KNOT ADJACENCY AND SATELLITES

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Abstract. A knot K is called n-adjacent to the unknot, if K admits a projection containing n *generalized crossings* such that changing any $0 < m \leq n$ of them yields a projection of the unknot. We show that a non-trivial satellite knot K is n-adjacent to the unknot, for some $n > 0$, if and only if it is n-adjacent to the unknot in any companion solid torus. In particular, every *model knot* of K is n-adjacent to the unknot. Along the way of proving these results, we also show that 2-bridge knots of the form $K_{p/q}$, where $p/q = [2q_1, 2q_2]$ for some $q_1, q_2 \in \mathbf{Z}$, are precisely those knots that have genus one and are 2-adjacent to the unknot.

Keywords: companion torus, model knot, n-adjacent to the unknot, n-trivializer.
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1. Introduction

The development of the theory of finite type knot invariants has led to the notion of *n-triviality* which is a multiplex unknotting operation. This notion was introduced independently by Gussarov and Ohyama ([G], [O]). Roughly speaking, a knot is n-trivial if it can be unknotted in $2^n - 1$ different ways by *multiple crossing changes*. The research in this paper is motivated by the following question: If a non-trivial satellite knot K is n-trivial is there a companion torus of K that is disjoint from all the crossing changes that exhibit K as n-trivial? In this paper, we are concerned with a stronger version of n-triviality where each set of multiple crossing changes is taken to be a set of *twist crossings* on two strings of the knot. In this case, using results of Lackenby([La]) and Scharlemann-Thompson ([ST1], [ST2]) we obtain an affirmative answer to the aforementioned question. In fact, we show that the generalized crossings involved can be taken to be disjoint from any companion torus of K. As a consequence, we obtain that if a non-trivial satellite knot K is n-adjacent to the unknot then it is n-adjacent to the unknot in any companion solid torus. In particular, any *model knot* of K is n-adjacent to the unknot. Along the way of proving these results, we also characterize 2-bridge

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knots of the form $K_{p/q}$, where $p/q = [2q_1, 2q_2]$ for some $q_1, q_2 \in \mathbf{Z}$, precisely as those knots that have genus one and are 2-adjacent to the unknot.

A generalized crossing of order $q \in \mathbf{Z}$ on an embedding of a knot K is a set C of $|q|$ twist crossings on two strings that inherit opposite orientations from any orientation of K. If K' is obtained from K by changing all the crossings in C simultaneously, we will say that K' is obtained from K by a generalized crossing change (see Figure 1). In particular, if $|q| = 1$, K and K' differ by an ordinary crossing change while if $q = 0$ we have $K = K'$. Note that a generalized crossing change can be achieved by $\frac{1}{q}$-surgery on a crossing circle, which is an unknotted curve that bounds an embedded disc $D \subset S^3$ such that K intersects $\mathrm{int}(D)$ exactly twice with zero algebraic intersection number.

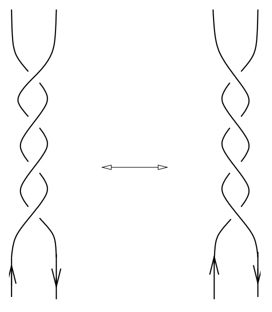

FIGURE 1. The knots K and K' differ by a generalized crossing change of order $q = -4$.

Definition 1.1. *We will say that K is n-adjacent to the unknot, for some $n \in \mathbf{N}$, if K admits an embedding containing n generalized crossings such that changing any $0 < m \leq n$ of them yields an embedding of the unknot. A collection of crossing circles corresponding to these crossings is called an n-trivializer. If all the generalized crossings used have order $+1$ or -1 (i.e. they are ordinary crossings), we will say that K is simply n-adjacent to the unknot. An n-trivializer that shows K to be simply n-adjacent to the unknot will be called a simple n-trivializer.*

Remark 1.2. Let V be a solid torus in S^3 and suppose that a knot K is embedded in V. Throughout the paper, we will use the term "K is n-adjacent to the unknot in V" to mean the following: There exists an embedding of K in V that contains n generalized crossings such that changing any $0 < m \leq n$ of them unknots K in V.

To state our result recall that if K is a non-trivial satellite with *companion knot* $\hat{K}$ and *model knot* P then: i) $\hat{K}$ is non-trivial; ii) P is geometrically essential in a standardly embedded solid torus $V_1 \subset S^3$; and iii) there is a homeomorphism $h : V_1 \longrightarrow V := h(V_1)$, such that $h(P) = K$ and $\hat{K}$ is the core of V.

Theorem 1.3. *Let K be a non-trivial satellite knot and let V be any companion solid torus of K. Then, K is n-adjacent to the unknot, for some $n > 0$, if and only if it is n-adjacent to the unknot in V.*

As a consequence of Theorem 1.3 we have the following:

Corollary 1.4. *A non-trivial satellite knot K is n-adjacent to the unknot, for some $n > 0$, if and only if any model knot of K is n-adjacent to the unknot in the standard solid torus V_1.*

The paper is organized as follows: In Section 2 we summarize some results that are used in the proof of Theorem 1.3. In Section 3 we study satellite knots of winding number zero that are n-adjacent to the unknot. In Section 4 we prove Theorem 1.3 and work out some corollaries. In Section 5, we show that a knot K of genus one is 2-adjacent to the unknot iff it is a 2-bridge knot of the form $[2q_1, 2q_2]$ for some $q_1, q_2 \in \mathbf{Z}$.

Note that a weaker version of Theorem 1.3 is generalized to a broader class of n-trivial knots in [K].

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2. Preliminaries

In this section we summarize some results that will be used in the proof of Theorem 1.3. We begin with the following theorem that summarizes some results from [ST1] and [La]. Part a) of the theorem is stated as Corollary 3.2 in [ST1]. Part b) is stated as Corollary 4.4 of [ST1] and also follows from Theorem 1.4(b) and Proposition 2.5 of [La].

Theorem 2.1. *Let K, K' be knots that differ by a generalized crossing change of order $q \in \mathbf{Z}$. Let L be a crossing circle for K corresponding to this generalized crossing and that K is a non-trivial satellite.*

a) If $|q| > 1$ and $\mathrm{genus}(K') \leq \mathrm{genus}(K) - 1$ then any companion torus of K can be isotoped in $S^3 \setminus K$ to be disjoint from L.

b) If $|q| = 1$ and $\mathrm{genus}(K') \leq \mathrm{genus}(K) - 2$ then any companion torus of K can be isotoped in $S^3 \setminus K$ to be disjoint from L.

To continue, note that if L is an n-trivializer for a knot K, since the linking number of K with each component of L is zero, K bounds a Seifert surface in the complement of L. We will need the following lemma:

Lemma 2.2. *Let L be an n-trivializer of a knot K. Suppose that S is a Seifert surface bounded by K in the complement of L and such that among all such surfaces S has minimum genus. Then,* $\mathrm{genus}(S) = \mathrm{genus}(K)$.

Proof: For simple n-trivializers the lemma is stated as Theorem 4.1 in [HL]; the proof relies on a result of Gabai (Corollary 2.4 of [Ga]). The argument for general n-trivializer's is essentially the same. The details are given in the proof of Theorem 3.1 of [KL]. □

We will apply Theorem 2.1 to the case when K' is the unknot. In this case, when $\mathrm{genus}(K) \geq 2$, for every q, we can isotopy a companion torus of K to be disjoint from the crossing circle L. When $\mathrm{genus}(K) = 1$ and $|q| = 1$, Theorem 2.1 can not be used anymore. Nevertheless, the following lemma guarantees that the same conclusion still holds in this situation.

Lemma 2.3. *Let K, K' be knots that differ by a generalized crossing of order $q \in \mathbf{Z}$. Let L be the corresponding crossing circle. Suppose that* $\mathrm{genus}(K') < \mathrm{genus}(K)$ *and let K_0 denote the 2-component link obtained by smoothing C in a way consistent with the orientation of K. Then, there exist Seifert surfaces Σ and Σ_0 of maximal Euler characteristic for K and K_0 respectively, $\Sigma \cap L = \emptyset$, such that Σ is obtained from Σ_0 by plumbing on an unknotted annulus with a $(2, 2q)$-torus link as its boundary and L as one of its small linking circles.*

Here by a *small linking circle* of an annulus A embedded in S^3, we mean an unknot which bounds a disk D such that $D \cap A$ is a proper non-boundary parallel arc in A.

Proof: For $|q| = 1$ the lemma is stated as Proposition 3.1 in [ST2]. The proof of that proposition uses Theorem 1.4 (of [ST2]) that gives a relation between the Euler characteristics of the triple (K, K', K_0). Theorem 6.4.3 of [Ka] states that the same relation holds when $|q| > 1$. Using this, the arguments used in the proof of Proposition 3.1 in [ST2] go through to give the lemma in the case that $|q| > 1$ (see also the proof of Theorem 6.4.2 of [Ka]). □

Corollary 2.4. *Let K be a non-trivial satellite knot which can be unknotted by a single generalized crossing change, and L be the corresponding crossing circle. Then any companion torus of K can be isotoped in $S^3 \setminus K$ to be disjoint from L.*

Proof: The case when $\mathrm{genus}(K) \geq 2$ is covered by Theorem 2.1. So we assume $\mathrm{genus}(K) = 1$. Let Σ be the genus one Seifert surface of K claimed to exist in Lemma 2.3. Then Σ is the plumbing of two annuli A_1 and A_2. One of them, say A_1, is unknotted with L as its small linking circle. Thus, K is contained in of a torus $T = \partial N$, where N is a tubular neighborhood of A_2. Since A_1 is unknotted, we may assume that $A_1 \subset N$. It is not hard to see that T is the inner most companion torus of K: every other companion torus of K can be isotoped to contain the solid torus N in one side. Thus, we can isotope every companion torus of K to be disjoint from L. □

3. Satellite knots with zero winding number

Throughout this section, we suppose that K is a non-trivial satellite knot and V is a companion solid torus of K, such that the winding number of K in V is zero.

3.1. **A technical lemma.** In this subsection we prove a technical lemma which will play a key role in our discussion in the next subsection.

Let S be a minimal genus Seifert surface of K. We assume that the intersection of S and $T = \partial V$ is transverse and the number of components of $S \cap T$ is minimal. Let α be a proper arc on S.

Lemma 3.1. *Suppose there is an isotopy of S^3, fixing K pointwise, which brings α to an arc α' in V, then we can isotopy α on the surface S, relative to $\partial S = K$, to a proper arc α'' in V.*

Proof: We may assume

(1) that $S \cap T$ is a collection of disjoint parallel copies of an essential simple closed curve on T;

(2) that $S \cap \overline{S^3 \setminus V}$ is a collection of disjoint parallel copies of a proper incompressible and boundary incompressible annulus in $\overline{S^3 \setminus V}$;

(3) that each proper component of $S \cap V$ in V is an annulus and is boundary incompressible in $V \setminus K$.

Since these points follow from well known facts in 3-dimensional manifold topology we only give a brief explanation. Point (1) follows immediately from the fact that T is incompressible in the complement of K. To see (2), note that since the winding number of K in V is zero, the intersection $S \cap T$ is homologically trivial in T. Thus, we can replace the components of $X := S \cap \overline{S^3 \setminus T}$ by annuli on T and then isotope them away from T to obtain a Seifert surface $S' \subset V$ for K. Since we must have genus$(S) =$ genus(S'), it follows that each component of S must be an annulus. Since $S \cap T$ is minimum, each component of X must be an essential annulus. Now the conclusion follows from the well known fact (see, for example [BZ]) that an essential annulus in a knot exterior is a cabling annulus, and any two such annuli must be parallel if they are disjoint. For (3), as in the argument for (2), we can see that if $S \cap V$ contains a proper component V (the component whose boundary lies on T), then it must be an annulus A. If this A is boundary compressible in $V \setminus K$, A can be isotoped on T to reduce $|S \cap T|$, which would contradict the assumption that $|S \cap T|$ is minimal.

Let $C_1, C_2, \ldots, C_{2r}$ be the components of $S \cap T$. In Figure 2, we draw a cross section of $S \cap T$, with $P, P', Q_1, \ldots, Q_{r-1}, Q'_1, \ldots, Q'_{r-1}$ denoting annuli on T such that their boundary components are $C_1 \sqcup C_2 \sqcup \cdots \sqcup C_{2r}$. We will denote these annuli $P, P', Q_1, \ldots, Q_{r-1}, Q'_1, \ldots, Q'_{r-1}$ by $A_1, A_2, \ldots, A_{2r}$ such that (i) $\partial A_i = C_i \sqcup C_{i+1}$, $i = 1, 2, \ldots, 2r \bmod r$, when $r > 1$; and (ii) $\partial A_1 = \partial A_2 = C_1 \sqcup C_2$, when $r = 1$.

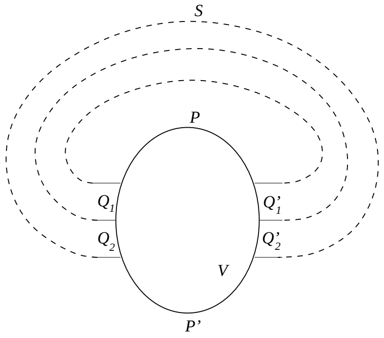

FIGURE 2. A cross section of T and $S \cap T$.

Claim: No closed component of $S \setminus T$ can contain an proper curve ξ, not necessarily simple, such that (i) $\partial \xi = \{x, y\}$, $x \in C_i$, $y \in C_{i+1}$; and (ii) ξ is path homotopic to a proper simple arc δ on A_i from x to y in the complement of K.

Proof of the Claim: As the first case, we assume that ξ lies outside of T. Then $A_i = P$ or P'. On the component A of $S \cap \overline{S^3 \setminus V}$ containing ξ, which is an annulus, β is path homotopic to a proper simple arc. This would imply that this component of $S \cap \overline{S^3 \setminus V}$ is boundary compressible in $\overline{S^3 \setminus V}$, and it leads to a contradiction to Point (2) above.

For the second case when ξ lies inside of T, there are two possibilities: $A_i = P$ or P', and $A_i = Q_j$ or Q'_j. In both possibilities, we may perform a surgery on S to get a new minimal Seifert surface S' for K. In the second possible case, in addition to S', the surgery on S may produce also a torus T'. By Point (3), this torus T' is incompressible in the complement of K. See Figure 3. The loop $\xi \cup \delta$ is seen on S' or T' and inessential in the complement of K. So it is inessential on S' or T'. Yet there is a loop on S' or T' (the core circle of A_i) intersecting $\xi \cup \delta$ at a single point. This is impossible and we complete the proof of the claim.

Now let α be a proper simple arc on S, which is isotopic in the complement of K to an arc α' lying inside of T. This isotopy gives rise to a map $f : D^2 \to S^3 \setminus K$,

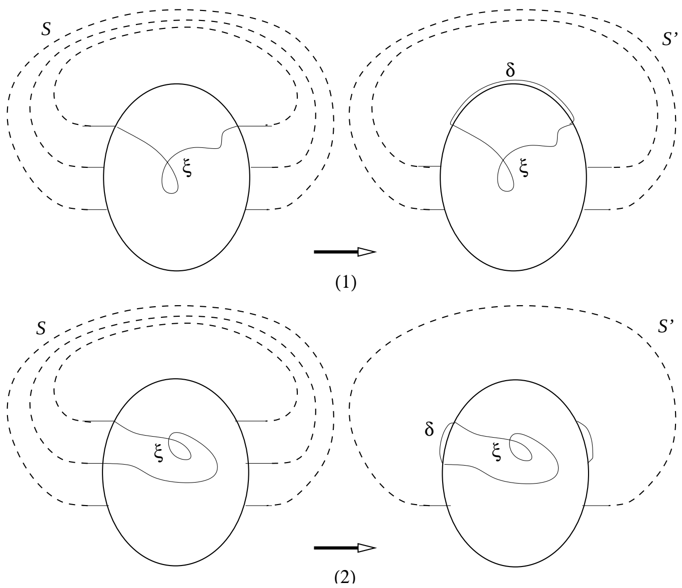

FIGURE 3. Two possibilities (1) and (2); two different kinds of surgery on S.

such that ∂D^2 is the union of two arcs, d_1 and d_2, meeting at their end points, with images under f being α and α', respectively. I.e. $f(d_1) = \alpha$ and $f(d_2) = \alpha'$.

Assume that f is in general position with respect to T, so that $f^{-1}(T)$ is a disjoint collection of proper arcs and simple loops. We will choose a proper arc a_2 in $f^{-1}(T)$, which cobounds a disk D with an arc a_1 on d_1, such that it is inner most in the sense that there is no more proper arc components of $f^{-1}(T)$ in D. See Figure 4. Then, $\beta := f(a_1)$ is a subarc of α, $\mathrm{Int}(\beta) \cap T = \emptyset$. Denote $\lambda := f(a_2)$, which is a curve on T. We will assume that $g = f|_D$ is in general position with respect to S, and it sends the inward normal vectors of a_1 in D to the normal vectors of S along β. We have $g(D) \cap K = \emptyset$.

Consider $g^{-1}(S)$, which consists of a_1 and a proper 1-submanifold of D whose boundary lies in $\mathrm{Int}(a_2)$. Let c_1 be an inner most proper arc in $g^{-1}(S)$ so that it cobounds a disk D' with a subarc c_2 of a_2 on D and $g(c_2)$ is contained in a certain A_j on T. See Figure 5. Note that $g(c_1)$ is a curve on S, which is path homotopic to $g(c_2)$ in the complement of K. Thus, by the claim above, it is impossible to have the two end points of $g(c_2)$ to lie on different boundary components of A_j. When the end points of $g(c_2)$ lie on the same boundary component of A_j, we may modify $g|_{a_2}$ by a path homotopy on T to push $g(c_2)$ out of A_j, and the number of intersection points of $g(a_2)$ with S is reduced.

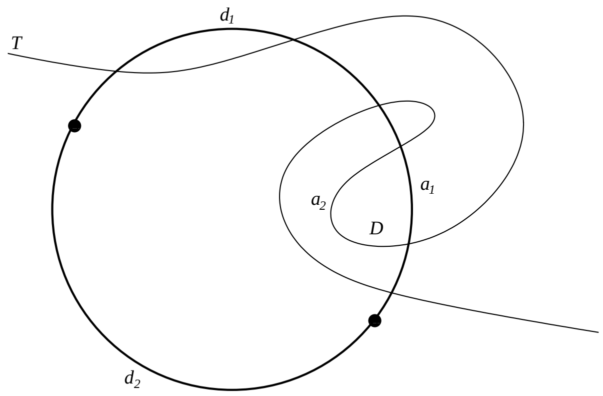

FIGURE 4. A inner most disk D bounded by a proper arc in $f^{-1}(T)$.

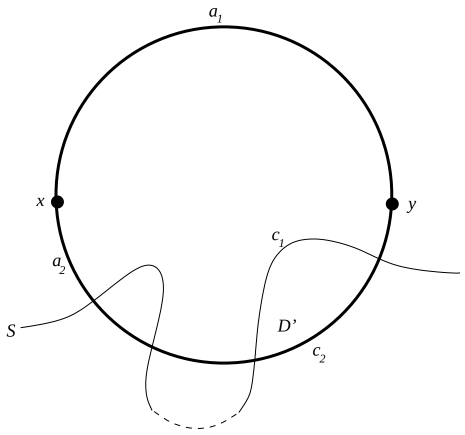

FIGURE 5. A inner most disk D' bounded by a proper arc in $g^{-1}(S)$.

So finally, we arrive at the following situation: There is a curve λ' on T contained entirely in some A_i such that (i) $\partial\beta = \partial\lambda' = \{x, y\}$; and (ii) β and λ' are path homotopic in the complement of K. Thus, by the claim, we must have $x, y \in C_i$.

The curve λ' is path homotopic in A_i to one of the curves shown in Figure 6. We write $\lambda' = \zeta * C_i^n$, $n \geq 0$, where ζ is the arc from x to y shown in Figure 6, C_i has y as its base point, and $*$ means to join paths. Then $\zeta^{-1} * \beta$ and C_i^n homotopic in the complement of K. But $\zeta^{-1} * \beta$ and C_i are both simple closed curves on S. So we conclude that $\zeta^{-1} * \beta$ and C_i^n are homotopic on S and either $n = 0$ or $n = 1$. When $n = 0$, β and ζ are path homotopic on S. When $n = 1$, β and $\overline{C_i \setminus \zeta}$ are path homotopic on S. In either cases, we may isotopy α on S to reduce the intersection of α and T. The finishes the proof of the lemma. $\square$

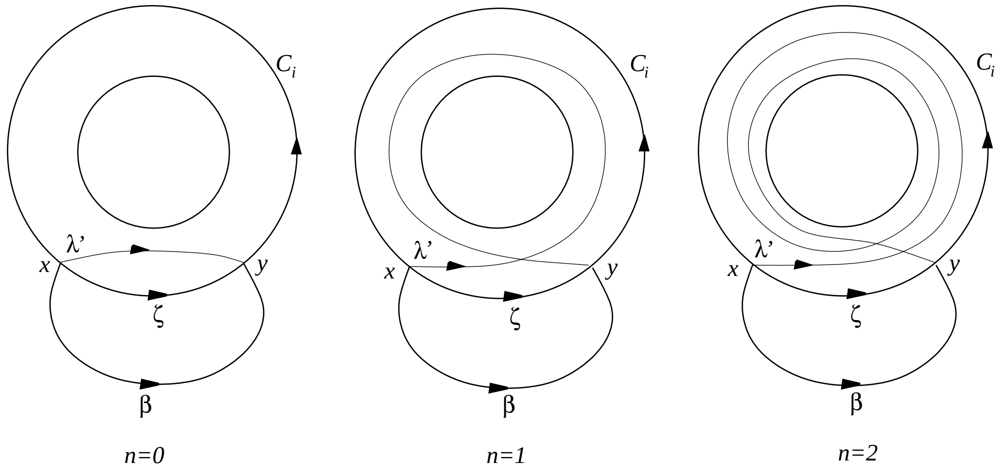

FIGURE 6. Possible forms of the curve λ', the arc ζ, and the arc β.

3.2. **Finding an n-trivializer in a companion solid torus.** We can now have the the following lemma, which will allows us to find an n-trivializer for K in any companion solid torus.

Lemma 3.2. *Let K be a non-trivial satellite and let V be a companion solid torus of K. Suppose that the winding number of K in V is zero. If K is n-adjacent to the unknot, for some $n > 0$, then, there exists an n-trivializer for K that lies in V.*

Proof: Let $L := \cup_{i=1}^{n} L_i$ be an n-trivializer of K and let $D_1, \ldots, D_n$ be crossing discs bounded by $L_1, \ldots, L_n$, respectively. Let S be a Seifert surface for K in the complement of L that has minimum genus. Then by Lemma 2.2, S is also a minimal genus Seifert surface of K. We may isotope S so that each $S \cap \mathrm{int}(D_i)$ is the union of an arc α_i and several closed components. The arc α_i is properly embedded on S. Since S is incompressible in the complement of L, after an isotopy we can arrange so that $S \cap D_i$ contains no closed curves that are inessential on D_i. Thus each closed component of $S \cap D_i$ has to be parallel to L_i on D_i. By replacing L_i with the closed component of $S \cap D_i$ that is innermost on D_i, we may assume that $S \cap D_i = \alpha_i$. Since twisting along α_i unknots K, it follows that α_i must be essential on S. Furthermore, the arcs $\alpha_1, \ldots, \alpha_n$ are disjoint with each other.

By Corollary 2.4, for each L_i, we can isotope the torus $T = \partial V$ in the complement of K to T' such that $T' \cap L_i = \emptyset$. Assume that T' intersects the disk D_i transversely. Since T' is disjoint with $L_i = \partial D_i$, each component of $T' \cap D_i$ is a simple closed curve in D_i. If a component of $T' \cap D_i$ bounds a disk in D_i which contains only one point in $K \cap D_i$, we would have the winding number of K in V to be ± 1. So every component of $T' \cap D_i$ either bounds a disk in D_i which is disjoint with $K \cap D_i$ or bounds a disk in D_i which contains $K \cap D_i$. In either cases, a further isotopy of T' in the complement of K will remove this component of $T' \cap D_i$. The reversed isotopy in the complement of K from this T' to T then will bring the arc α_i into V. Thus, by Lemma 3.1, we can isotopy each α_i on the minimal genus Seifert surface S, relative to $\partial S = K$, to an proper arc α_i' in V.

On S, let α and β be two proper 1-submanifolds whose intersection is transverse. Suppose that there is an isotopy of S that reduces the geometric intersection $|\alpha \cap \beta|$. Then there will be a disk D on S such that $D \cap (\alpha \cup \beta) = \partial D$, $D \cap \alpha$ and $D \cap \beta$ are subarcs in the interior of α and β, respectively. This is a well-known fact (see, for example, Proposition 3.10 in [FLP] or Lemma 3.1 in [HS]).

We apply this fact to $\{\alpha_1, \dots, \alpha_n\}$ and $S \cap T = C_1 \sqcup \cdots \sqcup C_r$ (see the proof of Lemma 3.1). Since α_1 can be made disjoint from $S \cap T$ by an isotopy of S relative to ∂S, we find a disk D between α_1 and $S \cap T$ as described above. We then use this D to define an isotopy of S relative to ∂S to remove a pair of intersection points of α_1 and $S \cap T$. This isotopy will not increase the intersection points of other α_i's with $S \cap T$. And it will also keep α_i's disjoint. So inductively, we have an isotopy on S relative to ∂S, which brings the entire disjoint collection of proper arcs $\{\alpha, \dots, \alpha_n\}$ to a disjoint collection of proper arcs $\{\alpha_1', \dots, \alpha_n'\}$ in V.

Finally, we construct a small disk D_i' in V whose intersection with S is α_i', for each i, and they are disjoint from each other. Let $L_i' = \partial D_i' \subset V$. Since $L' = \cup_{i=1}^{n} L_i'$ is isotopic to L in the complement of K, it is an n-trivializer for K that lies in V. $\square$

4. The proof of the main result

Here we finish the proof of Theorem 1.3 and Corollary 1.4.

Proof of Theorem 1.3: The "if" direction of the statement is clear. To prove the "only if" direction, suppose that K is a non-trivial satellite that is n-adjacent to the unknot. Let V be any companion solid torus of K. Let $\hat{K}$ denote the core of V and set $T := \partial V$. We need the following:

Claim. The winding number of K in V is zero.

Proof of Claim. By Corollary 2.4, there exists a component $L_1 \subset L$ that can be isotoped to be disjoint from T. Let D_1 be a crossing disc bounded by L_1. After an isotopy in the complement of K, $D_1 \cap T$ will consist of a collection of curves, none of which bounds a disc in D_1 in the complement of K. Let C be a component of $D_1 \cap T$. If C is boundary parallel on D_1 then it can be eliminated by an isotopy in the complement of K so that L_1 is still disjoint from T. If all components of

$D_1 \cap T$ are boundary parallel, we will have D_1 disjoint from T after an isotopy in the complement of K. Then D_1 is contained in V. Since a satellite with non-zero winding number cannot be unknotted by crossing changes in $\mathrm{Int}(V)$, we conclude every component of $D_1 \cap T$ bounds a disc on D_1 that contains exactly one point of $D_1 \cap K$. It follows that the winding number of K in V is ± 1. Since K was assumed to be a non-trivial satellite we conclude that K is a composite knot and T is the follow-swallow torus. But then the crossing change realized by L_1 occurs within a summand of K and it cannot unknot K. This contradicts the fact that L_1 is part of an n-trivializer and it finishes the proof of the claim.

Let us now finish the proof of the theorem. The claim above allows us to assume that the winding number of K in V is zero. By Lemma 3.2, K admits an n-trivializer L' in V. Now each of the surgeries along the sublinks of L' that unknot K must turn it into a knot that is isotopically trivial in $\mathrm{Int}(V)$. For, otherwise the knot obtained from K after any of these surgeries will still have $\hat{K}$ as a companion and it can't be the unknot. Thus K is n-adjacent to the unknot in V. $\square$

Proof of Corollary 1.4: Let P be any model of K in a standard solid torus $V_1 \subset S^3$ and let $h :\longrightarrow S^3$ the satellite embedding. If P is n-adjacent to the unknot in V_1 and $L \subset V_1$ is an n-trivializer then $h(L)$ is an n-trivializer for K in V. Conversely, by Theorem 1.3 and its proof, if K is n-adjacent to the unknot then any n-trivializer, say L, can be isotoped in $V := h(V_1)$ and exhibits K n-adjacent to the unknot in V. But then the crossing circles $h^{-1}(L_1), \dots, h^{-1}(L_n)$ form an n-trivializer for P in V_1. $\square$

There exist many criteria in terms of the finite type knot invariants or polynomial invariants that detect n-adjacency to the unknot. For example, in [AK] it is shown that if a knot is n-adjacent to the unknot, for some $n \geq 3$, then all the finite type invariants of order $< 2n-1$ and the Alexander polynomial are trivial. More recently, criteria that detect simple 2-adjacency to the unknot were obtained by N. Askitas and A. Stoimenow ([AS]) in terms of the HOMFLY polynomial, and by the second named author of this paper and Z. Tao in terms of the Kauffman polynomial. Due to the computational complexity of the invariants involved, these criteria become harder to test for knots that are non-trivial satellites. The results of this paper, reduce the problem of deciding whether a non-trivial satellite K is n-adjacent to the unknot to deciding the same problem for a model knot of K. In particular, we have the following:

Corollary 4.1. *Suppose that P is a knot that is not n-adjacent to the unknot. Then, no satellite that is modeled on P is n-adjacent to the unknot.*

Proof: This follows from Theorem 1.3 and the fact that if P is n-adjacent to the unknot in the solid torus then it is n-adjacent to the unknot in S^3. □

5. Knots of genus one

We finish this paper by taking a look at genus one knots that are n-adjacent to the unknot, for $n > 1$. In fact, we will obtain a characterization of knots of genus one which are 2-adjacent to the unknot.

Consider 2-bridge knots of the form $K_{p/q}$, where $p/q = [2q_1, 2q_2]$ in Conway's notation (see, for example, [BZ]). Such a knot is formed by plumbing two unknotted, $2q_1$ and $2q_2$ twisted annuli, and taking the boundary of the resulting genus one surface. Obviously, this is a genus one knot which is 2-adjacent to the unknot. The orders of the two generalized crossing changes are q_1 and q_2, respectively.

Theorem 5.1. *A genus one knot K is 2-adjacent to the unknot if and only if $K = K_{p/q}$, $p/q = [2q_1, 2q_2]$, for some integers q_1, q_2.*

It is clear that we only need to prove the "only if" part. So we suppose K is a genus one knot and it is 2-adjacent to the unknot. Let $L = L_1 \cup L_2$ be a 2-trivializer of K of order (q_1, q_2). By Lemma 2.2, we have a Seifert surface S of K in the complement of L and the genus of S is one. We may assume that the crossing disks D_1, D_2, with $\partial D_1 = L_1$ and $\partial D_2 = L_2$, intersect S along essential proper arcs α_1, α_2, respectively.

Lemma 5.2. (See also [HL].) *The arcs α_1, α_2 are not parallel on S.*

Proof: If α_1 and α_2 were parallel to each other on S, L_1 and L_2 would cobound an annulus in the complement of K. Then perform both $1/q_1$-surgery on L_1 and $1/q_2$ surgery on L_2 would be the same as doing $1/(q_1 + q_2)$-surgery on L_1 or L_2. Since K is nontrivial, we would have two distinct surgeries on L_1 under which S does not remain of minimal genus.

Since a twist along L_1 unknots K, it follows that the 3-manifold $M := S^3 \setminus \eta(K \cup L_1)$ is irreducible. S gives rise to a properly embedded surface in M that minimizes the Thurston norm in its homology class. Corollary 2.4 of [Ga], applied to M and $T := \partial\eta(L_1)$, implies that there can be at most one Dehn filling of T (or equivalently at most one surgery along L_1) under which S doesn't remain a minimum genus surface for K. This contradiction finishes the proof of the lemma. □

Next we use Lemma 2.3 to K and L_1. This lemma gives us another genus one Seifert surface Σ of K in the form of the plumbing of annuli A_1 and A_2, such that A_1 is unknotted, $2q_1$-twisted, and its small linking circle is L_1. Without loss of generality, we may assume that $D_1 \cap \Sigma = \alpha_1$. So we may cut open both S and Σ along α_1. For Σ, we get the annulus A_2 from this surgery. For S, we get another annulus A_2' from this surgery. The annuli A_2 and A_2' have the same boundary K_0, and we may assume that they are disjoint. Thus $T = A_2 \cup A_2'$ is a torus which bounds a solid torus in S^3. Assume that the core circle of A_2 (and the core circle of A_2') is a (m, l) curve on T, where $l \geq 0$ is the winding number in the longitude direrction and m is the winding number in the meridian direction on T.

The arc α_2 on A_2' should have its end points on different boundary components of A_2' by Lemma 5.2. We may pick a possible α_2 and all other possible α_2's are obtained by Dehn twist along A_2'. See Figure 7.

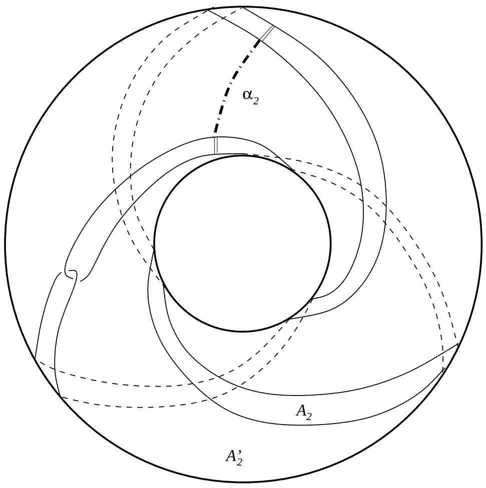

FIGURE 7. A possible position for the arc α_2 on the annulus A_2'.

Let us perform a generalized crossing change at L_1, which will unhook the clasp seen on A_2 in Figure 7. We then shrink the two seperated clasp ends along A_2 until they meet the ends of α_2 (the places on A_2 marked by double lines in Figure 7). Denote the subarc of the core circle of A_2 between the double line marks, which does not run throught the clasp, by β. Then we may get a simple closed curve $J = \alpha_2 \cup \beta$. The curve J intersects the core circle of A_2' only once. Let J be a (a, b) curve on T. We have $|mb - la| = 1$.

If we perform generalized crossing changes at both L_1 and L_2, K will be changed to the unknot. This is possible only when $(a, b) = (0, \pm 1)$ or $(a, b) = (\pm 1, 0)$. Otherwise, J would have a non-zero framing in the solid torus V bounded by T and generalized crossing changes at both L_1 and L_2 would change K into a $[2r, 2s]$ knot in V for $rs \neq 0$. Such a knot can not be unknotted in S^3.

When $(a, b) = (0, \pm 1)$, V has to be unknotted in S^3. Notice that we must have $m = \pm 1$. If $l = 0$, the knot K would be trivial. So we may assume that $l \geq 1$. Thus, we can have one possible choice of α_2 as shown in Figure 8. Any other choices of α_2 are obtained by applying a power of the Dehn twist of along the core of A_2' to this particular α_2. From this fact, we see that this particular α_2 is the only one with $(a, b) = (0, \pm 1)$. In Figure 8, we can see that the arc α_2 can be isotoped to the arc γ in A_2'. Furthermore, γ and an arc on A_2, which is the intersection of A_2 with its small linking disk, cobound a disk whose interior is disjoint with T. Thus the generalized crossing change at L_2 is the same as a generalized crossing change at a small linking circle of A_2. This implies that K is a 2-bridge knot of the form $[2q_1, 2q_2]$.

When $(a, b) = (\pm 1, 0)$, we have $l = 1$ and $m \neq 0$. We can argue as in the previous case: First, we find a unique choice of α_2. This choice of α_2 will force V to be unknotted in S^3. And then the knot K will be a 2-bridge knot of the form $[2q_1, 2q_2]$. This finishes the proof of Theorem 5.1. □

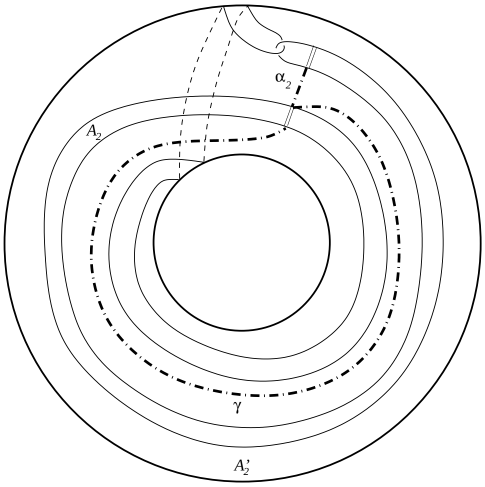

FIGURE 8. This is a knot of the form $[2q_1, 2q_2]$.

Corollary 5.3. *The only genus one knots that are simply 2-adjacent to the unknot are the two trefoils and the figure eight.*

Proof: This corollary corresponds to the case of $q_1 = \pm 1$ and $q_2 = \pm 1$ of Theorem 5.1. □

Note that, as observed by T. Stanford, if a knot K is simply 2-adjacent to the unknot, then we have $a_2(K) = 0$ or ± 1, where a_2 is the second coefficient of the Alexander-Conway polynomial. Using this observation, we see that the knot 5_2 is not simply 2-adjacent to the unknot since $a_2(5_2) = 2$. By Theorem 5.1, 5_2 is not 2-adjacent to the unknot since it is the 2-bridge knot $[2,3]$. Apparently, no way is known to detect this using knot invariants.

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