ON KNOT ADJACENCY

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Abstract. A knot K is called n-adjacent to the unknot if it admits a projection that contains n disjoint single crossings such that changing any $0 < m \leq n$ of these crossings, yields a projection of the unknot. Using a result of Gabai ([Ga]] we characterize knots that are n-adjacent to the unknot as these obtained from the unknot by n "finger moves" determined by a certain kind of trivalent graphs (Brunnian Suzuki n-graphs). Using this characterization we derive vanishing results about abelian invariants as well as Vassiliev invariants of knots that are n-adjacent to the unknot. Finally, we partially settle a conjecture of [K-L].

Keywords: n-adjacency, Brunnian graphs, sutured 3-manifolds, Suzuki graphs, Seifert surfaces, Vassiliev invariants.

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1. Introduction

A knot is called n-adjacent to the unknot if it admits a projection containing n crossings such that changing any $0 < m \leq n$ of these crossings, yields a projection of the unknot. In particular, the unknotting number of such a knot is at most one. The purpose of the present paper, is to study these knots and understand their topological properties.

The right framework to examine the notion of n-adjacency to the unknot is via the more general notion of n-triviality. This notion was introduced by M. Gusarov ([G]) and Y. Ohyama ([O]) and was shown in [G] (see also [N-St]) to capture precisely the vanishing of finite type knot invariants of degree $< n$; by definition (see Definition 2.1), any knot which is n-adjacent to the unknot is in fact n-trivial. Since n-triviality is defined in terms of combinatorics of knot projections, one would like to relate it to the topology of the knot complement and understand the geometric properties that are detected by it. This, in turn, would lead to an understanding of the vanishing of Vassiliev

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and Jones-type knot invariants in terms of topological properties of the knot complement. Classical results about the topology of the knot complement captured by the Alexander polynomial, suggest that in the quest of such information one should study Seifert surfaces spanned by n-trivial knots. In [K-L], it was conjectured that n-trivial knots admit Seifert surfaces that look "simple" to nilpotent invariants of the surface complement. In the present paper, among other things, we settle this conjecture for many classes of knots that are n-adjacent to the unknot.

The paper has two main parts: In the first part, roughly speaking, we characterize knots that are n-adjacent to the unknot as these obtained from the unknot by doing n "finger moves" determined by certain type of graphs ("Brunnian Suzuki n-graphs"). In the second part, we study Seifert surfaces spanned by knots that are n-adjacent to the unknot. Here is a more detailed account of the contents of the paper:

In section 2, we state some of the basic definitions that will be used in subsequent sections and we prove a theorem (Theorem 2.2) that characterizes simultaneous "nugatory crossings" on projections of the unknot. The proof of this theorem uses a result of Gabai ([Ga]) to study the effect of simultaneous crossing changes on the knot genus. In section 3, we introduce the notion of Suzuki graphs and discuss their relation to disc-band presentations of knots. Our work here is inspired by [Su] where a knot K is realized as the boundary of an immersed disc that is obtained from a standardly embedded disc $D \subset \mathbf{R}^3$ by performing a number of 'finger moves" on ∂D. The information on how these finger moves are performed is encoded by a trivalent graph; the Suzuki graph underlying K. This graph consists of the boundary of the disc D together with a number of edges; one for each finger move performed on ∂D. In section 4, we use Theorem 2.2 to obtain a characterization of the spatial graphs that underlay the Suzuki presentations of knots that are n-adjacent to the unknot. See Proposition 4.3 and Theorem 4.4. As already mentioned, n-adjacency to the unknot implies the vanishing of Vasilliev invariants of orders $< n$. Our work in this section leads to the following stronger result:

Theorem 1.1. *If K is n-adjacent to the unknot for $n \geq 3$ then it is $(2n-1)$-trivial and hence all of its Vassiliev invariants of degree less than $2n-1$ vanish.*

Our characterization of Suzuki graphs underlying n-adjacent knots ($n \geq 3$) leads to certain "nice" embeddings of such knots (called canonical embeddings). In section 5 we show that a canonical embedding gives rise to a Seifert surface of K that shares a common Seifert matrix with a surface of the unknot. Using these surfaces we show the following:

Theorem 1.2. *Let K be a knot that is n-adjacent to the unknot for some $n \in \mathbf{N}$. If $n \geq 3$, then the Alexander polynomial of K is trivial.*

Nontrivial fibered or alternating knots are known to have nontrivial Alexander polynomial (see [Ro] and [C], respectively). Combining this with Theorem 1.2 we obtain the following:

Corollary 1.3. *Let K be a knot that is fibered or alternating. If K is n-adjacent to the unknot for some $n \geq 3$, then K is the unknot.*

Theorem 1.2 implies that a knot K that is n-adjacent to the unknot, for some $n \geq 3$, is S-equivalent to the unknot([Le]). A knot that is S-equivalent to the unknot admits a Seifert surface that looks simple to abelian invariants of the surface complement. In [K-L] the notion of S-equivalence was generalized; the notion of n-hyperbolicity was defined to describe knots that bound surfaces that look simple to nilpotent invariants. More precisely, a knot is called n-hyperbolic, if it bounds a *regular* Seifert surface that looks like a disc to the first $(n+1)$ terms of the lower central series of π_1 of the surface complement. See section 6 for the precise definition. In [K-L] it is shown that there is a function $l(n)$, with $l(n) \to \infty$ as $n \to \infty$, such that if K is n-hyperbolic then it is $l(n)$-trivial. Knots that are n-hyperbolic ($n \geq 1$) are easily seen to have trivial Alexander polynomial. On the other hand, for every $n \in \mathbf{N}$, there exist n-trivial knots with nontrivial Alexander polynomial. In view of these results it seems reasonable to conjecture the following: There is a function $m(n)$, with $m(n) \to \infty$ as $n \to \infty$, such that if K is an n-trivial knot with trivial Alexander polynomial then it is $m(n)$-hyperbolic. Theorem 1.2 makes knots that are n-adjacent to the unknot suitable candidates for this conjecture. Indeed in section 6, we verify the conjecture for many classes of knots that are n-adjacent to the unknot.

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2. Simultaneous and nugatory crossing changes

Let $D = D(K)$ be a projection (a knot diagram) of a knot K and let C be a set of crossings on D. We will use K_C to denote the knot represented by the diagram obtained from D by changing all the crossings in C simultaneously.

Definition 2.1. *[Gusarov- Ohyama] A knot K is said to be n-trivial iff it has a projection $D = D(K)$ with n disjoint sets of crossings $C_1, \ldots, C_n$ such that for any non-empty subset $I \subset \{1, \ldots, n\}$, K_{C_I} is the unknot, where $C_I = \cup_{i \in I} C_i$. If every C_i consists of single crossing then K is called n-adjacent to the unknot.*

Let K_+, K_- and K_0 denote three links related by the skein moves of Figure 1. A crossing disc for a crossing of a knot K, is a disc D whose interior intersects K in precisely two points of opposite orientation. Let $K_1 := \partial(D)$. Notice that the knot in S^3 obtained from K_+ (resp. K_-) by doing -1 surgery (resp. +1 surgery) on K_1 is K_- (resp. K_+).

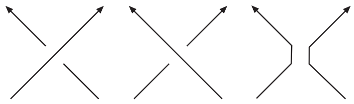

FIGURE 1. The skein links K_+, K_- and K_0.

A set of crossings, say C, in a projection of a knot K will be called *nugatory* if K and K_C are isotopic knots. In particular, a single crossing of a knot K is called *nugatory* if K_+ and K_- are isotopic knots.

The following theorem will be used in our characterization of knots that are n-adjacent to the unknot that will be given in Section 4. To prove the theorem we use a result of [Ga], that predicts the behavior of minimal genus surfaces under Dehn surgery, as used by Scharlemann and Thompson in [S-T].

Theorem 2.2. *Let $\mathcal{C}_n$ be a collection of n single crossings on a projection of the unknot K, and let $D_1, \ldots, D_n$ be crossing discs corresponding to the crossings in $\mathcal{C}_n$. For $i = 1, \ldots, n$, let $K_i := \partial(D_i)$. Suppose that every non-empty subset $C \subset \mathcal{C}_n$ is a nugatory set of crossings. Then, K bounds an embedded disc in the complement of $K_1 \cup \ldots \cup K_n$.*

Proof : Let $L := K_1 \cup \ldots \cup K_n$ and let $\eta(K)$ and $\eta(L)$ denote a regular neighborhood of K and L respectively. Since the linking number of each component of L and K is zero, K bounds a Seifert surface in the complement of L. Among all such surfaces let S be one of minimal genus. We will show that S is a minimal genus surface for K in S^3; thus, since K is the unknot, S has to be a disc.

Let $M := S^3 \setminus (\eta(K) \cup \eta(L))$. For $i = 1, \ldots, n$, there is $q_i \in \{-1, 1\}$ such that q_i surgery on K_i changes the crossing corresponding to D_i. Let $M(q_i)$ denote the 3-manifold obtained from M by q_i surgery on K_i and let $K(q_i)$ denote the image of K in $M(q_i)$.

Suppose that $n = 1$. Then $M = S^3 \setminus (\eta(K) \cup \eta(K_1))$. If M is reducible, every essential 2-sphere in M must intersect D_1 and an innermost disc argument shows that K_1 bounds a disc, say D, disjoint from K. Thus $K \cup K_1$ is the unlink of 2-components and S must be a disc. Suppose now that M is irreducible and let $M(\infty)$ denote the 3-manifold obtained from M by ∞ surgery (trivial surgery) on K_1. By Corollary 2.4 of [Ga], S must remain of minimal genus in at least one of $M(q_1)$, $M(\infty)$. Since both of $K(q_1)$, K are unknots we conclude that S must be a disc.

Suppose inductively that for every $0 < m < n$ and every collection of m crossings of K that satisfies the hypothesis of the statement of the theorem the conclusion of the theorem is true. We will prove that the conclusion is true for $m = n$.

If M is reducible, as above, we conclude that at least one of the knots $K_1, \dots, K_n$, say K_n, bounds a disc, say D, disjoint from $K \cup K_1 \cup \dots \cup K_{n-1}$. By induction K bounds a disc in the complement of $K_1 \cup \dots \cup K_{n-1}$. Since this disc can easily be chosen to be disjoint from D the conclusion follows. Suppose M is irreducible and let $M(\infty)$ denote the 3-manifold obtained from M by ∞ surgery on K_n. Also let $K(q_n)$ (resp. K) denote the image of K in $M(q_n)$ (resp. $M(\infty)$). The induction hypothesis implies that each of $K(q_n)$ and K bound a disc in the complement of $K_1 \cup \dots \cup K_{n-1}$. Thus, the genus of each of $K(q_n)$ and K in the complement of $K_1 \cup \dots \cup K_{n-1}$ is zero. On the other hand, by Corollary 2.4 of [Ga], S must remain of minimal genus in at least one of $M(q_n)$, $M(\infty)$. Thus S must be a disc. □

3. Knots, Graphs and Maps.

In this section we define categories of objects which we will need in the rest of the paper, we establish the relevant notation and we define certain maps between these categories. To begin with, we will use $\mathcal{K}$ to denote the set of all isotopy classes of oriented knots.

Consider $\mathbf{R}^3$ parameterized by rectangular coordinates (x, y, z), and let

$$R := \{ \ (x, y, 0) \mid x, y \in \mathbf{R} \ \}.$$

Let $n \in \mathbf{N}$. An *n-graph* G, will be a trivalent graph in $\mathbf{R}^3$ with $2n$ vertices, consisting of a planar circle S that lies on the plane R, and edges $a_1, \dots, a_n$ attached to S. See Figure 2. Unless otherwise stated, throughout the paper, our convention will be the following: Each of $a_1, \dots, a_n$ lies on R leaving it only at small arcs so that G doesn't have self intersections. A *weighted n-graph* will be an n-graph

$$G := S \cup (\cup_{i=1}^{n} a_i),$$

such that S and each edge a_i is oriented and each a_i has been assigned a pair (w_i, z_i), where $w_i \in \mathbf{Z}$ and $z_i \in \{+1, -1\}$.

Next we will consider regular projections of weighted n-graphs on the plane R. The only singular points of such projections are double

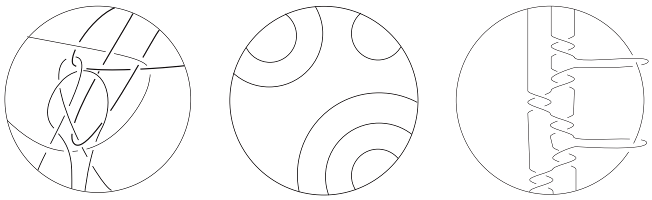

FIGURE 2. An example of a 5-graph, a 6-graph and a 3-graph.

points such that no vertex of the graph projects on a double point. Let G_n denote the set of all such projections of weighted n-graphs. On the set G_n we will consider the equivalence relation generated by the local moves of Figure 3. The drawings in each of the moves of Figure 3, indicate parts of projections of weighted n-graphs that differ only locally as indicated. Move I involves only a piece of a weighted edge a_i, while in moves II-M2 one of the arcs involved may be a piece of the circle S.

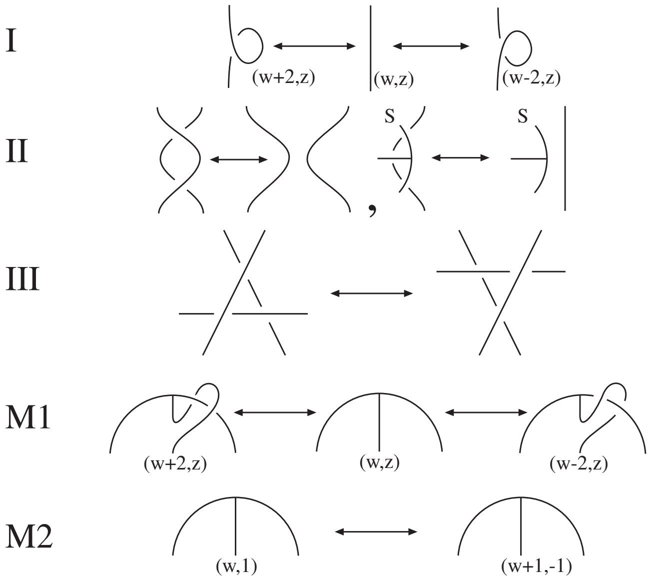

FIGURE 3. Local moves on projections of weighted n-graphs.

Definition 3.1. *The set G_n modulo the equivalence relation generated by the moves of Figure 3 will be denoted by $\mathcal{G}_n$. An element in $\mathcal{G}_n$ will be called a Suzuki n-graph.*

To continue, we need to define a map

$$\mathcal{G}_n \xrightarrow{k} \mathcal{K}.$$

Let $\mathbf{G} \in \mathcal{G}_n$ be a Suzuki n-graph and let $G_R \in G_n$ be a representative of $\mathbf{G}$. We obtain a weighted n-graph $G = S \cup (\cup_{i=1}^{n} a_i)$ as follows: For each double point (crossing) of G_R we push a small part of the arc over crossing slightly off R in $\mathbf{R}^3$. Since each a_i is oriented we can speak of its initial and terminal point. We can construct a knot, say $K(G)$, out of G as follows: Following the blackboard framing, first we thicken the interior of each a_i to a band and then we add w_i half twists to the band. The interiors of bands corresponding to different edges should be disjoint. Finally, at the end of each band corresponding to the initial (resp. terminal) point of the underlying edge we create a *foot* (resp. a *hook*) as indicated in Figure 4. The hook will consist of a positive or a negative clasp according to whether z_i is equal to $+1$ or -1. It is not hard to see that the effect on $K(G)$ of any of the moves of Figure 3 is knot isotopy. Thus we may define $k(\mathbf{G})$ to be the isotopy class of $K(G)$. We will say that G is the underlying n-graph of $K(G)$ and that $\mathbf{G}$ is the underlying Suzuki n-graph of $k(\mathbf{G})$. By abusing our notation, whenever it is convienient, we will confuse between $K(G)$ and its isotopy class $k(\mathbf{G})$.

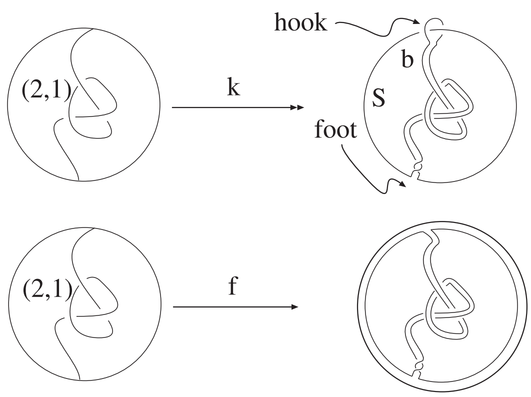

FIGURE 4. Constructing the knot $K(G)$ and the link $f(G)$.

Let $G_n^2 \subset G_n$ denote the set of projections of all weighted n-graphs for which all the weights w_i are even numbers. Notice that by the move M2 of Figure 3 every element in $\mathcal{G}_n$ can be represented by an element of G_n^2. Let $G = S \cup (\cup_{i=1}^{n} a_i)$ be an n-graph obtained from a projection in G_n^2 by pushing a small arc slightly off the plane R in $\mathbf{R}^3$ for each double point of the projection. We obtain a surface, Σ_G, by thickening S and each of the $a_i's$ to a band following the blackboard framing and then adding w_i half twists to the band that results from a_i. The boundary $f(G) := \partial(\Sigma_G)$ is a link of $n+2$ components (see the lower picture of Figure 4). We will say that G is *zero framed* if the components of $f(G)$

have zero pairwise linking numbers and the framing on each component defined by parallel copies on R (self linking number) is zero.

Next, we need to define some special classes of Suzuki graphs.

Definition 3.2. *Let $G = S \cup (\cup_{i=1}^{n} a_i)$ be an n-graph. For $i = 1, \dots, n$, let p_i, q_i denote the vertices on S to which a_i is incident. The complement $S - \{p_i, q_i\}$ consists of two disjoint arcs, say S_{i1} and S_{i2}. We will say that G is* admissible *iff for every pair (i, j), with $i \neq j$, both p_j, q_j belong on one of S_{i1}, S_{i2}. A Suzuki n-graph $\mathbf{G}$ will be called admissible if every n-graph representing it is admissible.*

Let $\mathbf{G}$ be an admissible Suzuki n-graph. We will say that $\mathbf{G}$ is *contained* if it can be represented by a weighted n-graph $G = S \cup (\cup_{i=1}^{n} a_i)$, such that the interiors of the a_i's do not intersect the interior of a planar disc $D \subset R$ bounded by S. For example, the left and middle picture of Figure 2. $\mathbf{G}$ will be called *standard*, if it is represented by a weighted n-graph all of whose weights are the form $(0, z_i)$ and such that it admits a projection on R so that the edges a_i are disjointedly embedded in D. The graph shown in the middle picture of Figure 2 is a standard 6-graph.

Definition 3.3. *Let $\mathbf{G}$ be a Suzuki n-graph, G be a weighted n-graph representing $\mathbf{G}$ and let $K(G)$ be the knot associated to G. We will say that $\mathbf{G}$ is* nugadjacent *iff for every G and $K(G)$ as above, if we let $\mathcal{C}_n$ be a set of n crossings containing exactly one crossing from each clasp of $K(G)$ then every non empty subset of $\mathcal{C}_n$ is nugatory.*

For an n-graph $G = S \cup (\cup_{i=1}^{n} a_i)$ and a subset $I \subset \{1, \dots, n\}$, let $G(I)$ denote the graph obtained by deleting all the edges in the set $\{a_i \mid i \in I\}$.

Definition 3.4. *Let $\mathbf{G}$ be an admissible Suzuki n-graph. We will say that $\mathbf{G}$ is* Brunnian *iff for every weighted n-graph G representing $\mathbf{G}$ and every non-empty subset $I \subset \{1, \dots, n\}$ the graph $G(I)$ is standard.*

4. Brunnian graphs and n-adjacency to the unknot.

It follows from [Su] that the map

$$\mathcal{G}_n \xrightarrow{k} \mathcal{K},$$

defined in the Section 3, is surjective. In this section we study the pre image $k^{-1}(\mathcal{K}_n)$, where $\mathcal{K}_n \subset \mathcal{K}$ denotes the set of knots that are n-adjacent to the unknot and provide a characterization of the graphs in this pre image. We begin with the following lemma.

Lemma 4.1. *A nugadjacent Suzuki n-graph is standard.*

Proof: Suppose that $\mathbf{G}$ is a nugadjacent Suzuki n-graph and let $G := S \cup (\cup_{i=1}^{n} a_i)$ be an n-graph representing $\mathbf{G}$. Let $K(G)$ denote the knot obtained from G as described in Section 3 and let $\mathcal{C}_n$ be a set of n crossings containing exactly one crossing from each hook point of $K(G)$. Also let $D_1, \ldots, D_n$ be crossings discs, one for each crossing in $\mathcal{C}_n$, and let $K_1 := \partial(D_1), \ldots, K_n := \partial(D_n)$. We may choose each K_i to be a small circle linking once around a clasp of $K(G)$. Let K' denote the knot obtained from $K(G)$ by changing all the crossings in $\mathcal{C}_n$ simultaneously. Since K is obtained from S by a number of finger moves, one along each a_i, K' is isotopic to S. This isotopy is realized as follows: For $i = 1, \ldots, n$ we push a small subarc of S containing one initial endpoint of a_i along a_i following the orientation of a_i. Thus $D_1, \ldots, D_n$ are also crossing discs for S. By Theorem 2.2 K bounds a disc in the complement of $K_1 \cup \ldots \cup K_n$. We may isotope Δ so that the intersection $\Delta \cap (\mathrm{int}(D_1) \cup \ldots \cup \mathrm{int}(D_n))$, consists of a collection of disjoint arcs that are properly embedded in Δ. Since K' is obtained from K by untwisting along $D_1, \ldots, D_n$, the complements of the links $K \cup K_1 \cup \ldots \cup K_n$ and $K' \cup K_1 \cup \ldots \cup K_n$ are homeomorphic. It follows that K' bounds a disc, say Δ', in the complement of $K_1 \cup \ldots \cup K_n$. Let $b_1, \ldots, b_n$ denote the arcs in $\Delta' \cap (\mathrm{int}(D_1) \cup \ldots \cup \mathrm{int}(D_n))$. Let $b := b_1 \cup \ldots \cup b_n$ and let $\eta(b)$ denote an open regular neighborhood of b in Δ'. The complement $\Delta' \setminus \eta(b)$ consists of $n+1$ disjoint discs. Let $B_1, \ldots, B_n$ denote disjoint regular neighborhoods of the components of $\Delta' \setminus \eta(b)$. We can arrange so that each B_i contains exactly one component of K_O; the $(n+1)$-component link obtained from K by resolving each of the crossings in $\mathcal{C}_n$ as indicated in the third picture of Figure 1. Clearly, performing the isotopy from K' to S described earlier backwards, isotopes $b_1, \ldots, b_n$ to $a_1, \ldots, a_n$ and thus $K' \cup (\cup_{i=1}^{n} b_i)$ to the n-graph G. It follows that any two of $a_1, \ldots, a_n$ must be geometrically unlinked and after a number of moves I-M2 of Figure 3, G becomes a standard graph. $\square$

In order to continue we need to introduce some terminology. Let $\mathbf{G}$ be a Brunnian Suzuki n-graph and let $G = S \cup (\cup_{i=1}^{n} a_i)$ be a zero framed n-graph in $\mathbf{R}^3$, representing $\mathbf{G}$. Recall that S lies on the plane R and it bounds a disc D in there while the arcs $a_1, \ldots, a_n$ leave R only at small arcs. We may isotope G such that the following are true:

i) The subgraph $S \cup (\cup_{i=1}^{n-1} a_i)$ is standardly embedded in $\mathbf{R}^3$ so that $\mathrm{int}(a_1), \ldots, \mathrm{int}(a_{n-1})$ are disjoint from $\mathrm{int}(D)$ and lie in $z > 0$ slightly above R. Furthermore any two of these arcs are geometrically unlinked.

iii) $\mathrm{int}(a_n)$ may have parts in both $z > 0$ and $z < 0$ and it intersects $\mathrm{int}(D)$ in an even number of pairwise oppositely signed points. If G is contained, then $\mathrm{int}(a_n)$ may be assumed to lie in $z > 0$ as well.

An n-graph G with the properties (i)-(ii) will be called *canonical*.

Definition 4.2. *Two Suzuki n-graphs $\mathbf{G}$ and $\mathbf{G}'$ are called m-similar, for some $m \in \mathbf{N}$, if $\mathbf{G}$ is represented by an n-graph G with the following property: G admits a projection that contains a collection of m disjoint sets of crossings, say $A_1, \ldots, A_m$, such that for every non empty $I \subset \{1, \ldots, m\}$, G_I represents $\mathbf{G}'$. Here G_I is the n-graph obtained from G by changing all the crossings in $\cup_{i \in I} A_i$ simultaneously.*

Proposition 4.3. *Let $\mathbf{G}$ be a Brunnian Suzuki n-graph that is not standard. Then $\mathbf{G}$ is $(n-1)$-similar to a standard graph.*

Proof: By definition a Brunnian n-graph $\mathbf{G}$ is admissible. Let $G = S \cup (\cup_{i=1}^{n} a_i)$ be an n-graph representing $\mathbf{G}$ and such that G is canonical. Thus, in particular, the algebraic intersection number of $\mathrm{int}(a_n)$ and $\mathrm{int}(D)$ is zero. Consider a regular projection of G on the plane R such that the images of $a_1, \ldots, a_{n-1}$ are disjointedly embedded in D; the disc bounded by S. By abusing the notation we will also use $a_1, \ldots, a_{n-1}$ to denote the projections of $a_1, \ldots, a_{n-1}$ on R. For $i = 1, \ldots, n-1$, let A_i denote the set of crossings between a_i and a_n where a_i under crosses a_n. First, we claim that each of $A_1, \ldots A_{n-1}$ is a non-empty set. For, suppose that one of them, say A_1, is empty. Then a_1 lies entirely above of the rest of the projection of G and it can be isotoped so that its interior lies in $z > 0$ without disturbing the rest of the projection. Since by the Brunnian property the graph G' obtained from G by deleting a_1 is standard, it follows that G is standard. However, this contradicts our assumption.

Next we show that the sets $A_1, \ldots A_{n-1}$ exhibit the $(n-1)$-similarity of G to a standard graph. For a non-empty subset of $I \subset \{1, \ldots, n-1\}$, let $\overline{I} := \{1, \ldots, n\} \setminus I$. We will argue that the n-graph G_I is isotopic to a standard graph. The fixed projection of G gives rise to a projection of G_I. Notice that on this projection the edges of G_I corresponding to I lie entirely above the rest of the projection. Thus they can be isotoped so that their interior lies in $z > 0$ without disturbing the rest of the projection. Let G_l (resp. G_u) denote the graph obtained from G_I by deleting all the edges corresponding to I (resp. to $\overline{I}$). Since $I \neq \emptyset$, G_l is a proper subgraph of G. Thus, by the Brunnian property, G_l is standard. Since $n \in \overline{I}$, by our assumption that G is canonical it follows that G_u is standard (in fact G_u is $G_{\bar{I}}$). Now it is not hard to see that the graph G_I is also standard. $\square$

To continue, let $\mathcal{BG}_n \subset \mathcal{G}_n$ denote the set of Brunnian Suzuki n-graphs and let k also denote the restriction of k on $\mathcal{BG}_n$. The following theorem says that the map

$$\mathcal{BG}_n \xrightarrow{k} \mathcal{K}_n,$$

is surjective.

Theorem 4.4. *Suppose $n \geq 3$. A knot K' is n-adjacent to the unknot iff there exists a Brunnian Suzuki n-graph $\mathbf{G}$, such that $K' = k(\mathbf{G})$.*

Proof: Suppose $\mathbf{G}$ is a Brunnian and let $G = S \cup (\cup_{i=1}^{n} a_i)$ be any representative of it. Then $k(\mathbf{G})$ is easily seen to be n-adjacent to the unknot. In fact, the embedding $K(G)$ satisfies the definition of n-adjacency to the unknot because of the following property:

Property 1: Let $\mathcal{C}_n$ denote a collection of crossings containing a single crossing from each of the clasps of $K(G)$. For every non-empty subset $I \subset \{1, \ldots, n\}$, the knot obtained from $K(G)$ by trivializing all the crossings in $\mathcal{C}_n$ corresponding to I is $K(G(I))$. (Recall that $G(I)$ denotes the graph obtained from G by deleting all the edges in $\{a_i \mid i \in I\}$.) Since $\mathbf{G}$ is Brunnian, $G(I)$ is a standard graph and hence $K(G(I))$ is the unknot.

Suppose now, conversely, that K' is n-adjacent to the unknot. It follows from [Su] (see also [Ya]) that there exists a weighted n-graph G such that: (i) we have $K' = K(G)$; (ii) a set of crossings, say $\mathcal{C}_n$, consisting from exactly one crossing from each clasp of $K(G)$ satisfies the definition of n-adjacency to the unknot; and (iii) Property 1 above is satisfied. After modifying by the move M1 of Figure 3 we may assume that G is zero framed. Since $\mathcal{C}_n$ satisfies the definition of n-adjacency to the unknot, $G(I)$ is nugadjacent for every $\emptyset \neq I \subset \{1, \ldots, n\}$. By Theorem 2.2 G is admissible and by Lemma 4.1 $G(I)$ is standard.

Now let $\mathbf{G}$ be the Suzuki n-graph represented by G. By our discussion above, $\mathbf{G}$ is Brunnian. By definition of the map k, clearly we have $K' = k(\mathbf{G})$. $\square$

We, now, turn our attention to Vassiliev invariants of knots that are n-adjacent to the unknot. Recall that, by definition, all the invariants of orders $\leq n-1$ vanish for such a knot. However, our analysis here allows us to obtain the following stronger result:

Theorem 4.5. *Let K' be a knot that is n-adjacent to the unknot for some $n \geq 3$. Then K' is $(2n-1)$-trivial and hence all its Vassiliev invariants of degrees less than $2n-1$ vanish.*

Proof: We may assume that K' is not the unknot since otherwise the conclusion is trivially true. Since K' is n-adjacent to the unknot, by

Theorem 4.4, there is a Brunnian n-graph $G = S \cup (\cup_{i=1}^{n} a_i)$ such that $K' = K(G)$. Assume that G is canonical as in the proof of Proposition 4.3.

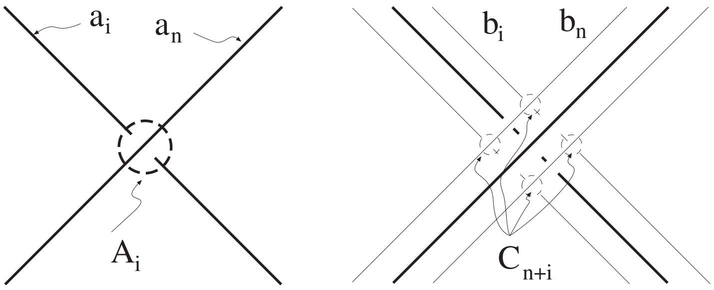

FIGURE 5. Crossings of bands resulting from crossings of arcs.

For $i = 1, \ldots, 2n-1$, define a set of crossings C_i as follows: For $i = 1, \ldots, n$, define C_i to be a set containing a crossing from the clasp of $K(G)$ corresponding to i. Moreover, let $A_1, \ldots, A_{n-1}$ be the sets of crossings on G, chosen in the proof of Proposition 4.3. Since K is the not the unknot by Lemma 4.1 G is not a standard graph. Thus, as argued in the proof of Proposition 4.3, each of $A_1, \ldots, A_{n-1}$ is non-empty. By our construction, each edge a_i is replaced by a band b_i in $K(G)$. Thus each crossing in some A_i gives rise to four crossings between b_i and a b_j, $j \neq i$, on the projection $K(G)$ (see Figure 5). For $i = 1, \ldots, n-1$, define C_{n+i} to be the set of crossings on $K(G)$ corresponding to A_i.

The sets $C_1, \ldots, C_{2n-1}$ are clearly disjoint; we claim that they exhibit the $(2n-1)$-triviality of K. To that end let $\emptyset \neq I \subset \{1, \ldots, 2n-1\}$ and let $C_I = \cup_{i \in I} C_i$. We must show K'_{C_I} is the unknot. This is trivially the case if there exists $x \leq n$ such that $x \in I$, but there is no $y > n$ with $y \in I$. Otherwise, it follows by Proposition 4.3 that K'_{C_I} is of the form $K(G^*)$ where G^* is a standard n-graph. Thus, K'_{C_I} is the unknot. $\square$

Remark 4.6. Theorem 4.5 is not true for $n = 2$. For example, both the trefoil knot and the figure-8 knot are 2-adjacent to the unknot but both have a nontrivial Vassiliev invariant of order two. The reason for this is that not every Suzuki 2-graph is admissible and Proposition 4.3 is not true for inadmissible graphs. It is not hard to see that an admissible 2-graph is, in fact, 1-similar to a standard graph. Thus one has the following: Suppose that $\mathbf{G}$ is an admissible Suzuki 2-graph. Then, the knot $k(\mathbf{G})$ is 3-trivial and thus all its Vassiliev invariants of order ≤ 2 vanish.

5. Abelian invariants

In this section we study the properties of abelian invariants of knots which are n-adjacent to the unknot. In particular, we are interested in the extent to which such invariants provide obstructions to a knot being adjacent to the unknot. We begin with the following:

Proposition 5.1. *Suppose that a knot K of genus g is n-adjacent to the unknot for some $n > g$. Then, one of the following is true:*
a) K is a twisted Whitehead double and $n = 2$.
b) We have $\Delta_K(t) = 1$, where $\Delta_K(t)$ denotes the Alexander polynomial of K.

Proof: For $n = 1$, the conclusion follows trivially since K is the unknot. If $n = 2$, then $g \leq 1$. By [S-T], a knot with genus and unknotting number one is a Whitehead double. Now the conclusion follows, since an untwisted double has trivial Alexander polynomial. Suppose now that $n \geq 3$ and that $\Delta_K(t) \neq 1$. Let $\nabla_K(z) = 1 + c_2 z^2 + c_4 z^4 + \dots c_{2k} z^{2k}$, denote the Conway version of the Alexander polynomial. Recall that $k \leq g$. Since $\nabla_K(z) \neq 1$, we have $c_{2i} \neq 0$, for some $1 \leq i \leq k$. Since c_{2i} is a Vassiliev invariant of order $2i$ ([B]), by Theorem 4.5 we must have $g \geq i \geq n$; which is a contradiction. Thus we must have $\Delta_K(t) = 1$. □

Next we will show the following stronger result:

Theorem 5.2. *If K is n-adjacent to the unknot for some $n \geq 3$, then $\Delta_K(t) = 1$.*

To prove Theorem 5.2, we will construct Seifert surfaces of K, using the properties of the Suzuki n-graphs underlying K, established in the previous sections. Let K be a knot that is n-adjacent to the unknot. It follows from Section 3 that $K = K(G)$, where G is a Brunnian n-graph. As in Section 4 we may assume that G is a canonical embedding. For each of the hooks of the embedding $K(G)$ we obtain a pair of bands attached to D as shown in Figure 6. If the graph G is contained, this process will lead to a Seifert surface $\Sigma(G)$, of K.

Lemma 5.3. *If G is a contained Brunnian n-graph, for some $n \geq 3$, then $K = K(G)$ has trivial Alexander polynomial.*

Proof: Let $\Sigma(G)$ be a genus n Seifert surface for K, obtained by the method described above. Let $u_1, v_1, \dots, u_n, v_n$ denote the symplectic basis of $H_1(\Sigma(G))$ represented by the cores of the bands of Figure 6. We claim that the Seifert matrix of $\Sigma(G)$ with respect to the ordered basis $\{v_1, \dots, v_n, u_1, \dots, u_n\}$, is of the form:

$$V = \begin{pmatrix} O & O \\ A & B \end{pmatrix} \text{ or } V = \begin{pmatrix} O & A \\ O & B \end{pmatrix}$$

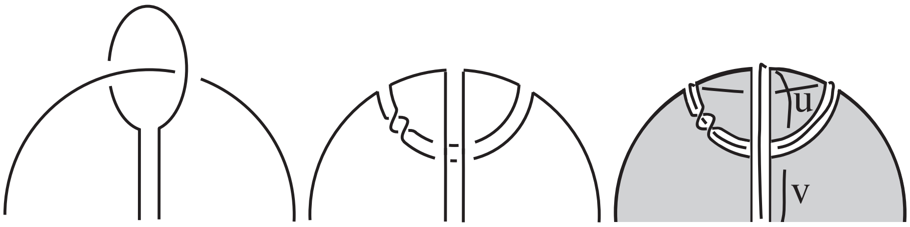

FIGURE 6. Modifying the hook parts of $K(G)$.

where each of A, B, O is an $n \times n$ matrix and all entries of O are zero. To see that, notice that since after deleting any of the edges $a_1, \ldots, a_{n-1}$ the resulting graph is standard, the linking number of v_n with any of $v_1, \ldots, v_{n-1}$ is zero. On the other hand, since G was adjusted to be zero framed, the self linking number of the v_i's is zero, for $i = 1, \ldots, n$. Finally, by our choice of G the linking number of any two of $v_1, \ldots, v_{n-1}$ is also zero. Since $\Delta_K(t) = \det(V - tV^T)$ ([Ro]), it follows that $\Delta_K(t) = 1$. $\square$

Let G be a canonical, Brunnian n-graph that is not contained. Then, the process outlined before the statement of Lemma 5.3 will produce an immersed surface of $K(G)$. Since the algebraic intersection number of $\mathrm{int}(a_n)$ and $\mathrm{int}(D)$ is zero, we can eliminate these intersection by attaching 2-handles to D. As Figure 7 indicates there might be a variety of ways to do that.

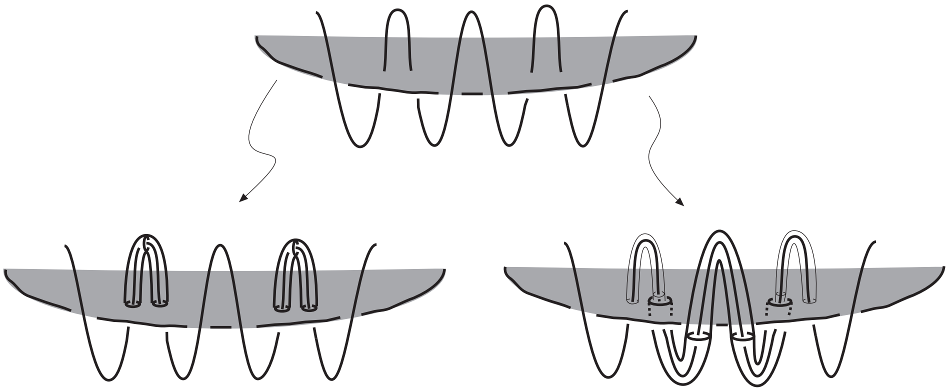

FIGURE 7. Two ways to attach 2-handles that remove the same set of intersections of a_n with the S-disc.

Lemma 5.4. *Let* $G = S \cup (\cup_{i=1}^{n} a_i)$ *be a canonical embedding of a Brunnian* n*-graph. Suppose that* $n \geq 3$*, and fix* $a_i \in \{a_1, \ldots, a_{n-1}\}$*. We can remove the intersections* $\mathrm{int}(a_n) \cap \mathrm{int}(D)$ *by attaching 2-handles whose cores have zero algebraic linking number with every* $a_j \neq a_i, a_n$*.*

Proof: Since G is Brunnian, for $i \in \{1, \ldots, n-1\}$ the graph, say $G(a_i)$, obtained from G by deleting a_i, is isotopic to the standard one. Since the algebraic intersection of $\operatorname{int}(a_n)$ and $\operatorname{int}(D)$ is taken to be zero it follows that the intersections of $\operatorname{int}(a_n)$ and $\operatorname{int}(D)$ in $G(a_i)$ may be eliminated in pairs of opposite sign by pushing subarcs of a_n above and below D. This means that there is a sequence of disjoint subarcs of a_n, say $\mathcal{B} = \{\gamma_1^i, \ldots, \gamma_t^i\}$, such that: (i) Each γ_j^i has both its endpoints on the same side of D; (ii) $\operatorname{int}(\gamma_j^i) \cap \operatorname{int}(D) = \emptyset$; (iii) in $G(a_i)$ each of $\gamma_1^i, \ldots, \gamma_t^i$ is geometrically unlinked from each a_k, $k \neq i, n$; and (iv)in $G(a_i)$, the arc obtained from a_n by an isotopy that removes the intersections with D corresponding to the endpoints of the arcs in $\mathcal{B}$, either is disjoint from $\operatorname{int}(D)$ or it contains a sequence of subarcs satisfying (i)-(iii).

The process described in (i)-(iv) above can be continued till all the intersections of $\operatorname{int}(a_n)$ and $\operatorname{int}(D)$ have been removed. The sequence of all the subarcs of a_n that satisfy (i)-(iv) at some stage of the unknotting process, will be called *an unknotting sequence with respect to* a_i. The arcs $\mathcal{B} = \{\gamma_1^i, \ldots, \gamma_t^i\}$ will be called the *initial* arcs of the unknotting sequence with respect to a_i. Notice that

$$\operatorname{lk}(\gamma_j^i, a_k) = 0,$$

for every $j = 1, \ldots t$ and $k \neq i, n$. For $j = 1, \ldots t$, let $\delta_j \subset D$ be an embedded arc connecting the endpoints of γ_j^i such that $\operatorname{int}(\delta_j) \cap \operatorname{int}(a_n) = \emptyset$ and δ_j is geometrically unlinked to a_n. Let $G_{\mathcal{B}}$ denote the graph obtained from G by replacing $\operatorname{int}(\gamma_j^i)$ with $\operatorname{int}(\delta_j)$ and then isotoping off D in the side opposite to which $\operatorname{int}(\gamma_j^i)$ lies.

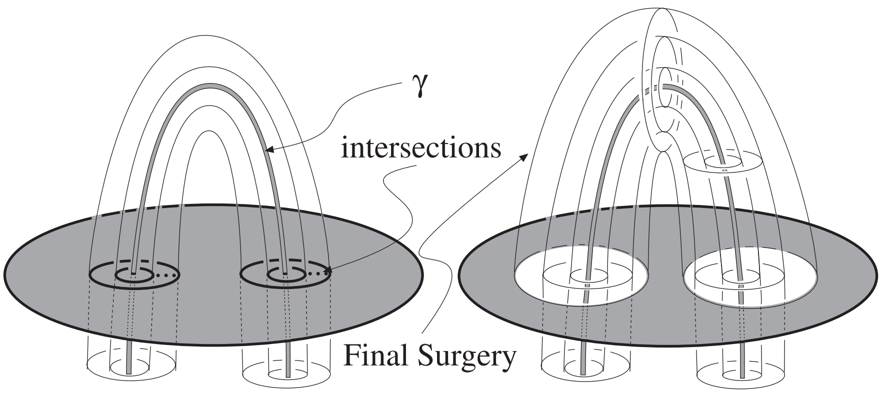

FIGURE 8

Inductively suppose that we have attached 2-handles, whose cores have zero algebraic linking number with every $a_j \neq a_i, a_n$, to remove the intersections $G_{\mathcal{B}} \cap \operatorname{int}(D)$. By isotopying these 2-handles back where the arcs in $\mathcal{B}$ where in the original graph G, we obtain an immersed surface for K. The self intersections of this surface occur in $\mathrm{int}(D)$ and are as indicated in Figure 8. Performing the final surgery shown in the right picture of Figure 8 results in an embedded Seifert surface of K with the desired properties. $\square$

Proof of Theorem 5.2: Let G be a canonical embedding of a Brunnian graph underlying the knot K. Let $\Sigma(G)$ denote the Seifert surface of Lemma 5.4 corresponding to the edge a_1. Construct an ordered basis $\Gamma(G) := \{v_2, u_2, v_1, u_1, \ldots, v_n, u_n, m_1, l_1, \ldots, m_r, l_r\}$ for $H_1(\Sigma^1(G))$ as follows: For $i = 1, \ldots, n$, u_i, v_i correspond to the band-hook pair of a_i as in Figure 6, and for $j = 1, \ldots, r$, m_j, l_j correspond to a pair of meridian-longitude of one of the 2-handles attached to $\mathrm{int}(D)$. Let

$$\theta : H_1(\Sigma(G)) \times H_1(\Sigma(G)) \longrightarrow \mathbf{Z},$$

denote the Seifert pairing of $\Sigma(G)$. Let G^* denote the graph obtained from G by changing all the under crossings of a_2 to over crossings; clearly G^* is a standard graph and thus $K(G^*)$ is the trivial knot. The surface $\Sigma(G)$ gives rise to a surface, say $\Sigma(G^*)$, for $K(G^*)$. Moreover $\Gamma(G)$ gives rise to symplectic basis, say $\Gamma(G^*)$, for $H_1(\Sigma(G^*))$. Let V (resp. V^*) denote the Seifert matrix of $\Sigma(G)$ (resp. $\Sigma(G^*)$) with respect to $\Gamma(G)$ (resp. $\Gamma(G^*)$). Since G represents a Brunnian Suzuki n-graph (and it was assumed to be zero framed)) we have

$$\theta(v_2, v_j) = 0,$$

for every $j = 1, \ldots, n$. By Lemma 5.4, we have

$$\theta(v_2, l_k) = 0,$$

for every $k = 1, \ldots, r$. Thus we obtain $V = V^*$ and

$$\Delta_K(t) = \det(V^* - tV^T) = \Delta_{K(G^*)}(t) = 1.$$

$\square$

Notice that the proof of Theorem 5.2 shows that a knot that is n-adjacent to the unknot for some $n \geq 3$ shares a common Seifert matrix with the unknot. Thus such a knot is S-equivalent to the unknot.

Remark 5.5. As already mentioned the class of knots that are n-adjacent to the unknot is a particular class of n-trivial knots. A more general class of n-trivial knots is this of *strongly n-trivial knots.* These are n-trivial knots in which each of the sets C_i of Definition 2.1 is either a single crossing or a set of twist crossings on two strings of K that inherit opposite orientations from any orientation of K. One can see that the results in Sections 4 and 5 generalize for strongly n-trivial knots. More specifically, with the notation of Section 3, first one can define generalized weighted n-graphs by allowing the weights of an n-graph $G = S \cup \cup_{i=1}^{n} a_i$ to be of the form $\{(w_i, z_i)\}_{i=1}^{n}$, where $w_i, z_i \in \mathbf{Z}$ (that is one removes the restriction that $z_i = \pm 1$). Given such a generalized weighted n-graph G, one can construct a knot $L(G)$ by doing finger moves of S along the edges $a_1, \ldots a_n$ in a way similar to this described in Section 3. The only difference is that now the "hook part" corresponding to the terminal point of an edge a_i will consist of $|z_i|$ clasps the sign of which is going to be determined by this of z_i. In this setting, Lemma 4.1 generalizes as follows: Suppose that $G := S \cup (\cup_{i=1}^{n} a_i)$ is a generalized weighted n-graph such that a knot $L(G)$, associated to G as above, is the trivial knot. For $i = 1, \ldots, n$, let C_i be a set of crossings that trivializes the clasps of $L(G)$ corresponding to the hook part of a_i. Suppose that the collection $\{C_i\}_{i=1}^{n}$, satisfies the definition of strong n-triviality. Then G is isotopic to a standard n-graph. Having this at hand, an argument similar to this in the proof of Theorem 4.4 shows that a knot K is strongly n-trivial if and only if there exists a generalized weighted Brunnian n-graph such that $K = L(G)$. Finally, the conclusions of Theorems 4.5 and 5.2 are true for strongly n-trivial knots and the proofs of the theorems go through with minor adjustments. Thus one has the following:

Theorem 5.6. *Suppose that K is a knot that is strongly n-trivial for some $n \geq 3$. Then, the following are true:*

a) K is $(2n-1)$-trivial and hence all its Vassiliev invariants of degrees less than $2n-1$ vanish.

b) We have $\Delta_K(t) = 1$.

6. Nilpotent invariants

We begin by recalling some terminology and notation from [K-L]. A Seifert surface Σ of a knot K, is called *regular* if it admits a spine Γ whose embedding in $\mathbf{R}^3$ induced by an embedding of Σ, is a bouquet of circles each of which is unknotted. The spine Γ is called a *regular* spine. It is not hard to see that if a Seifert surface S of genus g contains a set $B = \{v_1, u_1, \ldots, v_g, u_g\}$ of simple, closed, non-separating curves such that i) each curve in B is unknotted; and ii) B represents a basis of $H_1(\Sigma)$ that is symplectic with respect to the usual intersection form, then S is regular. A regular spine can be easily constructed from B. Each of the sets $\{v_1, \ldots, v_g\}$ and $\{u_1, \ldots, u_g\}$ is called *an 1/2- basis.* Set $\pi := \pi_1(S^3 \setminus \Sigma)$ and let $B^* = \{x_1, y_1, \ldots, x_g, y_g\}$ denote elements in π representing a basis of $H_1(\mathbf{R}^3 \setminus \Sigma)$ that is dual to the one of $H_1(\Sigma)$ represented by B. For a subset $A \subset B$, let $\mathcal{G}_A$ denote the normal subgroup of π generated by the subset of B^* corresponding to A. Moreover, let π_A (resp. ϕ_A) denote the quotient $\pi/\mathcal{G}_A$ (resp. the quotient homomorphism $\pi \longrightarrow \pi/\mathcal{G}_A$). If $A = \emptyset$ set $\pi_A := \pi$. For $n >$, let $\pi_A^{(1)} := \pi_A$ and $\pi_A^{(n)} := [\pi_A^{(n-1)},\ \pi_A]$.

Definition 6.1. *([K-L, Definition 3.1]) Let $n \in \mathbf{N}$. A regular Seifert surface Σ is called n-hyperbolic, if it contains an 1/2-basis $\mathcal{A}$ represented by circles in a regular spine Γ with the following property: There is an ordering, $v_1, \dots, v_g$, of the elements in A such that either $\phi_{A_{i-1}}([v_i^+])$ or $\phi_{A_{i-1}}([v_i^-])$ lies in $\pi_{A_{i-1}}^{(n)}$. Here $A_k = \{x_1, y_1, \dots, x_k, y_k\}$ for $k = 1, \dots, g$ and A_0 is the empty set. The boundary of such a surface will be called an n-hyperbolic knot.*

One can see that the Alexander polynomial on an n-hyperbolic knot ($n > 1$) is trivial. On the other hand, it is known that for every $n \in \mathbf{N}$, there are n-trivial knots with non-trivial Alexander polynomial. The following conjecture asserts that the only obstruction to an n-trivial knot being n-hyperbolic is the Alexander polynomial.

Conjecture 6.2. *There exists an increasing function $m : \mathbf{N} \to \mathbf{N}$ such that the following is true: If K is a knot with trivial Alexander polynomial that is n-trivial for some $n \in \mathbf{N}$, then it is $m(n)$-hyperbolic.*

To state our main result in this section we need some notation and terminology. Let $G = S \cup (\cup_{i=1}^n a_i)$ be a canonical embedding of a Brunnian n-graph. Recall that S bounds a disc D on the plane R. Let $p_1, p_2, \dots, p_s, p_{s+1}$ denote the points in $a_n \cap D$, in the order met following the orientation of a_n. A subarc of a_n inherits an orientation from that of a_n; thus we can talk of its initial and terminal point. For $i = 1, \dots s$, let γ_i denote the subarc with initial point p_i and terminal point p_{i+1}. The arcs in $\mathcal{A} := \{\gamma_1, \dots \gamma_s\}$ are partitioned into four types:

- Type I: Arcs both of whose endpoints approach $\mathrm{int}(D)$ from $z > 0$;
- Type II : Arcs both of whose endpoints approach $\mathrm{int}(D)$ from $z < 0$;
- Type III_1: Arcs with their initial point approaching D from $z > 0$ and their terminal point from $z < 0$; and
- Type III_2: Arcs with their initial point approaching D from $z < 0$ and their terminal point from $z > 0$.

Definition 6.3. *A canonical n-graph G is called simple if either it is contained or the following are true: i) The set $\mathcal{A}$ doesn't contain any arcs of type III_1 or III_2; and ii) for every $\alpha \in \{a_1, \dots, a_{n-1}\}$ the arcs in $\mathcal{A}$ that are geometrically linked to α are either all of type I or all of type II. A knot K' is called simply n-adjacent to the unknot if K' can be represented in the form $K(G)$, with G a simple Brunnian n-graph.*

Our main result in this section is the following:

Theorem 6.4. *Let K' be a knot that is simply n-adjacent to the unknot, for some $n > 1$. Then, K' is $(n-1)$-hyperbolic.*

In [K-L], it is shown that a knot that is n-hyperbolic for all $n \in \mathbf{N}$ is, in fact, n-trivial for all $n \in \mathbf{N}$. A proof of Conjecture 6.2 would imply the converse statement. Thus one would have that a knot is n-hyperbolic for all $n \in \mathbf{N}$ iff n-trivial for all $n \in \mathbf{N}$ (the conjecture was stated in this form in [K-L]). To that respect, Theorem 6.4 implies that a knot that is simply n-adjacent to the unknot for all $n \in \mathbf{N}$ is n-hyperbolic for all $n \in \mathbf{N}$. In fact, a stronger statement is known in this case. Howards and Luecke showed that a knot that is n-adjacent to the unknot for all $n \in \mathbf{N}$ is the unknot ([HL]).

Next we prove the lemmas needed to complete the proof of Theorem 6.4. For a simple closed curve u on a Seifert surface Σ we will use u^+ and u^- to denote the curves obtained by pushing u in the complement of Σ, in the two sides of Σ.

Lemma 6.5. *Let K be a knot that is n-adjacent to the unknot for some $n > 1$ and let G be a canonical embedding of a Brunnian n-graph with $K = K(G)$. If G is contained then K is $(n-1)$-hyperbolic.*

Proof: Let $\Sigma = \Sigma(G)$ be the Seifert surface of K constructed as indicated in Figure 6. Let $\Gamma(G) := \{v_1, u_1, \dots, v_n, u_n\}$ denote the basis of $H_1(\Sigma)$ corresponding to the band-hook pairs of $K(G)$ such that the $\frac{1}{2}$-basis $\{v_1, \dots, v_n\}$, corresponds to the cores of the bands. Let $\{x_1, y_1, \dots, x_n, y_n\}$ denote the basis of $H_1(\mathbf{R}^3 \setminus \Sigma)$ which is dual to $\Gamma(G)$. Σ is a regular surface with $\Gamma(G)$ a regular spine so the conclusion holds for $n = 2$. For $n > 2$ the pairwise linking numbers (as well as the self linking numbers) of the edges $a_1, \dots, a_n$ of G will vanish. By the construction of Σ, for every $i = 1, \dots, n$, one of the push off's v_i^+ or v_i^-, say v_i^+, will satisfy $\mathrm{lk}(v_i^+, z) = 0$, for every $z \in \Gamma(G)$. Fix $i \in \{1, \dots, n\}$ and let W_i denote the word represented by v_i^+ in π; W_i is a word in $x_1, \dots, x_n$. For $j \in \{1, \dots, n\}$ with $j \neq i$, let S_{ij} denote the subset of letters in W_i containing all appearances of x_j. Let $\mathcal{S}_i := \{S_{ij} \mid j \neq i\}$. Since G is Brunnian, and by the proof of Lemma 4.1, we can see that the following is true: For every non-empty subset $S \subset \mathcal{S}_i$, the word obtained from W_i by deleting all the letters in the union of all the sets in S simultaneously, is the identity in π. By Theorem 4.2 of [N-St], it follows that $W_i \in \pi^{(n-1)}$. Thus $\{v_1, \dots, v_n\}$, satisfies the definition of $(n-1)$-hyperbolicity. □

Let $G = S \cup (\cup_{i=1}^{n} a_i)$ be a simple Brunnian n-graph that is not contained. By definition, the arcs in $\mathcal{A}$ are type I or type II and every $\alpha \in \{a_1, \dots, a_{n-1}\}$ is geometrically linked to arcs of one type only. Let $B \subset \{a_1, \dots, a_{n-1}\}$ denote the subset of α's that are linked to only arcs of type I. If $B = \emptyset$, then G can be isotoped to be contained. Suppose that $B \neq \emptyset$. We can isotope the interior of each type I arc in $z < 0$. This isotopy eliminates intersection points of $\mathrm{int}(a_n) \cap \mathrm{int}(D)$ and introduces intersections of $\mathrm{int}(D)$ with the interiors of some of the arcs in B. By further isotopy we can assure that: i) $\mathrm{int}(a_n) \cap \mathrm{int}(D) = \emptyset$; and ii) the interior of every arc $b_i \in B$ intersects $\mathrm{int}(D)$ at two points of opposite sign, say r_i, t_i. Furthermore, the subarc of each b_i with endpoints r_i, t_i is unknotted and subarcs corresponding to different b_i's are geometrically unlinked. We have shown the following:

Lemma 6.6. *Let $G = S \cup (\cup_{i=1}^n a_i)$ be a simple Brunnian n-graph that is not contained. There exists a subset $\emptyset \neq B \subset \{a_1, \ldots, a_{n-1}\}$ such that after an isotopy of G that doesn't change the knot type of $K(G)$:*

(i) $\mathrm{int}(a_n) \cap \mathrm{int}(D) = \emptyset$.

(ii) The interior of an arc $b_i \in B$ intersects $\mathrm{int}(D)$ *at two points r_i, t_i of opposite sign. The subarc $\hat{b}_i \subset b_i$ with $\partial(\hat{b}_i) = \{r_i, t_i\}$ is unknotted and subarcs corresponding to different b_i's are geometrically unlinked.*

(iii) All the crossings between b_i and a_n occur along $\hat{b}_i$.

Let G be a simple Brunnian n-graph G isotoped so that (i)-(iii) of Lemma 6.6 are satisfied. We can attach 2-handles, $h_1, \ldots, h_r$ to D along the arcs $\hat{b}_1 \ldots \hat{b}_r$ respectively, to obtain a Seifert surface $\Sigma(G)$ for $K(G)$. Let $\Gamma(G) := \{v_1, u_1, \ldots, v_n, u_n, m_1, l_1, \ldots, m_r, l_r\}$ a spine of $\Sigma(G)$ obtained as follows: For $j = 1, \ldots, n$, v_j, u_j correspond to the band-hook pair of a_j and for $k = 1, \ldots, r$, m_k, l_k correspond to a pair of meridian-longitude of one of the 2-handles attached to D. We can see that $\Sigma(G)$ is a regular surface and $\Gamma(G)$ is regular spine of $\Sigma(G)$.

Lemma 6.7. *For every $k = 1, \ldots, r$, one of l_k^+, l_k^- represents an element in $\pi^{(n-1)}$. Furthermore for every $a \in \{a_1, \ldots, a_n\} \setminus B$, one of v_a^+, v_a^- represents an element in $\pi^{(n-1)}$. Here, v_a denotes the element in $\Gamma(G)$ corresponding to the band part of a.*

Proof: We prove the assertion for l_k; the argument in the remaining cases is similar. By Lemma 6.6 and our construction of $\Gamma(G)$, for every $k = 1, \ldots, r$ and $z \in \Gamma(G)$, one of l_k^+, l_k^-, say l_k^+, satisfies $\mathrm{lk}(l_k^+, z) = 0$. Fix $k = 1, \ldots, r$, and let E_k denote the complement of $\{b_k\}$ in $\{a_1, \ldots, a_n\}$. For a non-empty set $E \subset E_k$, let Σ_E denote the Seifert surface obtained from $\Sigma(G)$ as follows: For every $a \in E$, if $a \notin B$, remove from $\Sigma(G)$ the dual pair of bands from $v_1, u_1, \ldots, v_n, u_n$ corresponding to a. If $a \in B$ then remove the dual pairs of bands from $v_1, u_1, \ldots, v_n, u_n$ and from $l_1, m_1, \ldots, l_n, m_n$ corresponding to a. Let l_k^E denote the image of l_k^+ on Σ_E. The Brunnian property of G implies that l_k^E represents the identity in $\pi_1(\mathbf{R}^3 \setminus \Sigma_E)$. An argument similar to this in the proof of Lemma 6.5 shows that l_k^+ lies in $\pi^{(n-1)}$. □

Proof of Theorem 6.4: Let K be a knot that is simply n-adjacent to the unknot, for some $n > 1$, and let G be a simple Brunnian n-graph with $K = K(G)$. If G is contained, the result follows from Lemma 6.5. Otherwise, isotope G to satisfy the properties of Lemma 6.6. Let $\Sigma(G)$ and $\Gamma(G)$ be a Seifert surface and the corresponding spine constructed out of G as described before the statement of Lemma 6.7. It follows by Lemma 6.7 that the ordered 1/2-basis $\{l_1, \ldots l_r, v_1, \ldots, v_n\}$ satisfies the definition of $(n-1)$-hyperbolicity. □

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