a disk in $M(\varepsilon_1, \varepsilon_2, \varepsilon_3)$, we have

$$((k_1, l_1)_T \cdot (p_1, q_1)_T, (k_2, l_2)_T, (k_3, l_3)_T) \cdot \emptyset = (-A^2 - A^{-2})((k_1, l_1)_T, (k_2, l_2)_T, (k_3, l_3)_T) \cdot \emptyset,$$

or, equivalently,

(6) ?ex1? $A^* \cdot \overline{L}_{k_1+p_1,l_1+q_1,k_2,l_2,k_3,l_3} + A^* \cdot \overline{L}_{k_1-p_1,l_1-q_1,k_2,l_2,k_3,l_3} = -(A^2+A^{-2}) \cdot \overline{L}_{k_1,l_1,k_2,l_2,k_3,l_3},$

where A^* denotes an unspecified integral power of A, by (3).

This relation together with the analogous relations corresponding to T_2 and T_3, imply that for any $w \in \mathbb{Z}^6$ the $\mathbb{Z}[A^{\pm 1}]$-submodule of $S(N, \mathbb{Z}[A^{\pm 1}])$ generated by elements of $w + V_{p_1,q_1,p_2,q_2,p_3,q_3}$ is finitely generated. This implies the statement of the proposition. □

For $v := (k_1, l_1, k_2, l_2, k_3, l_3)$, let

$$w_i(v) := q_i k_i - p_i l_i \text{ and } c_i(v) := |w_i(v)|,$$

for $i = 1, 2, 3$. Let

$$c(v) := c_1(v)/p_1 + c_2(v)/p_2 + c_3(v)/p_3$$

and let

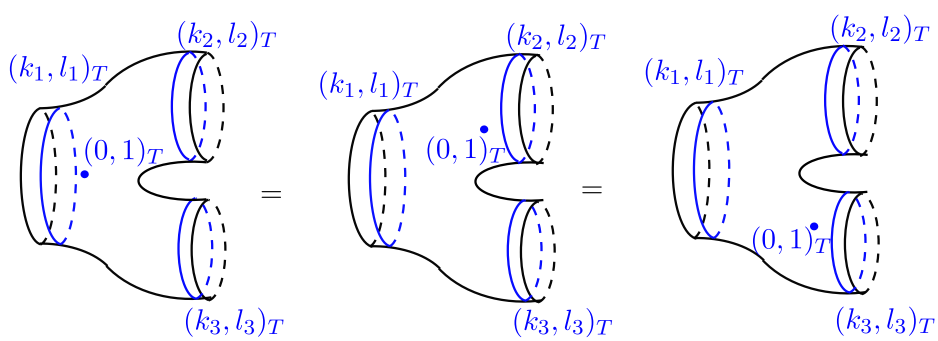
$$C(v) := (c(v), -c_1(v)) \in \mathbb{Z}^2,$$

equipped with lexicographical order. We call $C(v)$ the <u>complexity</u> of v.

⟨p.reduction1⟩ **Proposition 4.3.** *For any $v = (k_1, l_1, k_2, l_2, k_3, l_3)$ such that either $c_2(v) > p_2$ or $c_3(v) > p_3$, the element $\overline{L}_v$ is a linear combination of elements $\overline{L}_{v'}$, with $C(v') < C(v)$.*

Proof. Let us prove first that $\overline{L}_v$, with $c_2(v) > p_2$, can be expressed as $\mathbb{Z}[A^{\pm 1}]$-linear combination of $\overline{L}_{v'}$, with v' of smaller complexity than v. The proof for $\overline{L}_v$, with $w_3(v) > p_3$, is completely analogous.

Since $(-k, -l)_T = (k, l)_T$, we can assume that $w_i(v) \geq 0$, for $i = 1, 2, 3$. In $\mathcal{S}(T, \mathbb{Z}[A^{\pm 1}])$ we have the following relation obtained by moving the S^1 fiber in N so that it lies near each of the three boundary components of $S_{0,3}$:



The blue loops are in $S_{0,3} \times \{1\}$ and the dots represent circle fibers.

By the first equality of this relation and (3), we get

$$A^* \cdot \overline{L}_{k_1,l_1,k_2,l_2-1,k_3,l_3} + A^* \cdot \overline{L}_{k_1,l_1,k_2,l_2+1,k_3,l_3} = A^* \cdot \overline{L}_{k_1,l_1+1,k_2,l_2,k_3,l_3} + A^* \cdot \overline{L}_{k_1 l_1-1,k_2,l_2,k_3,l_3}.$$

Let $v = (k_1, l_1, k_2, l_2, k_3, l_3)$ and apply above relation to $\overline{L}_v$. By moving the second term to the right and shifting l_2 by 1 we obtain

$$\overline{L}_v = A^* \cdot \overline{L}_{k_1,l_1+1,k_2,l_2+1,k_3,l_3} + A^* \cdot \overline{L}_{k_1,l_1-1,k_2,l_2+1,k_3,l_3} - A^* \cdot \overline{L}_{k_1,l_1,k_2,l_2+2,k_3,l_3}.$$

Recall that $p_1, p_2 \geq 1$. Since $w_1, w_2 \geq 0$,

$$c(k_1, l_1 \pm 1, k_2, l_2 + 1, k_3, l_3) \leq c(k_1, l_1, k_2, l_2, k_3, l_3),$$

because going from the right to the left side decreases c_2 by p_2, while it increases c_1 by at most p_1.

Consequently, the c-value,

$$c(v) = \frac{c_1(v)}{p_1} + \frac{c_2(v)}{p_2} + \frac{c_3(v)}{p_3}$$

does not increase. If the above inequality is actually an equality, then by the above reasoning

$$c_1(k_1, l_1 \pm 1, k_2, l_2 + 1, k_3, l_3) > c_1(k_1, l_1, k_2, l_2, k_3, l_3),$$

and, hence,

$$C(k_1, l_1 \pm 1, k_2, l_2 + 1, k_3, l_3) < C(k_1, l_1, k_2, l_2, k_3, l_3).$$

Finally,

$$C(k_1, l_1, k_2, l_2 + 2, k_3, l_3) < C(k_1, l_1, k_2, l_2, k_3, l_3),$$

because $c_2(v) = w_2(v) = k_2q_2 - l_2p_2 > p_2 > 0$ (by the assumption of the proposition) implies $|k_2q_2 - (l_2 + 2)p_2| < |k_2q_2 - l_2p_2|$. □

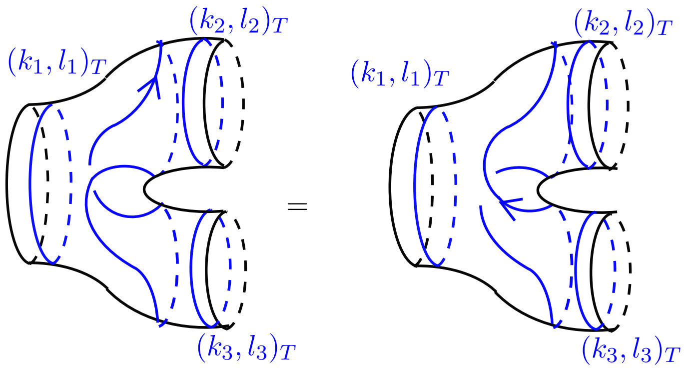
The next ingredient we need for the proof of Theorem 4.1 is the following:

⟨p.reduction2⟩ **Proposition 4.4.** *Assume that $|\varepsilon_1| + 1 > \max(|\varepsilon_2 - 1| + |\varepsilon_3|, |\varepsilon_2| + |\varepsilon_3 + 1|)$. Then for any v with $c_1(v) \geq 2|q_1| + 2p_1$, $\overline{L}_v$ can be expressed as a linear combination of elements $\overline{L}_{v'}$, with $C(v') < C(v)$.*

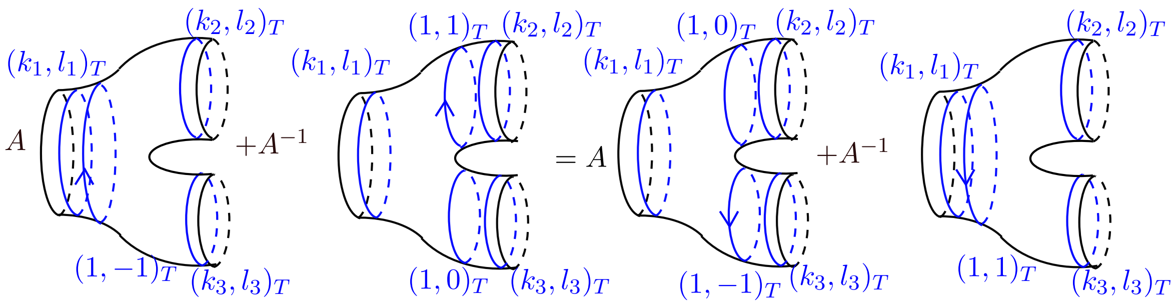
Proof. Recall that without loss of generality, we assumed $q_1 \geq 0$ (see Equation 5). Let $v \in \mathbb{Z}^6$ be such that

(7) $\boxed{\texttt{e.q1p1}}$
$$c_1(v) \geq 2q_1 + 2p_1.$$

As in the proof of Proposition 4.3, we will assume that $w_i(v) \geq 0$ for $i = 1, 2, 3$, without loss of generality. We have the following identity of skein elements in N, using the relation R_5 introduced above:



and, by resolving the crossings, we obtain



By applying the product-to-sum formula (3) we can turn each of the diagrams above into a A^*-weighted sum of $\overline{L}$-terms. In particular, the first diagram on the left equals $A^*\overline{L}_{v_1} + A^*\overline{L}_{v_2}$, where

$$v_1 := (k_1 + 1, l_1 - 1, k_2, l_2, k_3, l_3) \text{ and } v_2 := (k_1 - 1, l_1 + 1, k_2, l_2, k_3, l_3),$$

and the second diagram on the left equals

$$A^*\overline{L}_{v_3} + A^*\overline{L}_{v_4} + A^*\overline{L}_{v_5} + A^*\overline{L}_{v_6},$$

where

$$v_3 := (k_1, l_1, k_2 + 1, l_2 + 1, k_3 + 1, l_3), \quad v_4 := (k_1, l_1, k_2 - 1, l_2 - 1, k_3 + 1, l_3),$$

and

$$v_5 := (k_1, l_1, k_2 + 1, l_2 + 1, k_3 - 1, l_3), \quad v_6 := (k_1, l_1, k_2 - 1, l_2 - 1, k_3 - 1, l_3).$$

The $\overline{L}$-terms on the right are:

$$v_7 := (k_1, l_1, k_2 + 1, l_2, k_3 + 1, l_3 - 1), \quad v_8 := (k_1, l_1, k_2 - 1, l_2, k_3 + 1, l_3 - 1),$$
$$v_9 := (k_1, l_1, k_2 + 1, l_2, k_3 - 1, l_3 + 1), \quad v_{10} := (k_1, l_1, k_2 - 1, l_2, k_3 - 1, l_3 + 1),$$
$$v_{11} := (k_1 + 1, l_1 + 1, k_2, l_2, k_3, l_3), \quad v_{12} := (k_1 - 1, l_1 - 1, k_2, l_2, k_3, l_3).$$

We are going to complete the proof of the proposition by showing that $C(v_i) < C(v_1)$, for $2 \leq i \leq 12$. As in the proof of Proposition 4.3, we assume that $w_1(v_1), w_2(v_1), w_3(v_1) \geq 0$.

Let us analyze vectors v_2, v_{11}, v_{12} since they differ from v_1 at their first two entries only. Note that going from v_1 to v_2 decreases w_1 by at most $2q_1 + 2p_1$ and, consequently, it decreases the value of c_1, by Eq. (7). Since the values of c_2 and c_3 remain unchanged, $C(v_2) < C(v_1)$.

Going from v_1 to v_{11} decreases w_1 by at most $2p_1$ and, hence, $C(v_{11}) < C(v_1)$, as in the argument above. Going from v_1 to v_{12} decreases w_1 by at most $2q_1$ and, hence, $C(v_{11}) < C(v_1)$, as well.

Note that the remaining vectors, v_i, for $i = 3, ..., 10$, have their first two components (k_1, l_1). Hence, going from v_1 to any of them decreases c_1/p_1 by $(p_1 + q_1)/p_1 = 1 + \varepsilon_1$. Therefore, by (7), it is enough to show that going from v_1 to one of these vectors increases $c_2/p_2 + c_3/p_3$ by at most $\max(|\varepsilon_2 - 1| + |\varepsilon_3|, |\varepsilon_2| + |\varepsilon_3 + 1|)$.

Note that going from v_1 to v_i for $i = 3, 4, 5, 6$, increases c_2/p_2 by at most $|q_2 - p_2|/p_2 = |\varepsilon_2 - 1|$ and it increases c_3/p_3 by at most $|q_3|/p_3$. Hence, the above condition holds.

Going v_1 to v_i for $i = 7, 8, 9, 10$, increases c_2/p_2 by at most $|q_2|/p_2 = |\varepsilon_2|$ and it it increases c_3/p_3 by at most $|q_3 + p_3|/p_3 = |\varepsilon_3 + 1|$. Hence, the above condition holds as well. □

Now we complete the proof of Theorem 4.1 as a corollary of Propositions 4.3 and 4.4.

Corollary 4.5. *If $e(M) = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \neq 0$, then $\mathcal{S}(M(\varepsilon_1, \varepsilon_2, \varepsilon_3))$ is a finitely generated $\mathbb{Z}[A^{\pm 1}]$-module.*

Proof. As before, without loss of generality we assume, that $e(M) > 0$, $\varepsilon_1 > 0$ and $\varepsilon_2, \varepsilon_3 < 0$. Let us first verify the condition

$$|\varepsilon_1| + 1 > \max(|\varepsilon_2 - 1| + |\varepsilon_3|, |\varepsilon_2| + |\varepsilon_3 + 1|),$$

of Proposition 4.4. The left side is

$$\varepsilon_1 + 1 = e(M) - \varepsilon_2 - \varepsilon_3 + 1 > 1 - \varepsilon_2 - \varepsilon_3,$$

while the right side is

$$\max(-\varepsilon_2 + 1 - \varepsilon_3, -\varepsilon_2 + |\varepsilon_3 + 1|).$$

Hence, the above inequality follows from the fact that $|\varepsilon_3 + 1|$ is either $-\varepsilon_3 - 1$ or is less than 1. Recall that $\mathcal{S}(M)$ is generated by the elements $\bar{L}_v$ for $v \in \mathbb{Z}^6$. By Propositions 4.3 and 4.4 (which holds by the above inequality), we see that $\mathcal{S}(M)$ is generated by the elements $\bar{L}_v$ satisfying

$$c_1(v) < 2|q_1| + 2p_1, \quad c_2(v) \leq p_2 \text{ and } c_3(v) \leq p_3.$$

There are finitely many of them. □

Combining Theorem 1.2 and an earlier result of the authors [DKS23, Theorem 1.1], leads to Corollary 1.4 which we restate here:

Corollary 1.4 . For any non-Haken Seifert fibered manifold M, we have

$$|X(M)| \leq \dim_{\mathbb{Q}(A)} \mathcal{S}(M, \mathbb{Q}(A)) \leq \dim_{\mathbb{C}} \mathbb{C}[\mathcal{X}(M)].$$

In particular, if $\mathcal{X}(M)$ is reduced then $\dim_{\mathbb{Q}(A)} \mathcal{S}(M) = |X(M)|$.

Proof. It is a direct consequence of Theorem 1.2, combined with an earlier result of the authors [DKS23, Theorem 1.1]. Although Theorem 1.2 implies tameness for irreducible 3-manifolds only, the above statement does hold the reducible Seifert 3-manifolds, $S^2 \times S^1$ and $\mathbb{RP}^3 \# \mathbb{RP}^3$. In the first case, $\dim_{\mathbb{Q}(A)} \mathcal{S}(S^2 \times S^1) = 1$ by [HP95] and in the latter,

$$\dim_{\mathbb{Q}(A)} \mathcal{S}(\mathbb{RP}^3 \# \mathbb{RP}^3) = \dim_{\mathbb{Q}(A)} \mathcal{S}(\mathbb{RP}^3) \cdot \dim_{\mathbb{Q}(A)} \mathcal{S}(\mathbb{RP}^3) = 4,$$

by [HP93] and [Prz00]. □

5. Character varieties of non-Haken Seifert manifolds

⟨sec:characters⟩ In this section we will focus on computing $|X(M)|$, and understanding the extent to which the character scheme $\mathcal{X}(M)$ is reduced, for all non-Haken Seifert fibered 3-manifolds M. As discussed in the beginning of the proof of Theorem 1.2 such a 3-manifold is either $\mathbb{RP}^3 \# \mathbb{RP}^3$ or of the form $M = M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3})$, where M is a rational homology sphere or (equivalently) we have non-zero Euler number (i.e. $\frac{q_1}{p_1} + \frac{q_2}{p_2} + \frac{q_3}{p_3} \neq 0$). We will work with these assumptions throughout the section.

We recall that a presentation of $\pi_1(M)$ is

$$\boxed{\texttt{eq:pres}}\ \pi_1(M) = \langle c_1, c_2, c_3, h \mid [c_i, h] = 1 = c_i^{p_i} h^{q_i} \text{ for } i = 1, 2, 3,\ c_1 c_2 c_3 = 1 \rangle. \tag{8}$$

Let $R(M) := \mathrm{Hom}(\pi_1(M), \mathrm{SL}_2(\mathbb{C}))$. A character χ of M is called abelian if it is the trace of a diagonal representation $\rho \in R(M)$ and central if ρ takes values in the center $\{\pm I\}$ of $SL(2, \mathbb{C})$. A character is called irreducible if it is the trace of an irreducible $\rho \in R(M)$.

5.1. **Abelian characters.** For any $n \times m$ matrix P, where $n \geq m$, let $gcdm(P)$ denote the greatest common divisor of all $m \times m$ minors of P. The following lemma will help in counting of abelian characters of $\pi_1(M)$:

⟨l.gcdm⟩ **Lemma 5.1.** *For any finite abelian group H with an $n \times m$ presentation matrix P*

$$|Hom(H, \mathbb{C}^*)| = |H| = gcdm(P).$$

Proof. By presenting H as a product of cyclic groups $\mathbb{Z}/k_1 \times ... \times \mathbb{Z}/k_\ell$ we see that the number of homomorphisms $H \to \mathbb{C}^*$ is $k_1 \cdot ... \cdot k_\ell$ and, hence, it coincides with $|H|$.

We say that P is diagonal if its only non-zero entries are of the form p_{ii} for some $1 \leq i \leq m$. If the presentation matrix P of H is diagonal then $|H| = gcdm(P)$. Note now that any $n \times m$ matrix P can be brought to a diagonal one (called its Smith form) by multiplying it by an $n \times n$-matrix on the left and an $m \times m$-matrix on the right. Since these operations do not change the isomorphism type of H nor $gcdm(P)$, it follows that $|H| = gcdm(P)$. □

We begin with a lemma that computes the number of abelian characters of M.

⟨lemma:abelianChar⟩ **Lemma 5.2.** *(a) The number of abelian characters of M is $\frac{1}{2}\left(|H_1(M,\mathbb{Z})| + |H_1(M,\mathbb{Z}/2\mathbb{Z})|\right)$.*

(b) We have

$$|H_1(M,\mathbb{Z})| = |p_1p_2q_3 + p_1q_2p_3 + q_1p_2p_3| = p_1p_2p_3|e(M)|,$$

and

$$|H_1(M,\mathbb{Z}/2\mathbb{Z})| = \begin{cases} 4 & \text{if } 2 \text{ divides } p_1,p_2,p_3, \\ 2 & \text{if } 2 \text{ divides } p_1p_2p_3|e(M)| \text{ but not all of } p_1,p_2,p_3, \\ 1 & \text{otherwise.} \end{cases}$$

Proof. (a) The statement is true for any rational homology sphere: Up to conjugation, an abelian representation of $\pi_1(M)$ has the form

$$\rho_\lambda(x) = \begin{pmatrix} \lambda(x) & 0 \\ 0 & \lambda(x)^{-1} \end{pmatrix},$$

where $\lambda \in \mathrm{Hom}(\pi_1(M),\mathbb{C}^*) \simeq H_1(M,\mathbb{Z})$. Two such representations ρ_λ and $\rho_{\lambda'}$ are conjugated if and only if $\lambda = (\lambda')^{\pm 1}$ (as maps $\pi_1(M) \to \mathbb{C}^*$). Since one has $\lambda = \lambda^{-1}$ if and only if λ takes values in $\{\pm 1\}$, that is, $\lambda \in H_1(M,\mathbb{Z}/2\mathbb{Z})$, the statement follows.

(b) Abelianizing the presentation of $\pi_1(M)$ we get the presentation

$$H_1(M,\mathbb{Z}) = \langle h, c_1, c_2 \mid q_ih + p_ic_i = 0, \text{ for } i = 1,2, \ q_3h - p_3c_1 - p_3c_2 = 0\rangle.$$

By Lemma 5.1 the order of $H_1(M,\mathbb{Z})$ is the absolute value of the determinant of the presentation matrix,

$$\begin{pmatrix} q_1 & p_1 & 0 \\ q_2 & 0 & p_2 \\ q_3 & -p_3 & -p_3 \end{pmatrix}$$

which gives the first formula, where the matrix is taken with respect to the ordered set of generators $\{h, c_1, c_2\}$.

For the second formula, note that for each pair (p_i, q_i) at most one of the components is even. So the matrix of above presentation is non-zero mod 2 and the dimension of $H_1(M,\mathbb{Z}/2\mathbb{Z})$ over $\mathbb{Z}/2\mathbb{Z}$ is at most 2. The dimension is non-zero if and only if the determinant of this matrix is 0 mod 2. For the dimension to be 2 one needs all of the 2×2 minors: $p_1q_2,\ q_1p_2,\ p_1p_2,\ -q_1p_3 - p_1q_3,\ -q_1p_3,\ -p_1p_3,\ -q_2p_3,\ -q_2p_3 - p_2q_3,\ p_2p_3$ to vanish in $\mathbb{Z}/2\mathbb{Z}$. The later happens if and only if p_1, p_2 and p_3 are even. □

?⟨def:excepCharac⟩? **Definition 5.3.** An abelian character χ of $M = M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3})$ is <u>exceptional</u> if it is the trace of a representation $\rho \in R(M)$, such that $\rho(h) = \pm I$ and $\rho(c_i) \neq \pm I$ for $i = 1,2,3$.

We will use x_M to denote the number of exceptional abelian characters of M.

prop:nbExcepCharac⟩ **Proposition 5.4.** *For $\{i, j, k\} = \{1, 2, 3\}$, let*

$$m_i := \gcd(m, 4p_j, 4p_k, 2q_ip_j, 2q_ip_k, 2(p_jq_k + q_jp_k)).$$

We have

$$x_M = \frac{1}{2}(m - m_1 - m_2 - m_3) + |H_1(M, \mathbb{Z}/2\mathbb{Z})|,$$

where $m = \gcd(p_1p_2p_3|e(M)|, 2p_1p_2, 2p_1p_3, 2p_2p_3)$.

Proof. First, note that

$$x_M = \frac{1}{2}\left|\{\varphi \in \mathrm{Hom}(\pi_1(M), \mathbb{C}^*) \mid \varphi(h) = \pm 1, \varphi(c_i) \neq \pm 1\}\right|,$$

since any such homomorphism ϕ is different from ϕ^{-1}, and each exceptional character comes from a diagonal representation ρ_ϕ with diagonal (ϕ, ϕ^{-1}), and ρ_ϕ is conjugated to $\rho_{\phi'}$ if and only if $\phi = \phi'^{\pm 1}$.

By inclusion-exclusion principle, the number of such homomorphisms φ is

$$m - m_1 - m_2 - m_3 + 2|H_1(M, \mathbb{Z}/2\mathbb{Z})|,$$

where m is the number of $\varphi \in \mathrm{Hom}(\pi_1(M), \mathbb{C}^*)$ with $\varphi(h) = \pm 1$, and m_i is the number of $\varphi \in \mathrm{Hom}(\pi_1(M), \mathbb{C}^*)$ such that $\varphi(h) = \varphi(c_i) = \pm 1$. (Note that if two of c_1, c_2, c_3 are mapped to ± 1, so is the third one). By Lemma 5.1, $m = |H_1(M)/(2h)|$. Eliminating c_3 from the presentation of $H_1(M)/(2h)$ that with respect to the ordered set of generators $\{h, c_1, c_2\}$ the presentation matrix is

$$P = \begin{pmatrix} q_1 & p_1 & 0 \\ q_2 & 0 & p_2 \\ q_3 & -p_3 & -p_3 \\ 2 & 0 & 0 \end{pmatrix}$$

for which

(9) e.gcdmP $$gcdm(P) = \gcd(p_1p_2p_3|e(M)|, 2p_1p_2, 2p_1p_3, 2p_2p_3).$$

The computations of m_1, m_2, m_3 are similar. For instance, $m_1 = gcdm(P_1)$, where

$$P_1 = \begin{pmatrix} q_1 & p_1 & 0 \\ q_2 & 0 & p_2 \\ q_3 & -p_3 & -p_3 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}.$$

Note that P_1 has $\binom{5}{2} = 10$ of 3×3-minors, four of which appear in Eq. (9). Taking into account the remaining six we obtain

(10) e.m1 $$m_1 = gcdm(P_1) = \gcd(m, 4p_2, 4p_3, 2q_1p_2, 2q_1p_3, 2(q_2p_3 + p_2q_3)).$$

Since the presentation (8) is preserved by the permutations of indices $1, 2, 3$, we obtain formulas for m_2 and m_3 by permuting indices in (10). The formula for x_M follows. $\square$

In the remaining of the subsection we give conditions under which we have $x_M = 0$ (respectively $x_M > 0$). This information is used in the next subsection to study when $\mathcal{X}(M)$ is reduced. Note that the conclusion of Proposition 5.5 (respectively, 5.6) holds for all for all q_1, q_2, q_3 coprime with p_1, p_2, p_3.

⟨p.excepCharac⟩ **Proposition 5.5.** *If $H_1(M, \mathbb{Z})$ is 2-torsion or if p_1, p_2, p_3 are weakly coprime, then $x_M = 0$.*

Proof. If $H_1(M, \mathbb{Z})$ is 2-torsion, then $x_M = 0$ because all abelian representations have values in $\{\pm I\}$ in this case.

To prove the remaining claim, assume, without loss of generality, that p_1 is coprime with p_2, p_3. Then $|H_1(M, \mathbb{Z})| = |p_1p_2q_3 + p_1q_2p_3 + q_1p_2p_3|$ is coprime with p_1, and any $\rho \in R(M)$ such that $\rho(h) = \pm I$, satisfies $\rho(c_1)^{p_1} = \pm I$. But since p_1 is coprime with $|H_1(M, \mathbb{Z})|$ we must have $\rho(c_1) = \pm I$ also. $\square$

On the other hand, we have:

⟨p.excepCharac2⟩ **Proposition 5.6.** *Suppose that, for some $\{i, j, k\} = \{1, 2, 3\}$, $d := gcd(p_i, p_j) > 2$ and $s := gcd(p_ip_j/d, p_k) > 2$. If either $d \neq 4$ or $s \neq 4$, then $x_M > 0$ (for all q_1, q_2, q_3 coprime with p_1, p_2, p_3).*

Proof. Without loss of generality assume that $i = 1, j = 2, k = 3$. Since $d \mid \frac{p_1p_2}{s}$, we have $v_1p_2 + v_2p_1 = \frac{p_1p_2}{s}$, for some $v_1, v_2 \in \mathbb{Z}$ or, equivalently,

(11) e.vp
$$\frac{v_1}{p_1} + \frac{v_2}{p_2} = \frac{1}{s}.$$

We claim we can choose $v_1, v_2 \in \mathbb{Z}$ so that $\frac{v_1}{p_1}, \frac{v_2}{p_2} \notin \frac{1}{2}\mathbb{Z}$. Assuming the claim for a moment, note we have an exceptional representation $\rho \in R(M)$, where $\rho(h) = I$ and $\rho(c_i) = \mathrm{Diag}(e^{2\pi v_i/p_i}, e^{-2\pi v_i/p_i})$, for $i = 1, 2$. Here, we utilize the fact that $s > 2$ and that it divides p_3.

Let us prove the above claim now: Note that integers

$$v_1' = v_1 + k\frac{p_1}{d}, \quad v_2' = v_2 - k\frac{p_2}{d}$$

provide another solution of (11) for every $k \in \mathbb{Z}$. We argue that $\frac{v_1'}{p_1}, \frac{v_2'}{p_2} \notin \frac{1}{2}\mathbb{Z}$, for some $k \in \mathbb{Z}$. Assume, for a contradiction, that this is not the case: Then $\frac{v_i}{p_i} \in \frac{1}{2}\mathbb{Z}$ for at least one $i = 1, 2$, say $\frac{v_1}{p_1} \in \frac{1}{2}\mathbb{Z}$. We have $\frac{v_1 + p_1/d}{p_1} \notin \frac{1}{2}\mathbb{Z}$, since $d > 2$. Consequently, $\frac{v_2 - p_2/d}{p_2} \in \frac{1}{2}\mathbb{Z}$ and, hence, $\frac{v_2 - 2p_2/d}{p_2} \notin \frac{1}{2}\mathbb{Z}$, implying $\frac{v_1 + 2p_1/d}{p_1} \in \frac{1}{2}\mathbb{Z}$. This means that $\frac{1}{d} \in \frac{1}{4}\mathbb{Z}$ implying that $d = 4$. That means that $s \neq 1, 2, 4$ contradicting (11), since $\frac{v_1}{p_1}$ and $\frac{v_2}{p_2} - \frac{1}{4}$ are in $\frac{1}{2}\mathbb{Z}$. $\square$

It may worth pointing out that if at most one of p_1, p_2, p_3 is even, then the assumptions of Propositions 5.5 and 5.6 are complementary. Specifically, we have:

Corollary 5.7. *If at most one of p_1, p_2, p_3 is even then $x_M = 0$ if and only if p_1, p_2, p_3 are weakly coprime.*

Proof. The implication ($\Rightarrow$) is given by Proposition 5.5. To prove the other implication, assume that p_1, p_2, p_3 are not weakly coprime. Without loss of generality, we assume that p_1 is coprime with neither p_2 nor p_3. Since at most one of p_1, p_2, p_3 is even, $d := gcd(p_1, p_2) > 2$. Let $s := gcd(p_1p_2/d, p_3)$. Since d divides p_2, and p_1, p_3 are not coprime, we also have $s > 1$. And since at most one of p_1, p_2, p_3 is even, $s > 2$ and neither d nor s is 4. Hence, $x_M \neq 0$ by Proposition 5.6. $\square$

One may show that when more than one p_i is even then the vanishing of x_M depends on p_1, p_2, p_3 and on the parities of q_1, q_2, q_3 only.

⟨ss.irred⟩ 5.2. **Irreducible characters.** We will denote by $X^{irr}(M)$ the set of irreducible $\mathrm{SL}_2(\mathbb{C})$-characters of M.

⟨prop:irredCharac⟩ **Proposition 5.8.** *We have*

$$|X^{irr}(M)| = p_1^+p_2^+p_3^+ + p_1^-p_2^-p_3^- - x_M,$$

where $p_i^+ = \lceil \frac{p_i}{2} \rceil - 1$ and $p_i^- = \lfloor \frac{p_i}{2} \rfloor$, and x_M is as above.

Proof. Let $X_2(M)$, $X_{-2}(M)$ denote the set of characters $\chi \in X(M)$, where $\chi(c_i) \neq \pm 2$, (for $i = 1, 2, 3$), and $\chi(h) = 2$, $\chi(h) = -2$, respectively. Since h is central in $\pi_1(M)$, for any irreducible representation $\rho \in R(M)$, we have $\rho(h) = \pm I$. Moreover, since $c_i^{p_i}h^{q_i} = 1$, the matrices $\rho(c_i)$ are of finite order. We claim that for an irreducible representation, $\mathrm{Tr}(\rho(c_i)) \neq \pm 2$. Indeed, if $\mathrm{Tr}(\rho(c_i)) = \pm 2$, then the finiteness of the order of $\rho(c_i)$ implies that $\rho(c_i) = \pm I$. Since $\pi_1(M)$ is generated by h, c_1, c_2, such ρ must be abelian and thus reducible.

Consequently,

$$X^{irr}(M) = X_2(M) \cup X_{-2}(M) - X_{ea}(M),$$

where $X_{ae}(M)$ is the set of the exceptional abelian characters, and hence

$$|X^{irr}(M)| = |X_2(M)| + |X_{-2}(M)| - x_M. \tag{12}$$

e.|X^irr|

For $p \in \mathbb{Z}_{>0}$, set

$$C_p^+ := \{\zeta + \zeta^{-1} \mid \zeta^p = 1, \zeta \neq \pm 1\}, \quad \text{and} \quad C_p^- := \{\zeta + \zeta^{-1} \mid \zeta^p = -1, \zeta \neq -1\}.$$

By the above discussion, there are functions

$$T_+ : X_2(M) \to C_{p_1}^+ \times C_{p_2}^+ \times C_{p_3}^+$$
$$T_- : X_{-2}(M) \to C_{p_1}^{\epsilon_1} \times C_{p_2}^{\epsilon_2} \times C_{p_3}^{\epsilon_3},$$

with $T_\pm(\chi) = (\chi(c_1), \chi(c_2), \chi(c_3))$, where $\epsilon_i = +$ if q_i is even and $\epsilon_i = -$ otherwise. Note that $|C^+_{p_i}| = p_i^+ = \lceil \frac{p_i}{2} \rceil - 1$ and $|C^-_p| = p_i^- = \lfloor \frac{p_i}{2} \rfloor$.

We will show that $T_\pm$ are bijections, which implies

$$|X_2(M)| = p_1^+ p_2^+ p_3^+ \text{ and } |X_{-2}(M)| = p_1^- p_2^- p_3^-,$$

and together with Eq. (12) completes the proof.

Each $\chi \in X_2(M)$ is either irreducible or exceptional abelian. In either case, $\rho(h) = I$ and χ is determined by its values on $F_2 = \langle c_1, c_2, c_3 | c_1c_2c_3 = 1\rangle$. Since F_2 is free of rank two, χ is determined by $(\chi(c_1), \chi(c_2), \chi(c_3)) \in \mathbb{C}^3$. Thus T_+ is 1-1. Furthermore, each such value in $\mathbb{C}^3$ determines a character of F_2. In particular, each value in $C^+_{p_1} \times C^+_{p_2} \times C^+_{p_3}$ corresponds to a character of $\langle c_1, c_2, c_3 | c_1^{p_1} = c_2^{p_2} = c_3^{p_3} = c_1c_2c_3 = 1\rangle$. Thus T_+ is onto. The same argument shows that T_- is a bijection.

□

5.3. **Reducedness.** In this subsection we investigate the reduced points of the character varieties $\mathcal{X}(M)$. For $\rho \in R(M)$, let $\mathrm{Ad}\,\rho : \pi_1(M) \to GL(sl_2)$ be the representation of $\pi_1(M)$ on the Lie algebra $sl_2(\mathbb{C})$ induced by ρ by conjugation. We recall that

$$H^1(M, \mathrm{Ad}\,\rho) = Z^1(M, \mathrm{Ad}\,\rho)/B^1(M, \mathrm{Ad}\,\rho),$$

where $Z^1(M, \mathrm{Ad}\,\rho)$ is the set of all maps $\varepsilon : \pi_1(M) \longrightarrow sl_2(\mathbb{C})$ such that:

(13) cocycle $$\varepsilon(xy) = \varepsilon(x) + \mathrm{Ad}\,\rho(x)\varepsilon(y), \quad \text{for all } x, y \in \pi_1(M).$$

and $B^1(M, \mathrm{Ad}\,\rho)$ is the subspace of consisting of maps

$$\varepsilon_A(x) = A - \mathrm{Ad}(\rho(x)) \cdot A \text{ for } x \in \pi_1(M),$$

for all $A \in sl_2(\mathbb{C})$.

Note that the cocycle condition (13) implies that

(14) e.cocycle-n $$\varepsilon(x^n) = (I + \mathrm{Ad}\,\rho(x) + \ldots + \mathrm{Ad}\,\rho(x^{n-1}))\varepsilon(x)$$

for any $x \in \pi_1(M)$ and for any $n > 0$ by induction on n. For the identity element $1 \in \pi_1(M)$, since $\varepsilon(1^2) = 2\varepsilon(1)$, we get

(15) e.cocycle-e $$\varepsilon(1) = 0.$$

Applying ε to $x^{-1} \cdot x = 1$ implies

(16) e.cocycle-inv $$\varepsilon(x^{-1}) = -\mathrm{Ad}\,\rho(x^{-1})\varepsilon(x).$$

The rest of the subsection is devoted to proving the following the following:

⟨thm:reduced⟩ **Theorem 5.9.** *If a Seifert manifold $M = M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3})$ with $e(M) \neq 0$ has no exceptional abelian characters, then $\mathcal{X}(M)$ is reduced.*

We have the following:

Corollary 5.10. *If M is a non-Haken Seifert manifold and either $H_1(M,\mathbb{Z})$ is 2-torsion or p_1, p_2, p_3 are weakly coprime, then $\mathcal{X}(M)$ is reduced.*

Proof. As recalled in the beginning of the section, M is either $\mathbb{RP}^3\#\mathbb{RP}^3$ or of the form $M = M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3})$, with $e(M) \neq 0$. Now the statement follows from Theorem 5.9 and Proposition 5.5. □

It is known that each point of $X(M)$ is represented by a completely reducible representation ρ [LM85, Sik12]. In particular, ρ can be taken to be either irreducible or diagonal. Furthermore, the tangent space $T_\rho \mathcal{X}(M)$ at an irreducible ρ is isomorphic to $H^1(M, \mathrm{Ad}\,\rho)$, see [LM85]. Similarly, $H^1(M, \mathrm{Ad}\,\rho) = 0$, for a diagonal representation ρ implies that $[\rho]$ is reduced in $\mathcal{X}(M)$, by [Sik12, Theorem 1] and [HP23, Lemma 21]. Therefore, Theorem 5.9 follows from Lemmas 5.11, 5.12, 5.13 and 5.14 below.

mma:centralReduced⟩ **Lemma 5.11.** *Let M be a rational homology sphere, and let χ be the character of a central representation in $R(M)$. Then χ is isolated in $X(M)$ and $\mathcal{X}(M)$ is reduced at χ.*

Proof. Since ρ is a central representation, $\mathrm{Ad}\,\rho$ is a trivial representation, and $H^1(M, \mathrm{Ad}\,\rho)$ is isomorphic to $H^1(M, \mathbb{C}) \otimes sl_2(\mathbb{C}) = 0$ since M is a rational homology sphere. □

Next we consider irreducible characters in $\mathcal{X}(M)$, for $M = M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3})$.

lemma:irredReduced⟩ **Lemma 5.12.** *The irreducible characters are isolated and reduced in $\mathcal{X}(M)$.*

Proof. The proof is similar to the argument in [BC06, Lemma 2.4]: Since ρ is irreducible, $B^1(M, \rho)$ is of dimension 3, so we only need to show that $Z^1(M, \mathrm{Ad}\,\rho) = 3$. We use again the presentation of $\pi_1(M)$ given in Equation 8. Since h, c_1, c_2, c_3 generate $\pi_1(M)$, the cocycle condition implies that any $\varepsilon \in Z^1(M, \mathrm{Ad}\,\rho)$ is determined by $H := \varepsilon(h)$ and $X_i := \varepsilon(c_i)$, for $i = 1, 2, 3$. Furthermore, by (16) we obtain $\varepsilon(c_i^{-1}) = -\mathrm{Ad}\,\rho(c_i^{-1})\varepsilon(c_i)$. Hence applying ε to the relations $[c_i, h] = 1$, and utilizing properties (13), (15), (16), we obtain

(17) commute $$X_i + \mathrm{Ad}\,\rho(c_i)H - \mathrm{Ad}\,\rho(c_i h c_i^{-1})X_i - \mathrm{Ad}\,\rho(c_i h c_i^{-1} h^{-1})H = 0.$$

Since ρ is irreducible and $\rho(h)$ commutes with $\rho(\pi_1(M))$, we have $\rho(h) = \pm I$ and the last equation reduces to

$$\mathrm{Ad}\,\rho(c_i)H - H = 0,$$

implying that H commutes with $\rho(c_i)$ for $i = 1, 2, 3$. Furthermore, since ρ is irreducible, H must be a scalar in sl_2 and, hence,

(18) e.H0 $$H = 0.$$

By applying ε to the relation $c_i^{p_i} h^{q_i} = 1$, we obtain

$$\varepsilon(c_i^{p_i}) + \mathrm{Ad}\,\rho(c_i^{p_i})\varepsilon(h^{q_i}) = 0,$$

which by (14) implies

$$\left(I + \operatorname{Ad}\rho(c_i) + \ldots + \operatorname{Ad}\rho(c_i^{p_i-1})\right) X_i = -\operatorname{Ad}\rho(c_i^{p_i})\left(I + \ldots + \operatorname{Ad}\rho(h^{q_i-1})\right) H, \tag{19}$$

and by (18),

(20) powers2
$$\left(I + \operatorname{Ad}\rho(c_i) + \ldots + \operatorname{Ad}\rho(c_i^{p_i-1})\right) X_i = 0. \tag{20}$$

Since $\rho(c_i)$ is of finite order for each i, it is diagonalizable with eigenvalues ζ_i, ζ_i^{-1} such that $\zeta_i^{p_i} = (\pm 1)^{q_i}$. We note that $\zeta_i \neq \pm 1$ for each i. Indeed, if say $\zeta_1 = \pm 1$, then the image of ρ is contained in the set of powers of $\pm\rho(c_2)$ implying that ρ is abelian. In a basis of a diagonalization of $\rho(c_i)$, $\operatorname{Ad}\rho(c_i)$ sends $X_i = \begin{pmatrix} x_i & y_i \\ z_i & -x_i \end{pmatrix} \in sl_2(\mathbb{C})$ to $\begin{pmatrix} x_i & \zeta_i^2 y_i \\ \zeta_i^{-2} z_i & -x_i \end{pmatrix}$. Since $\zeta_i^{2p_i} = 1$, we get $1 + \zeta_i^2 + \ldots + \zeta_i^{2p_i-2} = 0$, which implies that that $\left(I + \operatorname{Ad}\rho(c_i) + \ldots + \operatorname{Ad}\rho(c_i^{p_i-1})\right) X_i$ is diagonal. Therefore, Equation (20) is satisfied if and only if X_i has diagonal zero, which is equivalent to $\operatorname{Tr}(\rho(c_i)X_i) = 0$.

Finally, applying ε to the last relation $c_1c_2c_3 = 1$, we obtain

(21) last
$$X_1 + \operatorname{Ad}\rho(c_1)(X_2) + \operatorname{Ad}\rho(c_1c_2)(X_3) = 0. \tag{21}$$

We can now describe the space of solutions to the last equation. Given $X_2, X_3 \in sl_2(\mathbb{C})$ such that

(22) e.cX
$$\operatorname{Tr}(\rho(c_2)X_2) = 0 = \operatorname{Tr}(\rho(c_3)X_3), \tag{22}$$

we have

$$X_1 = -\operatorname{Ad}\rho(c_1)(X_2) - \operatorname{Ad}\rho(c_1c_2)(X_3).$$

Note that since $\rho(c_2), \rho(c_3) \neq \pm I$, the space of solutions (X_2, X_3) of Eq. (22) is 4-dimensional. However, we claim that the space $Z^1(M, \operatorname{Ad}\rho)$ has dimension at most 3 implying $H^1(M, \operatorname{Ad}\rho) = 0$ as desired.

To show the last claim it is enough to prove that the condition $\operatorname{Tr}(\rho(c_1)X_1) = 0$ does not hold for all solutions $(X_2, X_3) \in sl_2(\mathbb{C})^2$ of Eq. (22). Suppose otherwise, for a moment, and take $X_3 = 0$. Then, $X_1 = -\operatorname{Ad}\rho(c_1)(X_2)$ and $\operatorname{Tr}(\rho(c_1)X_1) = 0$ becomes $\operatorname{Tr}(\rho(c_1)X_2) = 0$. Since the map $M_2(\mathbb{C}) \to M_2(\mathbb{C})^*$ sending A to $(B \to Tr(AB))$ is injective, the above implies that $\rho(c_1)$ is a linear combination of $\rho(c_2)$ and the identity matrix. Consequently, $\rho(c_1)$ and $\rho(c_2)$ commute and, hence, $\rho(c_3) = \rho(c_2^{-1}c_1^{-1})$ commutes with them as well. Since $\rho(h)$ commutes with $\rho(c_i)$'s, this would imply that ρ is abelian, and, hence, not irreducible. This contradiction finishes the proof of the claim and the lemma. □

Next we will study reducedness of abelian non-central characters. We first look at characters such that $\chi(h) \neq \pm 2$.

ma:abelianReduced1⟩ **Lemma 5.13.** *If $\rho \in R(M)$ is diagonal with $\rho(h) \neq \pm I$, then $H^1(M, \operatorname{Ad}\rho) = 0$ and χ_ρ is reduced in $\mathcal{X}(M)$.*

Proof. We follow the strategy used in Lemma 5.12: This time, since ρ is abelian non-central, $\dim B^1(M, \operatorname{Ad}\rho) = 2$. Let us compute the dimension of $Z^1(M, \operatorname{Ad}\rho)$:

As before, an element $\varepsilon \in Z^1(M, \operatorname{Ad}\rho)$ is determined by $H = \varepsilon(h)$ and $X_i = \varepsilon(c_i)$ for $i = 1, 2, 3$. Let

$$\rho(h) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \text{ and } \rho(c_i) = \begin{pmatrix} \mu_i & 0 \\ 0 & \mu_i^{-1} \end{pmatrix}, \text{ for } i = 1, 2, 3,$$

where $\lambda \neq \pm 1$, and furthermore

$$H = \begin{pmatrix} u & v \\ w & -u \end{pmatrix}, \text{ and } X_i = \begin{pmatrix} x_i & y_i \\ z_i & -x_i \end{pmatrix}.$$

Applying ε to the equations $[c_i, h] = 1$ we obtain (17) which now reduces to

$$X_i - \operatorname{Ad}\rho(h)X_i = H - \operatorname{Ad}\rho(c_i)H.$$

Note that the above matrices have zeros on their diagonals. By comparing the off-diagonal entries, we obtain

$$y_i = \frac{\mu_i^2 - 1}{\lambda^2 - 1} v, \;\; z_i = \frac{\mu_i^{-2} - 1}{\lambda^{-2} - 1} w.$$

Next applying ε to the equations $c_i^{p_i} h^{q_i} = 1$ we are led again to (19), which for the diagonal entires reduce to

(23) e.xpq $$x_i = -\frac{q_i}{p_i} u.$$

Finally, applying ε to $c_1 c_2 c_3 = 1$ once again gives Equation (21) and looking at the diagonal entries, we get $x_1 + x_2 + x_3 = 0$, which becomes

(24) e.sumx $$-\left(\frac{q_1}{p_1} + \frac{q_2}{p_2} + \frac{q_3}{p_3}\right) u = -e(M)u = 0.$$

Since we assumed that $e(M) \neq 0$, the last equation implies that u vanishes. Therefore, ε is entirely determined by v, w, and $\dim Z^1(M, \operatorname{Ad}\rho) = 2$, implying $\dim H^1(M, \operatorname{Ad}\rho) = 0$. $\square$

Finally, the last lemma of the subsection treats the case of diagonal characters with $\chi(h) = \pm 2$.

epCharacNonreduced⟩ **Lemma 5.14.** *Let $\rho \in R(M)$ be diagonal with $\rho(h) = \pm I$. Then $H^1(M, \operatorname{Ad}\rho) = 0$, if ρ is non-exceptional, and $\dim H^1(M, \operatorname{Ad}\rho) = 2$ otherwise.*

Proof. If ρ maps two of c_1, c_2, c_3 to $\pm I$ then so it does the third one and ρ is central. Hence, ρ is non-exceptional and the statement follows from Lemma 5.11. Therefore, we assume without loss of generality, that either $\rho(c_1) = \pm I$ and $\rho(c_2) \neq \pm I$, or that ρ is exceptional.

Similarly to the proof of Lemma 5.13, we have $\dim B^1(M, \operatorname{Ad}\rho) = 2$ and we compute $\dim Z^1(M, \operatorname{Ad}\rho)$. As before, $\rho(c_i) = \operatorname{Diag}(\mu_i, \mu_i^{-1})$, for $i = 1, 2, 3$. We also keep the notations for the entries of H, X_1, X_2, X_3. As in the proof of Lemma 5.12, applying ε to relation $[c_2, h] = 1$ gives $\operatorname{Ad}\rho(c_2)(H) - H = 0$. Hence H commutes with $\rho(c_2)$ and thus it is diagonal, i.e. $v = w = 0$. (Now the 1-cocycle condition is satisfied for $[c_i, h] = 1$, $i = 1$ and 3, as well.)

As in the proof of Lemma 5.13, the relations $c_i^{p_i} h^{q_i} = 1$, by looking at diagonals imply Eq. (23) again:

$$x_i = -\frac{q_i}{p_i} u \text{ for } i = 1, 2, 3.$$

If $\rho(c_i) \neq \pm I$ then $\rho(c_i^{p_i} h^{q_i}) = I$ is automatically satisfied on off-diagonal entries. However, if $\rho(c_1) = \pm I$ then (19), and the diagonality of H implies $y_1 = z_1 = 0$. Hence, the 1-cocycle conditions for the relations $[c_i, h] = 1$ and $c_i^{p_i} h^{q_i} = 1$, define a 7-dimensional space of cocycles (determined by parameters: u, y_i, z_i for $i = 1, 2, 3$) for exceptional ρ's and a 5-dimensional space of cocycles (determined by u, y_i, z_i for $i = 2, 3$) for non-exceptional ρ's.

From the last relation $c_1 c_2 c_3 = 1$, we get Eq. (24) again and, hence, $u = 0$. Consequently x_1, x_2, x_3 vanish as well and by looking at the off-diagonal entires of (21), we have

$$y_1 + \mu_1^2 y_2 + \mu_1^2 \mu_2^2 y_3 = 0 = z_1 + \mu_1^2 z_2 + \mu_1^2 \mu_2^2 z_3.$$

If $\rho(c_1) = \pm I$, then $X_1 = 0$ by the above discussion and the last equations reduce to

$$y_2 + \mu_2^2 y_3 = 0 = z_2 + \mu_2^{-2} z_3.$$

In either case, the above parameters are related by three linearly independent equations stemming from of the relation $c_1 c_2 c_3 = 1$. Now $Z^1(M, \operatorname{Ad}\rho)$ is given by all linear maps $\varepsilon : \pi_1(M) \to sl_2(\mathbb{C})$ satisfying the 1-cocycle conditions corresponding to the defining relations of $\pi_1(M)$ (since the 1-cocycle conditions corresponding to the products of the defining relations are linear combinations of those). Consequently, $\dim Z^1(M, \operatorname{Ad}\rho)$ is either 4 or 2-dimensional, depending on whether ρ is exceptional or not. Since $\dim B^1(M, \operatorname{Ad}\rho) = 2$, the statement follows. □

?⟨ss.basis⟩?

5.4. **Bases for $\mathbb{C}[\mathcal{X}(M)]$ and $\mathcal{S}(M, \mathbb{Q}(A))$.** Let $S_{-1}(M) := S(M, \mathbb{Z}[A^{\pm 1}]) \otimes_{\mathbb{Z}[A^{\pm 1}]} \mathbb{C}$, where the $\mathbb{Z}[A^{\pm 1}]$-module structure of $\mathbb{C}$ is given by sending A to -1. By Przytycki-Sikora [PS00], $S_{-1}(M)$ has the structure of a $\mathbb{C}$-algebra that is isomorphic to the coordinate ring $\mathbb{C}[\mathcal{X}(M)]$ of $\mathcal{X}(M)$, through the isomorphism $\psi : S_{-1}(M) \to \mathbb{C}[\mathcal{X}(M)]$, sending any framed link $L = L_1 \cup \ldots \cup L_k$ in M to $(-1)^k t_L$ where $t_L = t_{L_1} \cdot \ldots \cdot t_{L_k}$. For another approach, up to nilpotents, see [Bul97]. Here, t_{L_i} is the trace function of L_i with its framing ignored. As we explained in [DKS23, Proposition 3.3], if $S(M, \mathbb{Q}[A^{\pm 1}])$ is tame and $\mathcal{X}(M)$ is reduced, a basis of $S_{-1}(M) \simeq \mathbb{C}[\mathcal{X}(M)]$ leads to a basis of $\mathcal{S}(M) = \mathcal{S}(M, \mathbb{Q}(A))$.

In this subsection we compute a basis of $\mathbb{C}[\mathcal{X}(M)]$ for $M = M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3})$. We will assume that p_1, p_2, p_3 are weakly coprime, since then Proposition 5.5 implies that there are no exceptional abelian characters and by Theorem 5.9, the character scheme $\mathcal{X}(M)$ is reduced. We will write y_M for the number of abelian characters of M, which was computed in Lemma 5.2. With the notation, $p_i^+ := \lceil \frac{p_i}{2} \rceil - 1$ and $p_i- = \lfloor \frac{p_i}{2} \rfloor$, of Subsection 5.2 we have:

⟨t:basis⟩ **Theorem 5.15.** *Assume that p_1, p_2, p_3 are weakly coprime, and let $\delta_M \in \{0,1\}$ denote the parity of y_M. Then the following set is a basis of $\mathbb{C}[\mathcal{X}(M)]$:*

$$\mathcal{B} = \{(t_h+2)t_{c_1}^{k_1}t_{c_2}^{k_2}t_{c_3}^{k_3} \mid 0 \le k_i < p_i^+\} \cup \{(t_h-2)t_{c_1}^{k_1}t_{c_2}^{k_2}t_{c_3}^{k_3} \mid 0 \le k_i < p_i^-\} \\ \cup \{t_h^i \mid 2 \le i < y_M + \delta_M\} \cup \mathcal{B}_0,$$

where

$$\mathcal{B}_0 = \{(t_h+2)t_{c_1}^{p_1^+}\},\ \textit{for } y_M \textit{ odd and } \mathcal{B}_0 = \{(t_h+2)t_{c_1}^{p_1^+}, (t_h-2)t_{c_1}^{p_1^-}\},\ \textit{for } y_M \textit{ even.}$$

Moreover, any collection of $\mathbb{Z}$-linear combinations of links in M that represent those functions in $\mathbb{C}[\mathcal{X}(M)] \simeq S_{-1}(M)$ is a basis of $S(M, \mathbb{Q}(A))$.

?⟨rk:betterbasis⟩? **Remark 5.16.** If we assume furthermore that p_1, p_2, p_3 are odd, then $p_i^+ = p_i^-$, for $i = 1,2,3$, and for y_M even one can replace the previous basis by

$$\mathcal{B}' = \{t_h^i t_{c_1}^{k_1}t_{c_2}^{k_2}t_{c_3}^{k_3} \mid 0 \le k_1 \le p_1^+, 0 \le k_2 < p_2^+, 0 \le k_3 < p_3^+, 0 \le i \le 1\} \cup \{t_h^i \mid 2 \le i < y_M\},$$

as it can be easily seen that they span the same space. The latter basis consists only of trace functions of links, rather than linear combinations of such functions.

For the proof of 5.15 we need the two preparatory lemmas below:

lemma:generatorH_1⟩ **Lemma 5.17.** *If p_1, p_2, p_3 are weakly coprime, then h generates $H_1(M, \mathbb{Z})$.*

Proof. It is enough to show that $H_1(M,\mathbb{Z})/\langle h \rangle = 0$. Using the ordered set of generators $\{h, c_1, c_2\}$, this group has presentation matrix Q that is identical to the matrix P in the proof of Lemma 5.4 except that the first entry of the 4th row is 1 instead of 2. We get

$$|H_1(M,\mathbb{Z})/\langle h \rangle| = gcdm(Q) = \gcd(p_1p_2q_3 + p_1p_3q_2 + p_2p_3q_1, p_1p_2, p_1p_3, p_2p_3).$$

Without loss of generality assume that p_1 is coprime with p_2 and with p_3. Then $gcd(p_1p_2, p_1p_3) = p_1$ and, hence, $gcd(p_1p_2, p_1p_3, p_2p_3) = 1$ implying that $H_1(M,\mathbb{Z})/\langle h \rangle$ is trivial. □

The second lemma we need is the following:

⟨lemma:polynomial⟩? **Lemma 5.18.** *Let $\mathbb{K}$ be a field of characteristic zero and let $P \in \mathbb{K}[X_1, \dots, X_n]$ be a polynomial of degree $\le d_i$ in the variable X_i, for $i = 1, ..., n$. If P vanishes on a subset of $\mathbb{K}^n$ of the form $S_1 \times \ldots \times S_n$, where $|S_i| = d_i + 1$, then $P = 0$.*

Proof. We prove the lemma by induction on n. The case $n = 1$ is classical. Assume the lemma is true for polynomials in n variables, and let $P \in \mathbb{K}[X_1, \dots, X_{n+1}]$ satisfy the hypothesis of the lemma. For each $z \in S_{n+1}$, the polynomial $P(X_1, \dots, X_n, z)$ has degree $\le d_i$ in each variable X_i, hence it is the zero polynomial. This implies that P considered in $\mathbb{K}(X_1, ..., X_n)[X_{n+1}]$ has at least d_{n+1} roots, implying that $P = 0$ by the classical case. □

Proof of Theorem 5.15. The last claim of the theorem follows from the first part by [DKS23, Proposition 3.3(b)]. Let us prove the first part.

First we note that $\mathcal{B}$ has the right cardinality by Proposition 5.8. So we only need to show that those trace functions are linearly independent. Since $t_h - 2, t_h + 2$ belong to first and second subset respectively, one can equivalently replace the third subset by

$$\{(t_h^2 - 4)t_h^i \mid 0 \leq i < y_M - 2 + \delta_M\},$$

without affecting the linear independence. We work with the latter version of the third subset.

Consider a linear combination F

$$F = \sum_{k_1,k_2,k_3} \lambda_{k_1,k_2,k_3}(t_h + 2)t_{c_1}^{k_1}t_{c_2}^{k_2}t_{c_3}^{k_3} + \sum_{k_1,k_2,k_3} \mu_{k_1,k_2,k_3}(t_h - 2)t_{c_1}^{k_1}t_{c_2}^{k_2}t_{c_3}^{k_3} + \sum_j \nu_j(t_h^2 - 4)t_h^j + a(t_h + 2)c_1^{p_1^+} + b(t_h - 2)c_1^{p_1^-},$$

for some coefficients, $\lambda_{k_1,k_2,k_3}, \mu_{k_1,k_2,k_3}, \nu_j, a, b \in \mathbb{C}$, (with $b = 0$ if y_M is odd) and assume that it is zero in $\mathbb{C}[\mathcal{X}(M)]$, i.e. it vanishes on $X(M)$.

Restricting F to the subspace

$$X_{2,\tau}(M) = \{\chi \in X(M) \mid \chi(h) = 2, \chi(c_1) = \tau\}$$

of $X(M)$ we get:

$$a\tau^{p_1^+} + \sum_{0 \leq k_i < p_i^+} \lambda_{k_1,k_2,k_3}\tau^{k_1}t_{c_2}^{k_2}t_{c_3}^{k_3} = 0 \text{ on } X_{+,\tau}(M), \tag{25}$$

e.X_+

since the other components of F vanish on $X_{2,\tau}(M)$. Now, recall from the proof of Proposition 5.8 that $X_2(M)$ contains characters taking all possible values $(\chi(c_1), \chi(c_2), \chi(c_3) \in C_{p_1}^+ \times C_{p_2}^+ \times C_{p_3}^+$. In other words, characters in $X_{+,\tau}(M)$ take all possible values $(\chi(c_2), \chi(c_3))$ in $C_{p_2}^+ \times C_{p_3}^+$ for every $\tau \in C_{p_1}^+$. Since for each $\tau \in C_{p_1}^+$, (25) is a polynomial of degree $< p_i^+ = |C_{p_i}^+|$, for $i = 2, 3$, which vanishes on $C_{p_2}^+ \times C_{p_3}^+$,

$$a\tau^{p_1^+} + \sum_{0 \leq k_1 < p_1^+} \lambda_{k_1,k_2,k_3}\tau^{k_1}t_{c_2}^{k_2}t_{c_3}^{k_3} = 0 \text{ on } X_{+,\tau}(M),$$

for every $\tau \in C_{p_1}^+, k_2, k_3$. However, since the above identity also holds for the trivial character and since the above expression is a polynomial in τ of degree $p_1^+ < |C_{p_1}^+| + 1 = |C_{p_1}^+ \cup \{2\}|$, we have $a = \lambda_{k_1,k_2,k_3} = 0$ for all k_1, k_2, k_3.

Similarly, if y_M is odd, restricting F to $X_{-2,\tau}(M) = \{\chi \in X(M) \mid \chi(h) = -2, \chi(c_1) = \tau\}$ we get $\mu_{k_1,k_2,k_3} = 0$ for any $0 \leq k_i < p_i^-$, and if y_M is even, we also get that $b = 0$, since in the previous argument we can use instead of the trivial character the abelian character such that $\chi(h) = -2$. In particular, F vanishes for $y_M = 1$.

For $y_M > 1$,

$$F = \sum_{0 \le j < y_M - 2 + \delta_M} \nu_j (t_h^2 - 4) t_h^j.$$

Since F vanishes for $y_M = 2$ as well, assume $y_M > 2$ now. Then $G := \frac{F}{t_h^2 - 4}$ is a polynomial in t_h of degree $y_M - 3 + \delta_M$ that vanishes on all of the abelian characters of $H_1(M)$ for which we have $\chi(h) \neq \pm 2$. By Lemma 5.17, $H_1(M)$ is generated by h and, hence, t_h takes y_M distinct values on abelian characters, including $y_M - 2 + \delta_M$ values which are not ± 2. This implies that all coefficients of G vanish, i.e. $\nu_j = 0$ for every j.

□

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