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KNOT AND 3-MANIFOLD INVARIANTS AND EXCEPTIONAL DEHN SURGERIES

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Contents

[1]These notes were written long before [KM] where property P is proven
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1. Introduction

For a knot K in S^3, let $V_K(t)$ denote the Jones polynomial of K ([J]) and let $\nabla_K(z)$ denote the Alexander-Conway polynomial of K. We consider the following question: To what extend do $V_K(t)$ and $\nabla_K(z)$ provide obstructions to the existence of non-trivial Dehn fillings of $S^3 \setminus K$ that yield simply connected 3-manifolds?

Let $v_i(K)$ denote the i-th derivative of $V_K(e^{-x})$ at $x = 0$ and let $c_i(K)$ denote the i-th derivative of $\nabla_K(z)$ at $z = 0$. The invariants $v_i(K)$ and $c_i(K)$ are known to be of finite type in the sense of [V]. In this paper we prove the following:

Theorem 1.1. *Let K be a knot in S^3. If at least one of $v_2(K)$, $v_3(K)$, $v_4(K)$, $c_2(K)$, $c_4(K)$ is non-zero then K has property P.*

A knot K is said to have property P if no non-trivial Dehn surgery of S^3 along K yields a simply connected 3-manifold. For simple homological reasons, this is equivalent to saying that $\pi_1(M_K^n) \neq \{1\}$ for every $n \neq 0$, where M_K^n denotes the homology 3-sphere obtained by $\frac{1}{n}$-Dehn surgery along K. Furthermore, by a result of Culler, Gordon, Luecke and Shalen ([CGLS]) it is equivalent to saying that $\pi_1(M_K^{\pm 1}) \neq \{1\}$. The property P conjecture, which by a result of Gordon and Luecke ([GL]) is implied by the Poincaré conjecture, states that this is true for every non-trivial knot.

Dehn surgery on knots has received much attention in the last decades and important progress has been made in many open questions. In particular, the property P conjecture has motivated a great deal of research and it is known for many classes of knots (see, for example, [GT] for references and discussion)[1] On the other hand, the Jones type and finite type knot invariants were introduced in the eighties and nineties respectively, and have since been studied extensively. Despite a lot of activity in both areas, before the results of this paper, there was nothing known about the relation of any of these invariants to the fundamental group of 3-manifolds obtained by surgery on knots, save for Casson's criterion; if $c_2(K) \neq 0$ then K has property P. This criterion is a consequence of the surgery formula of Casson's invariant of $\mathbf{Z}$-homology 3-spheres ([AM]). The Casson invariant $\lambda(M)$ counts algebraically the number of conjugacy classes of irreducible representations of $\pi_1(M)$ into $SU(2)$ and it vanishes if $\pi_1(M)$ admits no non-trivial such representation. In particular, $\lambda(M) = 0$ if $\pi_1(M) = \{1\}$.

It should be pointed out here that Theorem 1.1 provides a new proof of Casson's criterion that is independent of the construction of a 3-manifold invariant using $SU(2)$ representation spaces. On the other hand, Theorem 1.1 can be used to construct new families of knots with property P (see Section 6).

In [O1], Ohtsuki defined a formal power series topological invariant $\tau(M) = \Sigma_{i=0}^{\infty} \lambda_i(M) h^i \in \mathbf{Q}[[h]]$, for oriented $\mathbf{Z}$-homology 3-spheres. The invariant $\tau(M)$ is extracted from the $SU(2)$ invariants of Reshetikhin and Turaev through an interpolation process and it is expected to correspond to the asymptotic expansion of Witten's Chern-Simons path integral. The invariants λ_n are known to be of finite type in the sense of [O] and, as shown by Murakami ([Mu]), one has $\lambda_0 = 1$ and $\lambda_1(M) = 6\lambda(M)$. On the other hand, attempts to relate known higher rank Casson type invariants (e.g. the $SU(3)$ invariant of Boden and Herald) to the Ohtsuki invariants have so far failed ([BoHKK]). Thus the question of whether there exist geometric interpretations, analogous to this for λ_1, for any of the higher order Ohtsuki invariants remains completely open. In particular, it is not known whether any higher order Ohtsuki invariant vanishes for simply connected 3-manifolds. As a first step towards this direction, we prove the following:

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Theorem 1.2. *Let M be a **Z**-homology 3-sphere obtained by surgery on a knot in S^3. If M simply connected then $\lambda_1(M) = \lambda_2(M) = 0$.*

Let us now describe more carefully the contents of the paper and the techniques used in the proof of Theorems 1.1 and 1.2. We should first point out that we only use very basic elements from the theory of finite type knot and 3-manifold invariants. On the other hand, we make essential use of powerful results from Dehn surgery ([CGLS]). One of our main tools is also an elegant characterization of homotopy 3-spheres in terms of their Heegaard diagrams that can be found in [RR]. We employ this characterization to obtain surgery presentations of homotopy 3-spheres that are suitable for the study of their finite type invariants. The results from [RR] needed in this paper, together with the notion of finite type invariants for **Z**-homology 3-spheres, are summarized in Section 2.

To explain our approach in more detail, for a framed link $L \subset S^3$ let $\chi(S^3, L)$ denote the 3-manifold obtained by surgery of S^3 along L. In Section 3, we consider the set $\mathcal{M}$ of 3-manifolds obtained by ± 1-surgery on knots in S^3 and the set $\mathcal{HM} \subset \mathcal{M}$ of homotopy 3-spheres in $\mathcal{M}$. Using a refinement of a technique of [BP], for a 3-manifold $M \in \mathcal{M}$ and a knot K such that $M = \chi(S^3, K)$, we construct a unit-framed, algebraically split link L, that satisfies certain technical conditions (see Lemma 3.2). These conditions, among other things, guarantee the following properties:

i) $M = \chi(S^3, L)$;

ii) the components of L are in one to one correspondence with the curves in a "nice" Heegaard diagram of M;

iii) for every sublink $L' \subset L$ we have $\chi(S^3, L') \in \mathcal{M}$.

Furthermore, for every sublink $L' \subset L$ there exists a knot $K_{L'}$ with $\chi(S^3, L') = \chi(S^3, K_{L'})$, and such that $K_{L'}$ can be obtained from K by a sequence of crossings changes; one for each component of $L \setminus L'$.

The link L is called a *nice link* associated to K and the Heegaard splitting of M corresponding to L is called a *nice Heegaard splitting*. At the end of section 3, we turn our attention to nice links and their corresponding Heegaard splittings associated to 3-manifolds in $\mathcal{HM}$.

The main result is, roughly speaking, the following: If $M \in \mathcal{HM}$ and L is a nice link with $M = \chi(S^3, L)$, there exists a collection of disjoint sets, say $\mathcal{R} := \{R_1, \dots, R_h\}$, of components of L such that the 3-manifolds obtained by surgery on sublinks of L obtained by removing unions of any of the sets in $\mathcal{R}$ are also in $\mathcal{HM}$. See Theorem 3.7 for the exact statement. The number $h(L)$, defined to be the smallest cardinality over all sets $\mathcal{R}$ that have the aforementioned properties, is an invariant of the link L. This invariant gives rise to a knot complexity. More precisely, given $K \subset S^3$ we define $\mu(K)$ to be, roughly speaking, the lowest $h(L)$ over all nice links L associated to K.

The proof of Theorem 3.7, which is based on a careful analysis of nice Heegaard splittings of 3-manifolds in $\mathcal{HM}$ using a result of [RR], is given in Section 7. The surgery presentations of Theorem 3.7 are very useful for studying finite type invariants of homotopy 3-spheres as well as for relating the question of property P of knots to the vanishing of certain Vassiliev invariants. In Sections 4, 5 and 6 we prove Theorem 1.1 and Theorem 1.2 assuming Theorem 3.7. In Section 4, we consider the set $\mathcal{HK}$ of knots K for which $\pi_1(M_K^{+1}) = \{1\}$ or $\pi_1(M_K^{-1}) = \{1\}$. Using our work in Section 3 we show the following: If $K \in \mathcal{HK}$ and $\mu(K) = 3$, then K admits a projection containing three disjoint sets of crossings such that changing all the crossings in any of these sets simultaneously results to the unknot. This, in turn, combined with an idea from [AsK] leads to the following:

Proposition 1.3. *Let $K \in \mathcal{HK}$ and let U denote the unknot. Let v be a Vassiliev invariant of order ≤ 4, with values in an abelian group $\mathcal{G}$. Then $v(K) = v(U)$.*

A consequence of the Cyclic Surgery Theorem of [CGLS] is that the only Dehn surgeries on a knot in S^3 that could yield a simply connected 3-manifold are these with framing ± 1. This combined with Proposition 1.3 yields the following theorem:

Theorem 1.4. *Let K be a non-trivial knot in S^3 and let U denote the unknot. Suppose that there exists a Vassiliev invariant v of order ≤ 4, with values in some abelian group $\mathcal{G}$, such that $v(K) \neq v(U)$. Then K has property P.*

Theorem 1.1 follows immediately from Theorem 1.4 and the fact that $v_2(K)$, $v_3(K)$, $v_4(K)$, $c_2(K)$, $c_4(K)$ are rational Vassiliev invariants for K of order at most four (see, for example, [BLi]). As a corollary of Theorem 1.4 and some properties of the finite type 3-manifold invariants we obtain the following:

Corollary 1.5. *Suppose that M is a homotopy 3-sphere obtained by surgery on a knot in S^3. Let λ be a finite type 3-manifold invariant of order ≤ 6, with values in an abelian group $\mathcal{G}$. Then $\lambda(M) = \lambda(S^3)$.*

Theorem 1.2 now follows from Corollary 1.5 and the fact that λ_1, λ_2 are finite type invariants of order three and six, respectively ([KrS]). Alternatively, Theorem 1.2 follows from Theorem 1.1 and the surgery formulas for λ_1, λ_2. The vanishing results of 3-manifold invariants are stated and proved in Section 5. The techniques used in the proofs of Proposition 1.3 and Theorem 1.4 lead to more general conditional property P criteria. For example, in Section 6 we show the following:

Corollary 1.6. *Let $m \in \mathbf{N}$ and assume that every knot $K_1 \subset S^3$ with $\mu(K_1) \leq m$ has property P. Suppose that K is a knot for which there is a Vassiliev invariant v of order $\leq m+2$, with values in some abelian group $\mathcal{G}$, such that $v(K) \neq v(U)$. Then K has property P.*

In Section 6, we discuss applications of Theorem 1.1 to the property P question for various classes of knots. In particular, we show that fibered knots of genus at most two have property P and we indicate how to use known techniques that produce knots with the same Vassiliev invariants of orders ≤ 4 to construct new families of knots with property P. We also remark that a result of [CGLS], that certain periodic knots have property P, can also be derived using Theorem 1.1.

Section 7 is devoted to the proof of Theorem 3.7. To briefly describe the idea of the proof let $M \in \mathcal{HM}$ and let L be a nice link with $M = \chi(S^3, L)$. Let $(S, \mathbf{f}, \mathbf{x})$ be a nice H-diagram associated to L, where S is the Heegaard surface and $\mathbf{f}, \mathbf{x}$ are complete systems of curves such that the curves in $\mathbf{f}$ are in one to one correspondence with the components of L. Let $H(\mathbf{x})$ denote the handlebody component of $M \setminus S$ whose meridian discs are bounded by the curves in $\mathbf{x}$. According to [RR], there exists an embedding $\phi : H(\mathbf{x}) \longrightarrow \mathbf{S^3}$ such that the curves in $\mathbf{f}$ bound disjoint oriented surfaces $S_1, \ldots, S_m$ in the complement $\phi(H(\mathbf{x}))$. To continue, for a collection of curves $\mathbf{x}' \subset \mathbf{x}$, let $H_{\mathbf{x}'}$ denote the handlebody obtained from $H(\mathbf{x})$ by cutting along the meridian discs corresponding to the curves in $\mathbf{x}'$ and let L' denote the link obtained from L by removing the components corresponding to $\mathbf{x}'$. Set $\mathbf{x}'' := \mathbf{x} \setminus \mathbf{x}'$. The diagram $(S, \mathbf{f}, \mathbf{x})$ can always be chosen so that there is collection of curves $\mathbf{f}' \subset \partial H_{\mathbf{x}'}$ such that the H-diagram $(\partial H_{\mathbf{x}'}, \mathbf{f}', \mathbf{x}'')$ represents $M_{L'} = \chi(S^3, L')$. The strategy of the proof of Theorem 3.7 is to show that actually $(S, \mathbf{f}, \mathbf{x})$ can be chosen so that there exist appropriate collections $\mathbf{x}' \subset \mathbf{x}$ for which the fixed embedding ϕ gives raise to embeddings $\phi_{\mathbf{x}'}$ such that the curves in $\mathbf{f}'$ bound disjoint oriented surfaces in the complement of $\phi_{\mathbf{x}'}(H_{\mathbf{x}'})$. Then, the results of [RR] guarantee that $M_{L'} \in \mathcal{HM}$. To construct these surfaces we need to analyze the geometric combinatorics of the surfaces $S_1, \ldots, S_m$ above. This is done by studying certain planar graphs that are obtained by intersecting $S_1, \ldots, S_m$ with certain embedded 3-spheres, called *a containing collection for* $\phi(H(\mathbf{x}))$. An important ingredient for this part of the argument is a result from Thurston's theory of curves on surfaces that characterizes the circumstances under which the geometric intersection of a pair of embedded curves on a surface can be reduced by isotopy on the surface. This result is recalled in the Appendix of the paper, where we also carry out a more detailed study of nice Heegaard splittings and we prove some technical results needed in Section 7.

Throughout the entire paper we work in the smooth or PL category.

2. Preliminaries

2.1. **Special Heegaard splittings of 3-manifolds.**

In this section, we summarize the results about Heegaard diagrams of 3-manifolds that we will need in the rest of this paper. For basic notions and definitions about links, 3-manifolds and braid groups the reader is referred to [Ro] and [B].

A *complete system* on a closed oriented surface S, is a set $\mathbf{x} = \{x_1, x_2, \dots, x_g\}$ of simple, closed, mutually disjoint curves, such that the result of cutting S along $\mathbf{x}$ is a 2-sphere with $2g$ holes. Given a pair of a closed surface and a complete system $(S, \mathbf{x})$, we can construct a solid handlebody $H(\mathbf{x})$ by attaching a thickened 2-disc, say D_i, along each x_i and gluing in a 3-ball to the 2-sphere resulting from this process. A *Heegaard diagram* (H-diagram) is a closed surface S with two complete systems $\mathbf{x}$, $\mathbf{y}$. An H-diagram, denoted by $(S, \mathbf{x}, \mathbf{y})$, defines a closed oriented 3-manifold

$$M(\mathbf{x}, \mathbf{y}) = H(\mathbf{x}) \cup_S H(\mathbf{y})$$

where $M(\mathbf{x}, \mathbf{y})$ has a handle decomposition such that the thickened discs attached to $\mathbf{x}$ (resp. $\mathbf{y}$) are the 2-handles (resp. 1-handles) and the 3-ball which forms $H(\mathbf{x})$ (resp. $H(\mathbf{y})$) is the only 3-handle (resp. 0-handle).

For simple closed curves $\alpha, \beta \subset S$, let $|\alpha \cap \beta|$ denote the number of points of intersection of α and β (geometric intersection). A genus g H-diagram $(S, \mathbf{x}, \mathbf{y})$, representing S^3,is called *standard* if and only if we have

$$|x_i \cap y_j| = \delta_{ij},$$

for every for $i, j = 1, \dots, g$. The Heegaard splitting $S^3 = A_g \cup_S B_g$, defined by a standard H-diagram will be called the *standard* Heegaard splitting of genus g for S^3. We will think of B_g as a 3-ball D^3 with a collection of 2-handles $h_1, \dots, h_g$, attached to ∂D^3 such that: For $i = 1, \dots, g$, the co-core of h_i is the meridional disc of B_g attached to y_i, and the core of h_i is an arc x_i^1 with $\partial x_i^1 \subset \partial D^3$ and such that $x_i = x_i^1 \cup x_i^2$, where x_i^2 is an arc on ∂D^3. We will say that a simple closed curve $\alpha \subset S$ runs *over* (resp. *under*) h_i if $|a \cap y_i| \neq \emptyset$ (resp. $|a \cap x_i^2| \neq \emptyset$).

Recall that, for a framed link $(L, \mathbf{f})$, $\chi(S^3, L)$ denotes the 3-manifold obtained by surgering S^3 along L. The framing $\mathbf{f}$ can be thought of as an m-tuple of integers (one for each component of L) or alternatively as a collection of simple closed curves, say $\{f_1, \dots, f_m\}$, such that f_i lies on the boundary of a tubular neighborhood of the i-th component of L and intersects the meridian exactly once. These two interpretations will be used interchangeably throughout the paper. We need the following theorem.

Theorem 2.1. *(*[BP]*, Theorems 4.1 and 4.2.) Let $(L, \mathbf{f})$ be an algebraically split, unit-framed, m-component link that is a pure plat-closure. There exists a standard Heegaard splitting $A_m \cup_S B_m$ of S^3, with a corresponding H-diagram $(S, \mathbf{x}, \mathbf{y})$, and a complete system of curves $\mathbf{f} := \{f_1, \dots, f_m\}$ on $S := \partial B_m$ such that:*

i) The H-diagram $(S, \mathbf{f}, \mathbf{x})$ represents $\chi(S^3, L)$.

ii) The framed link whose components are the curves in $\mathbf{f}$ and the framings are determined by parallel curves on S is $(L, \mathbf{f})$.

iii) The H-diagram $(S, \mathbf{f}, \mathbf{y})$ is standard. Thus, we have $|y_i \cap f_j| = \delta_{ij}$.

iv) The geometric intersection number $|x_i \cap f_i|$ is odd while the corresponding algebraic intersection number is equal to the framing of L_i. Furthermore, for $j \neq i$, $|x_i \cap f_j|$ is even while the corresponding algebraic intersection number is zero.

In this paper we will be concerned with special classes of algebraically split unit-framed pure plat-closures (see Definition 3.3). The Heegaard diagrams associated to such links turn out to have additional properties that are needed in this paper. The technical results establishing these properties are proven in the Appendix to the paper (Section 8). Very often, throughout the paper, we will be dealing with H-diagrams of homotopy 3-spheres. The following theorem, a proof of which is given by Rêgo and Rourke in [RR], characterizes such 3-manifolds in terms of their corresponding H-diagrams:

Theorem 2.2. *(*[RR]*, Theorem 1.) An H-diagram $(S, \mathbf{x}, \mathbf{y})$ represents a homotopy 3-sphere if and only if there exists an embedding of $H(\mathbf{y})$ in S^3 such that the curves $x_1, x_2, \dots, x_m$ bound disjoint, orientable surfaces $S_1, S_2, \dots, S_m$ in $\overline{S^3 \setminus H(\mathbf{y})}$.*

2.2. **Invariants of homology 3-spheres.**

We begin by recalling from [O] the definition of *finite type* invariants for $\mathbf{Z}$-homology spheres. An m-component link L in a $\mathbf{Z}$-homology sphere M will be called *admissible*, if it is ± 1-framed and in the case that $m > 1$, L is algebraically split (i.e. any two components of L have zero linking number). For such a link $L \subset M$, we will denote by $\chi(M, L)$ the $\mathbf{Z}$-homology 3-sphere obtained from M by surgery on L.

Let $\mathcal{G}$ be an abelian group and let $\mathcal{N}$ be the $\mathcal{R}$-module spanned by the set of homeomorphism classes of oriented $\mathbf{Z}$-homology spheres. For $M \in \mathcal{N}$ and an admissible link $L \subset M$, define $[M,\ L] \in \mathcal{N}$ by

$$[M,\ L] := \sum_{L' \subset L} (-1)^{\# L'} \chi(M, L') \ ,$$

where L' ranges over all sublinks of L (including the empty one), and $\# L'$ denotes the number of components of L'. Let $\mathcal{N}_n$ be the subspace of $\mathcal{N}$ spanned by the quantities $[M,\ L]$ for all M and all links L with $\# L \geq n$. A $\mathcal{G}$-valued invariant of $\mathbf{Z}$-homology 3-spheres defines a linear functional $f : \mathcal{N} \to \mathcal{R}$.

Definition 2.3. ([O]) *The invariant f is finite type of order $\leq n$ iff it induces a linear functional $f : \mathcal{N}/\mathcal{N}_{n+1} \to \mathcal{R}$.*

3. Manifolds obtained by surgery on knots

For a unit-framed knot $(K, \pm 1) \subset S^3$, let $M_K = \chi(S^3, K)$; clearly M_K is a $\mathbf{Z}$-homology 3-sphere. In this section, we will be concerned with the set of 3-manifolds

$$\mathcal{M} = \{ M_K \mid (K, \pm 1) \subset S^3 \}.$$

First, we refine a technique of [BP], to establish relations between special surgery presentations and Heegaard splittings for 3-manifolds in $\mathcal{M}$. At the end of the section, we turn our attention to the 3-manifolds in $\mathcal{M}$ that are homotopy 3-spheres and state a result (Theorem 3.7) that provides special surgery presentations for homotopy 3-spheres. These presentations are very useful for studying finite type invariants of homotopy 3-spheres as well as for relating the question of property P of knots to the vanishing of certain Vassiliev invariants.

3.1. **Plats, admissible links and nice Heegaard splittings.**

Let B_n denote the braid group on n strings and let $\{\sigma_i \mid i = 1, \dots, n-1\}$ denote the standard set of generators for B_n. Moreover, let P_n denote the pure braid group on n strings and let $\{a_{ij} \mid 1 \leq i < j \leq n\}$ denote the standard set of generators for P_n; that is, $a_{ij} = \sigma_j \cdots \sigma_{i-1}\sigma_i^2\sigma_{i-1}^{-1} \cdots \sigma_j^{-1}$ (see upper left picture of Figure 1). For $1 \leq i < j \leq n$, define $\bar{a}_{ij} = \sigma_j \cdots \sigma_{i-1}^{-1}\sigma_i^2\sigma_{i-1} \cdots \sigma_j^{-1}$ (see lower left picture of Figure 2). Throughout the paper, we will refer to $a_{ij}^{\pm 1}$ and $\bar{a}_{ij}^{\pm 1}$ as *elementary pure braids*.

Consider $\mathbf{R}^3$ parameterized by rectangular coordinates (x, y, z), and let

$$R := \{\ (0, y, z) \mid y \geq 0,\ z \geq 0\ \}.$$

Given a link L in $S^3 = \mathbf{R}^3 \cup \{\infty\}$, there exists $m \in \mathbf{N}$ and $b \in B_{2m}$, so that L can be represented as the $2m$-plat closure of b with respect to the height function on the z-axis (see [B]). This means that there is projection $P : \mathbf{R}^3 \to R$, such that $P(L)$ consists of:

a) A collection A (resp. B), of m arcs whose endpoints lie on the line $z = 1$ (resp. $z = 2$) and whose interiors lie in $\{z < 1\} \cap R$ (resp. $\{z > 2\} \cap R$) and contain the m minima (resp. maxima) of $P(L)$;

b) a collection C of $2m$-arcs, connecting the endpoints of A to the corresponding endpoints of B and whose interiors lie in the zone $\{1 < z < 2\}$. These arcs represent a braid $b \in B_{2m}$.

It will be convenient for us to assume that the maxima (resp. minima) of $P(L)$ lie on the line $\{z = 2 + \epsilon\} \cap R$ (resp. $\{z = 1 - \epsilon\} \cap R$), for some fixed $\epsilon > 0$. Also assume that $b = ps$ where $p \in P_{2m}$ and $s \in B_{2m}$ is a permutation braid.

Lemma 3.1. *An oriented knot $K \subset S^3$ can be represented as the plat-closure of a braid $b = ps$ such that, with the notation as above, the following are true:*

i) We have $p \in P_{2n}$, $s \in B_{2n}$ and $s = \sigma_2\sigma_4 \cdots \sigma_{2n-2}$.

ii) The braid p is a product of elementary pure braids such that $a_{ij}^{\pm 1}$ or $\bar{a}_{ij}^{\pm 1}$ appears in p only if the i-th and j-th strand in K have opposite orientations.

Proof: Let us begin with a projection $P(K)$ on R, that realizes K as the plat-closure of a braid $p_1 s_1 \in B_{2m}$, where $p_1 \in P_{2m}$ and $s_1 \in B_{2m}$. We may assume that p_1 is expressed in terms of the standard generators of P_{2m} and their inverses. We may also assume that $s_1 = \sigma_2 \sigma_4 \cdots \sigma_{2m-2}$ (see Lemma 3.3 of [BP]).

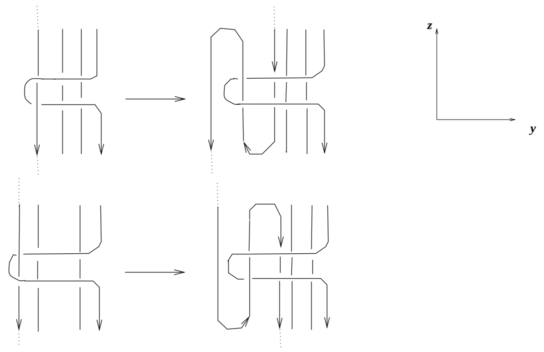

FIGURE 1. Pushing the crossings on oppositely oriented strings.

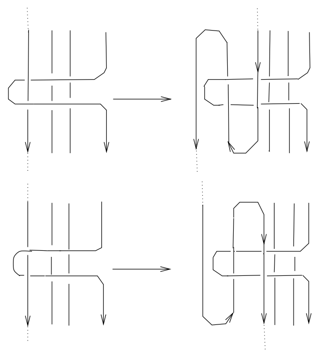

FIGURE 2. Introducing the elementary pure braids $\bar{a}_{ij}^{\pm 1}$.

We can arrange so that all appearances of the pure braid generators in p_1 occur in different z-heights. We will say that the appearance of a generator $a_{ij}^{\pm 1}$ in p_1 is *improper* if the orientations of the i-th and j-th arc in C, induced by the orientation of K, are the same (i.e. both arcs are oriented upwards or downwards). Let $i(p_1)$ denote the number of improper appearances of pure generators in p_1. If $i(p_1) = 0$, property (ii) of the statement of the lemma is already true; otherwise let a_{rt} denote the improper generator that occurs at the lowest z-height.

First we modify the r-th and t-th string, in $1 < z < 2$, by one of the stabilization moves shown in Figures 1 and 2. This should be done in such a way so that: a) The arc containing the newly created local minimum can be pushed below $z = 1$ so that it intersects the interior of exactly one of the arcs in A as indicated in the left hand side picture of Figure 3; and b) the newly created local maximum can be pushed on $z = 2 + \epsilon$ without intersecting the interiors of any of the arcs in B. One can see that one of the four moves of Figures 1, 2 would always do the job.

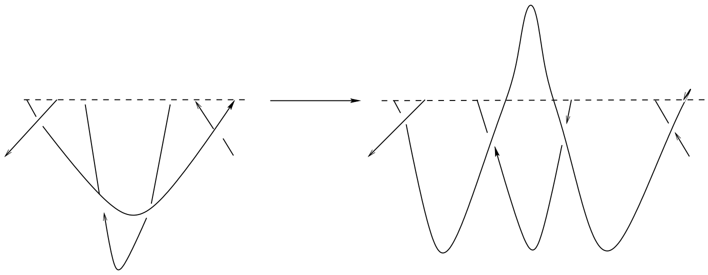

FIGURE 3. Adjusting the local minima.

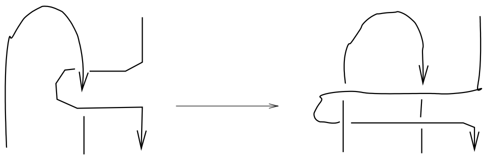

FIGURE 4

Next, we perform the isotopy shown in Figure 3 (the dashed line indicates the line $z = 1$). This isotopy inserts a factor of $\sigma_{2i-2}\sigma_{2i}^{-1}$ in s. At this stage, K is realized as the plat-closure of a braid $p_2 s_2 \in B_{2m_1}$, with $m_1 > m$ and

$$s_2 = \sigma_2\sigma_4 \cdots \sigma_{2i-2}\sigma_{2i}^{-1}\sigma_{2i+2} \cdots \sigma_{2m_1-2}.$$

By a local isotopy, we can replace $p_2 s_2$ by

$$p_2\sigma_2\sigma_4 \cdots \sigma_{2i-2}\sigma_{2i}\sigma_{2i}^{-2}\sigma_{2i+2} \cdots \sigma_{2m_1-2} = p_2\sigma_{2i}^{-2}s.$$

Then, we modify by the isotopy of Figure 4 and we push the newly created maximum of Figure 3 on $z = 2+\epsilon$. Now K has been realized as a plat-closure of $p_2\sigma_{2i}^{-2}s \in B_{2m_1}$. Clearly, $p_2\sigma_{2i}^{-2} \in P_{2m_1}$ and $i(p_2\sigma_{2i}^{-2}) < i(p_1)$. Thus the conclusion follows by induction. $\square$

The following lemma allows us to pass from surgery on unit-framed knots to surgery on admissible links that are pure plat-closures.

Lemma 3.2. *Let K be a unit-framed knot in S^3 and let $R := \{ (0, y, z) \in \mathbf{R}^3 \mid y \geq 0,\ z \geq 0 \}$. There exists an admissible m-component link $L := L_1 \cup L_2 \cup \ldots \cup L_m$ and a projection $P(L) \subset R$ with the following properties:*

a) $P(L)$ represents L as a pure plat with respect to the height on the z-axis.

b) If $m > 1$, the components $P(L_2), \ldots, P(L_m)$ bound disjointedly embedded discs on R each of which is pierced twice by $P(L_1)$.

c) $P(L_i)$ is embedded for every $i = 1, \ldots, m$.

d) We have $\chi(S^3, K) = \chi(S^3, L)$.

Proof: Let $P(K)$ be a projection on R realizing K as a plat-closure of a braid $ps \in B_{2n}$, with p and s as in Lemma 3.1. Let $k - 1$ be the number of elementary pure braids appearing in p. For each such braid, we apply the moves of link calculus of Figure 5 to obtain a k-component link $\hat{L} =: \hat{L}_1 \cup \ldots \cup \hat{L}_k$, where $\hat{L}_1$ denotes the component corresponding to K. The framing of a newly added component corresponding to

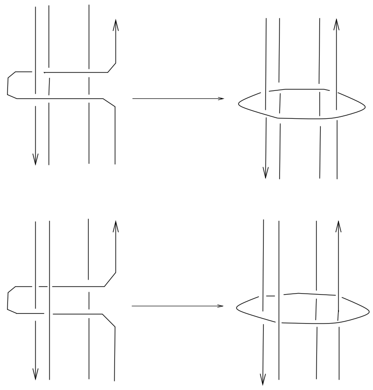

FIGURE 5. Changing the surgery presentation.

an elementary pure braid is $+1$ (resp. -1) according to whether we have a_{ij}^{-1} or $\bar{a}_{ij}^{-1}$ (resp. a_{ij} or $\bar{a}_{ij}$). The projection $P(\hat{L}) \subset R$, obtained from $P(K)$ by performing the modifications of Figure 5, represents $\hat{L}$ as a $2k$-plat with respect to the height on the y-axis. Since $\hat{L}$ is a k-component link, $P(\hat{L})$ is a pure-plat. Clearly, $\hat{L}$ is admissible and it satisfies condition (b). Since it is obtained from K by Kirby calculus it also satisfies condition (d). By rotating $P(\hat{L})$ on R we arrange so that it represents $\hat{L}$ as a plat closure of a braid $p_1 \in P_{2k}$, with respect to the height on the z-axis. Thus $\hat{L}$ satisfies condition (a).

Next further modify $\hat{L}$ so that condition (c) is satisfied. To begin notice that for $i \neq 1$, $P(L_i)$ is embedded. The projection $P(\hat{L})$ has the following property: For $k = 1, \ldots, m-1$, the $(2k-1)$-th and $(2k)$-th endpoint of p_1 on the line $z = 1$ (and on $z = 2$) belongs on $P(\hat{L}_{k+1})$ while the $(2m-1)$-th and $(2m)$-th endpoint of p_1 belongs on $P(\hat{L}_1)$. We may suppose that p_1 is expressed in terms of the elementary braids in P_{2k}. If p_1 doesn't contain appearances of elementary braids $a_{kl}^{\pm 1}, \bar{a}_{kl}^{\pm 1} \in P_{2m}$ such that both of the k-th and l-th strand belong on $\hat{L}_1$ then $P(\hat{L}_1)$ is embedded and there is nothing to prove. Otherwise, notice that for every elementary braid with both the strands on $P(\hat{L}_1)$ the two strands will be oppositely oriented. Each such elementary braid can be eliminated by applying the moves of Figure 5. The resulting projection represents link L that has the desired properties. $\square$

Definition 3.3. *An admissible link $(L, \mathbf{f})$ is called* nice *iff L can be represented as a pure plat-closure so that properties (a)-(c) in the statement of Lemma 3.2 are satisfied. The component L_1 is called the* special component *of L. The components $L_i, L_j \subset L$, $i, j \neq 1$, will be called parallel, iff they cobound an embedded annulus that is disjoint from $\bar{L} := L \setminus (L_i \cup L_j)$.*

Let $(K, \pm 1)$ be a unit-framed knot. By the proof of Lemma 3.2, given a projection $P(K)$ representing K as a plat-closure so that properties (i)-(ii) of Lemma 3.1 are satisfied, one constructs a nice link $(L, \mathbf{f})$ such that $\chi(S^3, K) = \chi(S^3, L)$. We will refer to L as a nice link *associated* to K. The proof of the following lemma follows immediately from the arguments in the proofs of Lemmas 3.1, 3.2.

Lemma 3.4. *Let K be a unit-framed knot and let L be the nice link associated to a projection $P(K)$ as above. For every proper sublink $L^{'} \subset L$ that contains the special component of L, there exists a knot $K_{L'}$ such that:*

i) $K_{L'}$ is obtained from K by a sequence of crossing changes on $P(K)$; one crossing change for each component in $L \setminus L^{'}$.

ii) We have $\chi(S^3, L^{'}) = \chi(S^3, K_{L'})$. In particular, $\chi(S^3, L^{'}) \in \mathcal{M}$.

3.2. **Homotopy 3-spheres.**

Now we turn our attention to nice links and Heegaard splittings corresponding to 3-manifolds in $\mathcal{M}$ that are homotopy 3-spheres. We define

$$\mathcal{HM} = \{M \in \mathcal{M} \mid \ \pi_1(M) = \{1\}\}.$$

Remark 3.5. By Corollary 2 of [CGoLuS], the only Dehn surgeries on a non-trivial knot in S^3 that could yield a simply connected 3-manifold are ± 1-surgeries. Furthermore, for a non-trivial knot K, at most one M_K^{+1}, M_K^{-1} can be simply connected. On the other hand, by Theorem 2 of [GL] such surgeries can never produce S^3. Thus, if there is a simply connected 3-manifold M obtained by Dehn surgery on a non-trivial knot $K \subset S^3$, and thus a counterexample to the Poincaré conjecture, then $M \in \mathcal{HM}$.

Definition 3.6. *For a finite set A we will use $\#A$ to denote the cardinality of A. Let $m > 1$. Let $(L, \mathbf{f})$ be an m-component nice link associated to a projection $P(K)$ of a unit-framed knot K and let L_1 denote the special component of L. A* division *of L is a partition of the set $\{L_2, \dots, L_m\}$ into mutually disjoint non-empty sets $\mathcal{R} := \{R_1, \dots R_h\}$, such that for every $1 \leq i \leq h$, one of the following is true:*

i) We have $\#R_i = 1$.

ii) We have $\#R_i > 1$ and any two components of L belonging in R_i are parallel.

iii) We have $\#R_i > 1$ and R_i is partitioned into two sets $R_i^{'}, R_i^{''}$ such that all the components of L belonging in $R_i^{'}$ (resp. in $R_i^{''}$) are parallel.

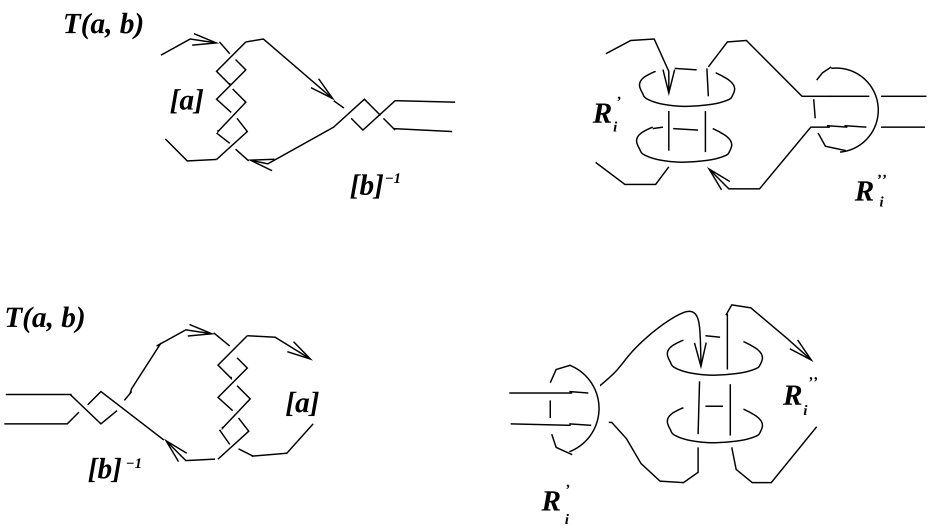

FIGURE 6. The tangle $T(a,b)$, $a,b \in \mathbf{Z}$, is the union of the rational tangles $[a]$ and $\frac{1}{[b]}$. The process of Figure 5, when applied to $T(a,b)$, gives rise to the subset $R_i := R_i' \cup R_i'' \subset L$, where $\#R_i' = \frac{1}{2}|a|$ and $\#R_i'' = \frac{1}{2}|b|$. The example in the top (resp. bottom) corresponds to the case $a=-2$, $b=1$ (resp. $a=-2$, $b=-1$).

Furthermore, the components in R_i are obtained from a tangle $T(a,b)$ of the projection $P(K)$ that has the form illustrated in Figure 6(a) and the corresponding part of the resulting projection of L is as illustrated in Figure 6(b).

The number $h := h(\mathcal{R})$ is called the height of the division.

Given a division of L and a subset $J \subset \{2, \dots, h\}$, set $R_J := \cup_{j \in J} R_j$. Also set $L_J := \cup_{X \in R_J} X$ and $\hat{L}_J := L_1 \cup L_J$. Clearly, L_J, $\hat{L}_J$ are sublinks of L; set $M_J := \chi(S^3, L_J)$ and $\hat{M}_J := \chi(S^3, \hat{L}_J)$. Also recall that for a sublink $L' \subset L$, $M_{L'}$ denotes the 3-manifold $\chi(S^3, L')$. The following theorem provides the surgery presentations of 3-manifolds in $\mathcal{HM}$, promised in the beginning of Section 3.

Theorem 3.7. *Let $M \in \mathcal{HM}$ and let K be a framed knot such that $M = \chi(S^3, K)$. Also let $(L, \mathbf{f})$ be a nice link associated to K. There exists a division $\mathcal{R}$ of L such that for every subset $J \subset \{2, \dots, h\}$ we have $M_J \in \mathcal{HM}$ and $\hat{M}_J \in \mathcal{HM}$.*

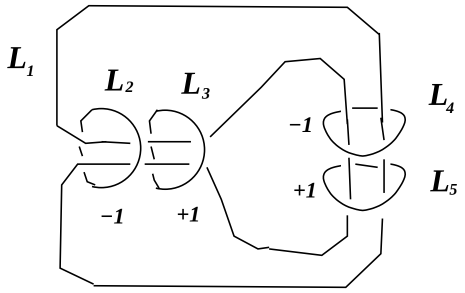

FIGURE 7. An example of a nice link L such that $S^3 = \chi(S^3,\ L)$. The division $\mathcal{R} = \{R_1, R_2\}$ where $R_1 := \{L_2,\ L_3\}$, $R_2 := \{L_4,\ L_5\}$ satisfies the conclusion of Theorem 3.7. However not every sublink of L gives S^3 by surgery. For example, the sublink $L_1 \cup L_2 \cup L_4$ gives a non simply-connected 3-manifold.

Remark 3.8. Let $M \in \mathcal{HM}$, K, L, and $\mathcal{R}$ be as in the statement of Theorem 3.7. Let $R_i \in \mathcal{R}$ be a set as described in (iii) of Definition 3.6. Let $P(K)$ denote the projection of K that gives rise to L. Let $J \subset \{2, \dots, h\}$ be a (possibly empty) subset with $i \notin J$ and let K_J denote the knot represented by the projection obtained from $P(K)$ by changing the crossings corresponding to the components of L that belong in R_J. Since $i \notin J$, K_J still contains the tangle $T(a_i, b_i)$ corresponding to R_i in the sense of Figure 6. Let $K_J^{[a_i]}$ denote the knot obtained from K by changing crossings in $T(a_i, b_i)$ to trivialize the rational tangle $[a_i]$. Also let K_J^i denote the knot obtained from K by changing all the crossings corresponding to the components in R_i; these crossing changes trivialize the whole $T(a_i, b_i)$. Notice that the knots $K_J^{[a_i]}$ and K_J^i are isotopic and that, with the notation of Lemma 3.4, we have $K_J^i = K_{L'}$ where $L' := L \setminus (R_J \cup R_i)$. On the other hand, with the notation of Theorem 3.7, we have $L' = \hat{L}_{J'}$ where J' denotes the complement of $J \cup \{i\}$ in $\{2, \dots, h\}$. Hence we have

$$\chi(S^3, K_J^{[a_i]}) = \chi(S^3, K_{L'}) = \chi(S^3, L') = \hat{M}_{J'} \in \mathcal{HM}. \tag{3.1}$$

Let $\mathcal{R}''$ denote the set of subsets of $\{L_2, \dots, L_m\}$ obtained from $\mathcal{R}$ by replacing R_i by R_i''. Moreover, for a non-empty subset $J \subset \{2, \dots, h\}$ let L_J'' (resp. $\hat{L}_J''$) denote the sublink of L_J (resp. $\hat{L}_J$) obtained by removing the components contained in R_i and set $M''_J := \chi(S^3, L''_J)$ (resp. $\hat{M}''_J := \chi(S^3, \hat{L}''_J)$). From (3.1), it follows that for every J as above we have $M''_J \in \mathcal{HM}$ and $\hat{M}''_J \in \mathcal{HM}$. That is, $\mathcal{R}''$ also satisfies the conclusion of Theorem 3.7.

Definition 3.9. *Let $(L,\ \mathbf{f})$ be a nice link and let $(\bar{L},\ \bar{\mathbf{f}})$ be a nice link that obtained from $(L,\ \mathbf{f})$ by one of the following moves:*

(1) Deletion of a unit-framed component that bounds an embedded disc in the complement of L.

(2) Addition of a pair of parallel unit-framed components whose framings are of opposite signs.

We will say that $(\bar{L},\ \bar{\mathbf{f}})$ is obtained from $(L,\ \mathbf{f})$ by an admissible move.

Let $(\bar{L},\ \bar{\mathbf{f}})$, $(L,\ \mathbf{f})$ be nice links such that $(\bar{L},\ \bar{\mathbf{f}})$ is obtained from $(L,\ \mathbf{f})$ by an admissible move. Note that if $(L,\ \mathbf{f})$ is associated to a knot K then $(\bar{L},\ \bar{\mathbf{f}})$ is also associated to K. In fact, if $P(K)$ is a projection of K that produces L via the process of Figure 5, it is easy to adjust $P(K)$ so that the resulting projection corresponds to $\bar{L}$. Thus, in particular we have $\chi(S^3,\ L) = \chi(S^3,\ \bar{L})$. We will need the following lemma:

Lemma 3.10. *Let $M \in \mathcal{HM}$ and let K be a framed knot such that $M = \chi(S^3, K)$. Also let $(L,\ \mathbf{f})$ be a nice link associated to K. Suppose that there exists a nice link $(\bar{L},\ \bar{\mathbf{f}})$ that is obtained from $(L,\ \mathbf{f})$ by a number of admissible moves and that satisfies the conclusion of the statement of Theorem 3.7. Then, either L has one component or $(L,\ \mathbf{f})$ also satisfies the conclusion of the statement of Theorem 3.7.*

Proof: Let $\bar{\mathcal{R}} := \{\bar{R}_1, \ldots \bar{R}_{\bar{h}}\}$ be a division for $\bar{L}$ that satisfies the conclusion of the statement of Theorem 3.7. We will say that a set $\bar{R}_i \in \bar{\mathcal{R}}$ is *incoherent* if there exists $\bar{R}_j \in \bar{\mathcal{R}}$ and a pair of components $\bar{L}_i, \bar{L}_j \subset \bar{L}$ corresponding to an inadmissible move as in (2) of Definition 3.9, such that $\bar{L}_i \in \bar{R}_i$ and $\bar{L}_j \in \bar{R}_j$. There exists a division of $\bar{L}$, say $\bar{\mathcal{R}}_1$, such that: i) Each of the sets in $\bar{\mathcal{R}}_1$ is a union of sets in $\bar{\mathcal{R}}$; and ii) $\bar{\mathcal{R}}_1$ doesn't contain incoherent sets. Let $\mathcal{R}$ be the set obtained as follows: First from every set in $\bar{\mathcal{R}}_1$ we throw away the components of $\bar{L}$ corresponding to inadmissible moves moves from L to $\bar{L}$. This yields a set $\bar{\mathcal{R}}'$ of sets of components of L. Then, we throw away all the empty sets from $\bar{\mathcal{R}}'$. If $\mathcal{R} = \emptyset$, then L has one component and in particular $M = S^3$ (compare with the example of Figure 7). If $\mathcal{R} \neq \emptyset$, then it is a division of L which clearly satisfies the conclusion of the statement of Theorem 3.7. $\square$

In the light of Lemma 3.10, given $M \in \mathcal{HM}$ and $(L,\ \mathbf{f})$ as in the statement of Theorem 3.7, it is enough to prove the conclusion of the theorem for any nice link that is obtained from L by admissible moves. This will be our strategy in the proof of Theorem 3.7 that is postponed until Section 7. In the next sections we will prove the main results of this paper assuming Theorem 3.7.

4. Surgery on knots and Vassiliev invariants

4.1. **A vanishing result for low order invariants.**

Let $\mathcal{G}$ be an abelian group. For $n \in \mathbf{N}$, let $\mathcal{V}_n$ denote the module of $\mathcal{R}$-valued Vassiliev invariants of order $\leq n$. Furthermore, for a knot $K \subset S^3$, let $M_K^{\pm}$ denote the $\mathbf{Z}$-homology 3-sphere obtained from S^3 by ± 1-surgery on K. In analogy with the set $\mathcal{HM}$ we consider the set of knots

$$\mathcal{HK} =: \{K \subset S^3 \ \mid \ M_K^{+} \in \mathcal{HM} \ \text{ or } \ M_K^{-} \in \mathcal{HM}\}.$$

As mentioned in Remark 3.5, for every $K \in \mathcal{HK}$ there is exactly one element of $\{M_K^{+}, M_K^{-}\}$ that belongs in $\mathcal{HM}$. Our goal in this section is to prove the following (Proposition 1.3 of the Introduction):

Proposition 4.1. *Let U denote the trivial knot. For every $K \in \mathcal{HK}$ and for every $v \in \mathcal{V}_4$, we have $v(K) = v(U)$.*

Before we can proceed with the proof of Proposition 4.1 we need to recall from [AsK] the notion of n-graphs and some of their properties. The reader is referred to [AsK] for more details.

Definition 4.2. *(*[AsK]*) An n-graph $G \subset \mathbf{R}^3$, is a trivalent graph with $2n$ vertices, consisting of a planar circle U and edges $a_1, \dots, a_n$ attached to U. The circle U bounds a standardly embedded disc $D \subset \mathbf{R}^3$ and the edges can intersect* $\mathrm{int}(D)$.

The n-graph $G = U \cup (\cup_{i=1}^n a_i)$ is called admissible, if for every pair of distinct indices $i, j \in \{1, \dots, n\}$, the points $\partial a_i \subset U$ are not separated by the points in ∂a_j. The n-graph G is called standard if it is admissible, the edges a_i are unknotted and unlinked to each other, and their interiors are disjoint from D.

For an n-graph $G = U \cup (\cup_{i=1}^n a_i)$, we may assign an integer z_i to one of the two vertices of a_i. Then we can construct a knot $K(G)$ as follows: For every edge a_i push U along a_i by a finger move and then introduce z_i clasps at the other end of a_i without introducing any twists away from the clasp; the sign of the clasps should be that of z_i. If G is the standard graph, then $K(G)$ is the unknot. For the rest of this subsection by an isotopy between n-graphs we will mean one that preserves the knot type of the corresponding knot.

Definition 4.3. *A knot K is called strongly n-trivial [2] iff it admits a projection that contains a collection of n disjoint sets of crossings $\{C_i\}_{i=1}^n$, such that: i) For every $0 < m \leq n$, changing all the crossings in any m of these sets simultaneously, yields a projection of the unknot; and ii) for $i = 1, \dots, n$, C_i consists either of a single crossing or is a set of twist crossings on two strings of K that inherit opposite orientations from any orientation of K.*

We will need the following lemma that can be proven using a result of Gabai (Corollary 2.4 of [Ga]). For the proof see Lemma 4.1 of [AsK] or alternatively the proof of Theorem 6.1 of [HoL].

Lemma 4.4. *Suppose that $G := U \cup (\cup_{i=1}^n a_i)$ is an n-graph such that a knot $K(G)$, associated to G as above, is the trivial knot. For $i = 1, \dots, n$, let C_i be a set of crossings that trivializes the clasps of $K(G)$ corresponding to a_i. Suppose that the collection $\{C_i\}_{i=1}^n$, satisfies the*

[2] In [AsK], the terminology n-adjacent to the unknot is used. Compare, Remark 5.2 of [AsK].

definition of strong n-triviality. Then G is isotopic to the standard n-graph.

Let $\mathcal{C} := \{C_1, \dots, C_{n+1}\}$ be a collection of $n+1$ disjoint sets of crossings on a projection of a knot K. For a non-empty subset $C \subset \mathcal{C}$, let

$$|C| := \sum_{C_i \in C} \# C_i,$$

and let K_C denote the knot obtained from K by changing all the crossings in all elements of C simultaneously. Recall that by the skein relation of the Vassiliev invariants ([V], [BLi]), for every $v \in \mathcal{V}_n$ we have

$$v(K) = \sum_{C} (-1)^{|C|}\, v(K_C), \tag{4.1}$$

where C ranges over all non-empty subsets of $\mathcal{C}$.

We close this subsection with two lemmas that will be needed in the proof of Proposition 4.1. The first lemma is a special case of Theorem 4.2 of [AsK]. For the convenience of the reader, however, we will give the proof here.

Lemma 4.5. *Suppose that K is strongly 3-trivial. Then, for every $v \in \mathcal{V}_4$, we have $v(K) = v(U)$.*

Proof: Let $P(K)$ be a projection of K containing a collection of sets $\mathcal{C} = \{C_1, C_2, C_3\}$, that satisfy the definition of strong 3-triviality. By [Su] (see also Section 3 of [AsK]), there exists a 3-graph

$$G = U \cup (\cup_{i=1}^{3} a_i),$$

with $P(K) = K(G)$, and such that, for $i = 1, 2, 3$, changing the crossings in C_i trivializes one of the three sets of clasps of $K(G)$. Next we choose two additional sets of crossings on $K(G)$ as follows: Let A_1 (resp. A_2) be the set of crossings on G where a_1 (resp. a_2) under crosses $a_2 \cup a_3$ (resp. $a_1 \cup a_3$). By construction, A_1 and A_2 give rise to sets of crossings C_4 and C_5, respectively, on $K(G)$. Set $\mathcal{E} = \{a_1, a_2, a_3\}$ and for a subset $\emptyset \neq E \subset \mathcal{E}$, let G_E denote the graph obtained from G by deleting the edges contained in E. By our assumption, G_E satisfies the hypotheses of Lemma 4.4. Thus, G_E is isotopic to a standard graph. Now let G_A denote any 3-graph obtained from G by changing the crossings in any of A_1, A_2, or $A_1 \cup A_2$, say in A_1. Let G_1 (resp. G_2) denote the 2-graph (resp. 1-graph) obtained from G_A by deleting the edge(s) corresponding to a_1 (resp. a_2 and a_3). Clearly, $G_1 = G_{E_1}$ (resp. $G_2 = G_{E_2}$) where $E_1 = \{a_1\}$ (resp. $E_2 = \{a_2, a_3\}$). By our discussion above, G_1 (resp. G_2) is isotopic to the standard 2-graph (resp. 1-graph). By our choice of A_1, it follows that G_A is also isotopic to the standard 3-graph. Thus, a knot obtained from $K(G)$ by changing the crossings in any of C_4, C_5 or $C_4 \cup C_5$ is the unknot. Now let $v \in \mathcal{V}_4$. By (4.1) we have

$$v(K) = \sum_C (-1)^{|C|}\, v(K_C),$$

where C ranges over all non-empty subsets of $\{C_1, C_2, C_3, C_4, C_5\}$. By our arguments above, is not hard to see that $K_C = U$, for every C. $\square$

Lemma 4.6. *Let $n > 3$. Suppose that a knot K admits a projection, say $P(K)$, containing $n+1$ disjoint sets of crossings $\{C_i\}_{i=1}^{n+1}$ such that changing all the crossings in any one of them results in a knot projection in which the remaining n sets of crossings satisfy the definition of strong n-triviality. Then, we can find a projection of K that contains a collection, say $\mathcal{C}$, of $n+1$ disjoint sets of crossings satisfying the same property as above, and another set of crossings C' such that:*

i) the crossings in C' are disjoint from these in $\mathcal{C}$;

ii) the knot $K_{C'}$ is the unknot;

iii) the knot $K_{C' \cup C}$ is the unknot for every subset $C \subset \mathcal{C}$.

Proof: By [Su], there exists an $(n+1)$-graph $G = U \cup (\cup_{i=1}^{n+1} a_i)$, with $P(K) = K(G)$, such that, for $i = 1, \ldots, n+1$, C_i consists of crossings from the set of clasps of $K(G)$ corresponding to a_i. By Lemma 4.4, a graph obtained from G by deleting at least two of the a_i's is the standard one; in particular, G is admissible. Let G_1 (resp. G_2) denote the $(n-1)$-graph (resp. 2-graph) obtained from G by deleting a_{n-1} and a_n (resp. $a_1, \ldots, a_{n-2}$). Consider the set of crossings on $a_{n-1} \cup a_n$, say A, where $a_{n-1} \cup a_n$ under cross any of $a_1, \ldots, a_{n-2}$. By construction, A gives rise to a set of crossings, say C', on $K(G)$. Let G_A denote the graph obtained from G by changing the crossings in A. Since G_1 and

G_2 are standard graphs, and by our choice of A, it follows that G_A is also standard and thus $K(G_A) = K_{C'}$ is the unknot. We have showed that C' satisfies (i) and (ii) of the statement of the lemma. Now the rest of the claim follows easily. $\square$

4.2. **Nice links with low division heights.**

For the proof of Proposition 4.1 we need the following lemma that treats nice links admitting divisions of height ≤ 2:

Lemma 4.7. *Let $M \in \mathcal{HM}$ and let K be a unit-framed knot such that $M = \chi(S^3, K)$. Let $(L, \mathbf{f})$ be a nice link associated to K. Suppose, moreover, that L admits a division of height less or equal than two. Then, K is the unknot.*

Proof: Recall that one starts with a projection of K, say $P(K)$, that has the properties of Lemma 3.1 and constructs L through the process described in the proof of Lemma 3.2 (see also discussion before the statement of 3.4). By this proof, K can be recovered from L as follows: Start with a projection, say $P(L)$, of $L := L_1 \cup \ldots \cup L_m$ that realizes it as a pure plat closure with the properties (a)-(b) of Lemma 3.2 and apply the reverse of the moves of Figure 5; one such move for every component $L_i \neq L_1$.

The projection $P(L_1)$ realizes L_1 as the plat-closure of the trivial 2-braid. Let $\mathcal{R}$ be a division for L with height $h(\mathcal{R}) \leq 2$. If $h(\mathcal{R}) = 0$ the conclusion is clearly true since $K = L_1$. Suppose that $h(\mathcal{R}) = 2$. Then $P(K)$ is obtained from $P(L_1)$ as follows: Consider two disjoint 3-balls B_1, B_2 such that, for $k = 1, 2$, $B_k \cap P(L_1)$ consists of two properly embedded, oppositely oriented arcs. If R_k is as in (i) or (ii) of Definition 3.6, in B_k modify $P(L_1)$ by giving the arcs $B_k \cap P(L_1)$ $\#R_k$-full twists. Otherwise, modify by inserting the tangle $T(\#R'_k,\ \#R''_k)$, where the notation is as in Figure 6. If $h(\mathcal{R}) = 1$, $P(K)$ is obtained by a similar modification of $P(L_1)$ along a single 3-ball. In both cases, $P(K)$ exhibits K as the sum of two algebraic tangles and thus K is an arborescent knot (see, for example, [Wu2]). Since arborescent knots are known to have property P ([Wu1]) it follows that K must be the unknot. $\square$

To continue, let K be a unit-framed knot and let $(L,\ \mathbf{f})$ be a nice link associated to K. Suppose the $\chi(L,\ \mathbf{f}) \in \mathcal{HM}$. Define $h(L)$ to be the smallest height over all divisions of L that satisfy the conclusion of Theorem 3.7. Define

$$\mu(K) := \min\{h(L) \mid L \text{ a nice link associated to } K\}.$$

Proof of Proposition 4.1: The proof will be by induction on $\mu(K)$. Suppose that $K \in \mathcal{HK}$ and $\mu(K) \leq 2$. Then we can associate to K a nice link that admits a division of height ≤ 2. By Lemma 4.7, K is the trivial knot. Thus the conclusion is trivially true.

Suppose, inductively, that the conclusion is true for all $K \in \mathcal{HK}$ such that $2 < \mu(K) < m$. Let $K \in \mathcal{HK}$ be a knot with $\mu(K) = m$. By assumption, $\pi_1(M_K) = \{1\}$ for some $M_K \in \{M_K^+, M_K^-\}$. Let $P(K)$ be a projection of K that realizes $\mu(K)$. By assumption, there is a nice link $L := L_1 \cup \ldots \cup L_s$, associated to $P(K)$, that admits a division of height m, say $\mathcal{R} := \{R_1, \ldots, R_m\}$, that satisfies the conclusion of Theorem 3.7. The components $L_2, \ldots, L_s$ correspond, in the sense of Lemma 3.4, to s distinct crossings on $P(K)$. Thus the sets in $\mathcal{R}$ correspond to m disjoint sets of crossings on $P(K)$. For $i = 1, \ldots m$, let C_i denote the set of crossings corresponding to R_i and set $\mathcal{C} := \{C_1, \ldots, C_m\}$. By our assumptions (see proof of Lemma 3.2 and Definition 3.6) for each C_i one of the following is true: i) C_i consists of a single crossing of $P(K)$; ii) C_i is a set of twist crossings on two strings of $P(K)$ that inherit opposite orientations from any orientation of $P(K)$; iii) C_i is the disjoint union of two sets of crossings, say C_i', C_i'', of $P(K)$ such that each of C_i', C_i'' is a set of twist crossings on two strings of $P(K)$ that inherit opposite orientations from any orientation of $P(K)$. By Remark 3.8, there is $C' \in \{C_i', C_i''\}$ so that if we let $R' \subset R_i$ denote the set of components of L corresponding to C', then the following is true: The set $\mathcal{R}'$ of subsets of $\{L_2, \ldots, L_m\}$ obtained from $\mathcal{R}$ by replacing R_i by R' also satisfies the conclusion of Theorem 3.7. Let $\mathcal{C}'$ denote the collection of sets of crossings obtained from $\mathcal{C}$ by replacing C_i with C'. By repeating this process as many times as necessary, we may assume that every $C_i \in \mathcal{C}$ consists either of a single crossing or it is a set of twist crossings on two strings of $P(K)$ that inherit opposite orientations from any orientation of $P(K)$.

To continue, for a subset $C \subset \mathcal{C}$, let K_C denote the knot represented by the projection obtained from $P(K)$ by changing all the crossings of the sets in C simultaneously. By our discussion above, we have $\pi_1(M_{K_C}) = \{1\}$ and $\mu(K_C) < \mu(K) = m$, for every non-empty subset $C \subset \mathcal{C}$.

Suppose for a moment that $\mu(K) \geq 5$, and let $v \in \mathcal{V}_4$. The set $\mathcal{C}$ will contain at least five sets of crossings. Applying (4.1) for these sets recursively, we can express $v(K)$ as an alternating summation, of an odd number of terms, of the form

$$v(K) = \sum \epsilon_i \, v(K^i),$$

where $\epsilon_i = \pm 1$ and each K^i is a knot with $\mu(K^i) \leq 4$ and such that $\pi_1(M_{K^i}) = \{1\}$. Thus, to finish the proof of the proposition it is enough to show that the conclusion holds for knots with $\mu(K) = 3$ and $\mu(K) = 4$.

Case 1: Suppose that $\mu(K) = 3$. We have $\pi_1(M_{K_C}) = \{1\}$ for every subset $C \subset \{C_1, C_2, C_3\}$. By Lemma 4.7, if $C \neq \emptyset$, the knot K_C is the unknot. Thus, K is strongly 3-trivial and the conclusion, in this case, follows from Lemma 4.5.

Case 2: Suppose that $\mu(K) = 4$. For every non-empty subset $C \subset \{C_1, C_2, C_3, C_4\}$, the knot K_C satisfies $\pi_1(M_{K_C}) = \{1\}$ and $\mu(K_C) \leq 3$. As in Case 1, we conclude that each of the knots K_C is strongly 3-trivial and by Lemma 4.5 we obtain that $v(K_C) = v(U)$ for every $v \in \mathcal{V}_4$. Since K satisfies the hypothesis of Lemma 4.6 there is a set of crossings C' that satisfies the conclusion of that lemma. Once again, by (4.1), for every $v \in \mathcal{V}_4$, we have

$$v(K) = \sum_{\bar{C}} (-1)^{|\bar{C}|} \, v(K_{\bar{C}}),$$

where $\bar{C}$ ranges over all non-empty subsets of $\{C_1, C_2, C_3, C_4, C'\}$ and $v(K_{\bar{C}}) = v(U)$. Thus, we conclude $v(K) = v(U)$. $\square$

For a knot $K \subset S^3$, the invariants $v_i(K)$ and $c_i(K)$ are $\mathbf{Q}$-valued Vassiliev invariants of order i ([BLi]); in fact one has $v_2(K) = -6c_2(K)$ and $c_3(K) = 0$. Combining this with Proposition 4.1 and the fact that $v_i(U) = c_i(U) = 0$, we obtain the following:

Corollary 4.8. *Suppose that for a knot $K \subset S^3$ we have $\pi_1(M_K^+) = \{1\}$ or $\pi_1(M_K^-) = \{1\}$. Then $v_2(K) = v_3(K) = v_4(K) = c_4(K) = 0$.*

We close this subsection by noting that the argument of the proof of Proposition 4.1 generalizes to show the following:

Proposition 4.9. *Let $m \in \mathbf{N}$ and let U denote the trivial knot. Suppose that if $K \in \mathcal{HK}$ and $\mu(K) \leq m$ then K is the unknot. For every $K \in \mathcal{HK}$ and for every $v \in \mathcal{V}_{m+2}$, we have $v(K) = v(U)$.*

Proof: The proof is an induction on μ. By assumption, the conclusion is trivially true for K with $\mu(K) \leq m$. Using (4.1) and an argument similar to this in the proof of Proposition 4.1, we can reduce the proof to the cases of $\mu(K) = m + 1$ and $\mu(K) = m + 2$. In the case that $\mu(K) = m + 1$ it follows that K is strongly $(m + 1)$-trivial and the conclusion follows by the argument in the proof of Theorem 4.4 of [AsK]. If $\mu(K) = m + 2$, the conclusion follows by Lemma 4.5 and (4.1) as in Case 2 in the proof of Proposition 4.1. □

5. Finite type invariants of 3-manifolds

5.1. A vanishing criterion.

In this section, we apply the results of Section 3 to derive vanishing results for finite type invariants of 3-manifolds in $\mathcal{M}$ and $\mathcal{HM}$. We begin with the following definition that refines the notion of the genus for 3-manifolds in $\mathcal{M}$.

Definition 5.1. *A Heegaard splitting (or H-diagram) of $\chi(S^3, L)$ associated to a nice link $(L, \mathbf{f})$ as in Theorem 2.1, is called* nice. *A 3-manifold M that admits a nice splitting is called a* nice 3-manifold. *For such an M we define the* nice genus *$ng(M)$, to be the smallest genus over all nice splittings of M.*

By Lemma 3.2 the nice genus is well defined for all 3-manifolds in $\mathcal{M}$.

Theorem 5.2. *Let $n > 0$. Let f be a finite type 3-manifold invariant of order $\leq n$, with values in an abelian group $\mathcal{G}$. Then, f vanishes on all $M \in \mathcal{M}$ if and only if it vanishes on every $N \in \mathcal{M}$ with $ng(N) \leq n$.*

Proof: Let $n > 0$, and let f be a finite type invariant of order $\leq n$ that vanishes on every $N \in \mathcal{M}$ with $ng(N) \leq n$. Inductively, suppose that $f(M) = 0$ for all $M \in \mathcal{M}$, with $ng(M) < m$, for some $m \geq n+1$. Let $M_1 \in \mathcal{M}$ with $ng(M_1) = m$. By Lemmas 3.2 and 3.4 we can find an m-component admissible link $(L, \mathbf{f})$ such that: i) $M_1 = \chi(S^3, L)$; ii) for every proper sublink $L^{'}$ of L the manifold $M_{L'} = \chi(S^3, L^{'})$ is also in $\mathcal{M}$; and iii) we have $ng(M_{L'}) < m$. By the induction hypothesis, $f(M_{L'}) = 0$. Since by Definition 2.3

$$f([S^3,\ L]) = 0,$$

we obtain that $f(M_1) = 0$, which finishes the proof of the inductive step and of the first claim in the statement of the theorem. $\square$

Let $\mathcal{G}$ be an abelian group and let $\mathcal{F}_6$ denote the $\mathcal{R}$-module of finite type invariants of order ≤ 6 for $\mathbf{Z}$-homology 3-spheres. Let $\lambda \in \mathcal{F}_m$. For $K \subset S^3$, define $v_\lambda(K) = \lambda(M_K^+)$ (alternatively, one could define $v_\lambda(K) = \lambda(M_K^-)$). By Proposition 3.9 of [GaLe], the assignment $\lambda \mapsto v_\lambda$ defines a map

$$\mathcal{F}_6 \to \mathcal{V}_4. \tag{5.1}$$

We are now ready to prove Corollary 1.5 of the Introduction.

Proof of Corollary 1.5: Suppose that M is a homotopy 3-sphere obtained by surgery on a knot in S^3, and let $\lambda \in \mathcal{F}_6$. By Corollary 2 of [CGoLuS], we must have $M \in \mathcal{HM}$ and thus $M = M_K^+$ or $M = M_K^-$ for some $K \subset S^3$. By (5.1), v_λ is a Vassiliev invariant of order ≤ 4 for K. Now the conclusion follows from Proposition 4.1. $\square$

5.2. **Low order Ohtsuki invariants.**

Let $\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$ be the sequence of invariants for $\mathbf{Z}$-homology 3-spheres defined by Ohtsuki in [O1]. It is known (see [KrS]) that λ_n

is a finite type invariant of order $3n$. Thus we may apply Theorem 5.2 to obtain a vanishing criterion for λ_n. We are particularly interested in the first two invariants; λ_1, λ_2. In this case, by Corollary 1.5 and the known fact that $\lambda_1(S^3) = \lambda_2(S^3) = 0$, we have:

Theorem 5.3. *Let M be a $\mathbf{Z}$-homology 3-sphere obtained by Dehn surgery on a knot in S^3. If $\pi_1(M) = \{1\}$, then $\lambda_1(M) = \lambda_2(M) = 0$.*

The invariants λ_1, λ_2 satisfy the following surgery formulae (see, for example, [LiW]).

$$\lambda_1(M_K^{\pm}) = \mp v_2(K), \tag{5.2}$$

$$\lambda_2(M_K^{\pm}) = \frac{\pm 1}{2} v_2(K) - \frac{\pm 1}{3} v_3(K) + v_2(K) + \frac{5}{3} v_2^2(K) - 60 c_4(K), \tag{5.3}$$

where $v_i(K)$ and $c_4(K)$ are as in Section 4. Thus, Theorem 5.3 also follows from (5.2), (5.3) and Corollary 4.8.

6. Applications to property P of knots

6.1. **Property P criteria.**

By combining a result of [CGLS] and Proposition 4.1 we are now ready to prove Theorems 1.1 and 1.4 of the Introduction.

Proof of Theorems 1.1 and 1.4: First of all, as already discussed, Theorem 1.4 immediately implies Theorem 1.1. Now let us complete the proof of Theorem 1.4. Suppose that $\pi_1(M_K^n) = \{1\}$ for some $n \in \mathbf{N}$. By Corollary 2 of [CGLS], we must have $n = +1$ or $n = -1$. Thus, the conclusion follows immediately from Proposition 4.1. $\square$

Similarly by combining Theorem 2 of [GL] and Proposition 4.9 we prove Corollary 1.6 of the Introduction:

Proof of Corollary 1.6: Recall the set $\mathcal{HK}$ and the complexity μ defined in Section 4. Suppose that $\pi_1(M_{K_1}^n) = \{1\}$ for some knot K_1 and some $n \in \mathbf{N}$. By Corollary 2 of [CGLS], we must have $n = +1$ or $n = -1$ and thus $K_1 \in \mathcal{HK}$. Now the conclusion follows by the assumption that every K with $\mu(K) \leq m$ has property P and by Proposition 4.9. $\square$

Example 6.1. In [St], Stoimenow calculates the invariants v_2, c_4, v_3 for knots with up to ten crossings. From his tables, we can see that for each such knot the value of at least one of v_2, c_4, v_3 is non zero. Thus, it follows from Theorem 1.1, that all knots up to ten crossings have property P.

Corollary 6.2. *A knot of genus $g \leq 2$ and non-trivial Alexander-Conway polynomial has property P.*

Proof: Recall that $\nabla_K(z) = 1 + c_2 z^2 + c_4 z^4 + \ldots + c_{2k} z^{2k}$, where $k \leq g$. Thus, if $\nabla_K(z) \neq 1$ and $g \leq 2$, at least one of c_2, c_4 must be non-zero. $\square$

Classes of knots in which the Alexander polynomial characterizes the unknot include *fibered* knots and *alternating* knots (see, for example, [BuZ]).

Corollary 6.3. *A knot of genus $0 < g \leq 2$ that is fibered or alternating has property P.*

Let us call a knot $K \subset S^3$ *special* if and only if there exists an invariant $v \in \mathcal{V}_4$, with values in some abelian group $\mathcal{G}$, such that $v(K) \neq v(U)$. Besides the examples discussed above, the class of *special* knots includes the positive or negative knots and many periodic knots (see the following remark). There exist many constructions that provide knots with the same Vassiliev invariants up to certain order or knots with the same Jones polynomial. These constructions, combined with Theorem 1.5 yield new families of knots with property P. We conclude this subsection with a result that describes an application along these lines.

For a braid $b \in B_n$, let $\widehat{b}$ denote the ordinary closure of b. Also for $m \in \mathbf{N}$, let $P_n^{(m)}$ denote the m-th term of the lower central series of the pure braid group P_n.

Corollary 6.4. *Suppose that $K \subset S^3$ is a special knot and let $b \in B_n$ such that $K = \widehat{b}$. Then, for every $m > 4$ and every $p \in P_n^{(m)}$ the knot represented by the closed braid $\widehat{bp}$ has property P.*

Proof: By the main result of [Sta] it follows that $v(K) = v(\widehat{bp})$, for every $v \in \mathcal{V}_4$, and every abelian group $\mathcal{G}$. Thus, the knot represented by $\widehat{bp}$ is *special*, for every $p \in P_n^{(m)}$ and every $m > 4$. The conclusion follows from Theorem 1.5. $\square$

6.2. **Knots with symmetries.**

A knot K is called periodic with period $n > 1$ if there exists an orientation preserving diffeomorphism $h : S^3 \to S^3$ of order n such that the set of fixed points f_h of h is a circle (i.e. $f_h \cong S^1$) disjoint from K and $h(K) = K$. By [Sm], f_h is the trivial knot and h is conjugate to a rotation with axis f_h. For a periodic knot K let $q_K = \mathrm{lk}(K, f_h)$. Periodic knots are shown to have property P in [CGLS]. It might be interesting to remark that property P also follows from Theorem 1.1 for all periodic knots with $q_K \not\equiv 1 \bmod p$ for some prime $p > 5$ dividing the period. To outline the proof of this statement, set $q = q_K$ and let $T_{p,q}$ denote the (p, q)-torus knot. Also let $a_{pq} = v_2(T_{p,q})$ and $c_{pq} = c_4(T_{p,q})$. A straightforward calculation, using the known formulas for $J_{T_{p,q}}$ and $\nabla_{T_{p,q}}$ ([J], [Ro]) shows that $a_{pq} \equiv -(q^2 - 1) \bmod p$ and that, if $q \equiv -1 \bmod p$, then c_{pq} is not divisible by p. By Theorem 3.1 of [Pr], for every Vassiliev invariant v of order $< p$ we have

$$v(K) \equiv v(T_{p,q}) \bmod p.$$

By examining these congruences we obtain that if $v_2(K) = 0$ then either $c_4(K) \neq 0$ or $q \equiv 1 \bmod p$. Thus property P follows by Theorem 1.1 and our assumption. We have shown the following:

Corollary 6.5. *Let K be a periodic knot. Suppose that there exists a prime number $p > 5$, dividing the period of K, such that $q_K \not\equiv 1 \bmod p$. Then K has property P.*

7. Nice links and nice H-diagrams of homotopy 3-spheres

Our goal in this section is to prove Theorem 3.7; we will use the notations and terminology

of Sections 2 and 3. We will also use the results and terminology given in the Appendix to the paper. We begin with the following:

Definition 7.1. *Let $M \in \mathcal{HM}$. A nice link $(L,\ \mathbf{f})$, with $M = \chi(S^3,\ L)$, will be called adequate if there exists an H-diagram of M associated to L that has properties (a)-(c) of the statement of Theorem A.5 and the properties of Proposition A.10, Corollaries A.11, A.12 and Lemmas A.13, A.14. An H-diagram satisfying these properties will be called adequate.*

By Theorem A.5, and the discussion following its proof, any nice link corresponding to M becomes adequate after a number of admissible moves. Thus, and in the light of Lemma 3.10, it is enough to prove Theorem 3.7 for adequate links. To give a brief outline of the strategy of the proof, let $M \in \mathcal{HM}$, let L be an adequate nice link such that $M = \chi(S^3,\ L)$ and let $(S, \mathbf{f}, \mathbf{x})$ be an adequate H-diagram of M associated to L. By Theorem 2.2, $H(\mathbf{x})$ can be embedded in S^3 so that the curves in $\mathbf{f}$ bound surfaces in $\overline{S^3 \setminus H(\mathbf{x})}$. By studying these surfaces we show there exists a division of L so that for every sublink $\hat{L}_J \subset L$ as in the statement of Theorem 3.7, there exist a corresponding nice H-diagram for $\hat{M}_J$ that satisfies the conclusion of Theorem 2.2. Thus, we will obtain that $\hat{M}_J \in \mathcal{HM}$ as desired.

7.1. **Systems of curves bounding surfaces in complements of handlebodies.**

In this subsection, first we discuss basic notions and introduce notation and terminology about nice Heegaard splittings. Then, we focus on nice H-diagrams of homotopy 3-spheres and state a technical result (Proposition 7.5) that is the main ingredient for the proof of Theorem 3.7. In the next subsection we will complete the proof of the theorem assuming Proposition 7.5; the remaining sections will be devoted to the proof of the proposition.

Let $H \subset S^3$ be an embedded handlebody and let x_1, x_2 be disjoint, simple closed curves on $S := \partial H$. Let $\alpha \subset S$ be a simple arc connecting x_1 to x_2 and such that $\mathrm{int}(\alpha) \cap (x_1 \cup x_2) = \emptyset$ and let N denote a regular neighborhood of $x_1 \cup \alpha \cup x_2$ on S. The boundary ∂N, has three components two of which are isotopic to x_1 and x_2. The third one, denoted by $x_1 \#_\alpha x_2$, is called *the band connected sum* of x_1 and x_2 along α. Now let $\mathbf{x}$ and $\hat{\mathbf{x}}$ be two complete systems of curves on $\partial(H)$. The system $\hat{\mathbf{x}}$ is said to be obtained from $\mathbf{x}$ by an *elementary slide*, if $\hat{\mathbf{x}}$ is obtained from $\mathbf{x}$ by replacing a curve $x_1 \in \mathbf{x}$ with a band connected sum $x_1 \#_\alpha x_2$, for some $x_2 \in \mathbf{x}$. It is well known that replacing a complete system of curves $\mathbf{x}$ in an H-diagram by one that is obtained from $\mathbf{x}$ by a number of elementary slides doesn't change the homeomorphism type of the 3-manifold defined by the H-diagram.

To continue, let $M \in \mathcal{HM}$. As discussed above, we will restrict ourselves to adequate nice links corresponding to M. Let $(L, \mathbf{f})$ be such a link, with $M = \chi(S^3, L)$, and let $(S, \mathbf{f}, \mathbf{x})$ be an adequate H-diagram for M corresponding to $(L, \mathbf{f})$, as constructed in Theorem A.5. With the notation as in the statement of Theorem A.5, we will view $\mathbf{f}$ and $\mathbf{x}$ on the boundary of the standard handlebody A_m whose meridian discs are bounded by the curves in $\mathbf{x}$. (In the statement of Theorems 2.1 and A.5 the curves are viewed on the boundary of $B_m := \overline{S^3 \setminus A_m}$.) We will think of A_m as a 3-ball B^3 with 2-handles $\hat{h}_1, \ldots, \hat{h}_m$ such that: The co-core of $\hat{h}_i$ is the meridional disc of A_m attached to x_i, the core of $\hat{h}_i$ is an arc y_i^1 with $\partial y_i^1 \subset \partial B^3$ and such that $y_i = y_i^1 \cup y_i^2$, where y_i^2 is an arc on $\partial(B^3)$. Let D_i^1, D_i^2 denote the discs on ∂B^3 on which $\hat{h}_i$ is attached (the *feet* of $\hat{h}_i$). By Theorem A.5, we have:

- (a_1) For every $j \neq i$, $f_i \cap y_j = \emptyset$.
- (a_2) The intersection $f_i \cap \hat{h}^j$, for $i \neq j$, is either empty or a collection $\mathcal{A}_{ij}$ of an even number of arcs on ∂A_m, such that: i) Each has one endpoint on ∂D_i^1 and the other on ∂D_i^2, its interior intersects x_j once and it is disjoint from y_j^1; ii) the elements in $\mathcal{A}_{ij}$ can be partitioned into pairs so that the arcs in each pair inherit opposite orientations from this of f_i.

- (a_3) The intersection $f_j \cap \hat{h}^j$ is a collection $\mathcal{A}_j$ of an odd number of arcs on ∂A_m, such that: i) Each has one endpoint on ∂D_j^1 and the other on ∂D_j^2; ii) there exists an arc $\alpha_j \in \mathcal{A}_j$ whose interior intersects each of x_j, y_j^1 once; iii) the interior of each arc in $\mathcal{A}_j \setminus \{\alpha_j\}$ intersects x_j once and it is disjoint from y_j^1; iv) the elements in $\mathcal{A}_j \setminus \{\alpha_j\}$ can be partitioned into pairs so that the arcs in each pair inherit opposite orientations from this of f_j.

Choose simple, disjoint curves $C_1, \dots C_m \subset \partial(B^3)$ such that C_i encircles $D_i^1 \cup D_i^2$ and separates it from all D_j^1, D_j^2, for $j \neq i$. By $(a_1) - (a_3)$, C_i can be chosen so that the following conditions are satisfied:

- (b_1) For $j \neq i$, the intersection $f_i \cap C_j$ is either empty or is a set of points $\{p_1^{ij}, \dots, p_{m_{ij}}^{ij}, q_1^{ij}, \dots, q_{m_{ij}}^{ij}\}$, such that, for $k = 1, \dots, m_{ij}$, the arc that connects p_k^{ij} to q_k^{ij} and runs over $\hat{h}^j$ contains an arc of $\mathcal{A}_{ij}$.
- (b_2) For $j = 1, \dots, m$, the intersection $f_j \cap C_j$ is a set of points $\{r_1^j, r_2^j, p_1^j, \dots, p_{m_j}^j, q_1^j, \dots, q_{m_j}^j\}$, such that: i) The arc that connects r_1^j to r_2^j and runs over $\hat{h}^j$ contains α_j; ii) for $k = 1, \dots, m_j$, the arc that connects p_k^j to q_k^j and runs over $\hat{h}^j$ contains an arc of $\mathcal{A}_j \setminus \{\alpha_j\}$; iii) for every $i \neq j$ such that $C_j \cap f_i \neq \emptyset$, the sets $\{p_1^{ij}, \dots, p_{m_{ij}}^{ij}\}$ and $\{q_1^{ij}, \dots, q_{m_{ij}}^{ij}\}$ lie on different components of $C_j \setminus \{r_1^j, r_2^j\}$; and iv) the sets $\{p_1^j, \dots, p_{m_j}^j\}$ and $\{q_1^j, \dots, q_{m_j}^j\}$ lie on different components of $C_j \setminus \{r_1^j, r_2^j\}$.

We need the following lemma the proof of which follows immediately from Proposition A.1 and Theorem A.5.

Lemma 7.2. *For every $j = 1, \dots, m$, the curve C_j can be chosen so that conditions $(b_1) - (b_2)$ are satisfied and $\mathbf{f}$ and C_j have no excess intersections. That is, $|f_i \cap C_j|$ is minimum in the isotopy classes of f_i, C_j, for all $f_i \in \mathbf{f}$.*

The intersections of $\mathbf{f}$ with a neighborhood of C_j are schematically illustrated in Figure 7. The dashed lines indicated the parts of f running over $\hat{h}^j$.

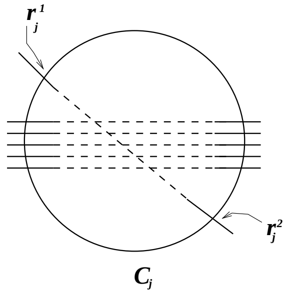

FIGURE 8. The intersections of $\mathbf{f}$ with a neighborhood of C_j.

In order to be able to state the main technical result needed for the proof of Theorem 3.7 we need to introduce some notation. We begin with the following definition:

Definition 7.3. *Let $(S, \mathbf{f}, \mathbf{x})$ be a nice H-diagram with $\mathbf{f} = \{f_1, \ldots, f_m\}$. Let the notation be as above and suppose that $\mathbf{f}$, $\mathbf{x}$ are embedded on $S := \partial A_m$. For a subset $I \subset \{2, \ldots, m\}$ let $\bar{I}$ denote the complement of I in $\{1, \ldots, m\}$. For a curve f_j, with $j \in \bar{I}$, let $f_I^j \subset S$ denote a curve obtained from f_j as follows: For every $i \in I$ and every pair $p_k^{ji}, q_k^{ji} \in \mathcal{P}_{ji}$, replace the interior of the arc $\alpha \subset f_j$ that connects p_k^{ji} to q_k^{ji} and runs over $\hat{h}^i$, with an arc $\beta \subset C_i$ such that $\partial(\alpha) = \partial(\beta)$. This can be done so that, after isotopying the arcs β away from C_i, f_I^j becomes embedded on S. Moreover, after an isotopy on S if necessary, the curves in $\mathbf{f}_I := \{f_I^j \mid j \in \bar{I}\}$ become disjoint. In particular, if $I = \emptyset$ then $\mathbf{f}_I = \mathbf{f}$.*

Remark 7.4. Although the collection $\mathbf{f}_I$ is not unique up to isotopy on S, by condition (a_2), the corresponding framed links whose components are the curves in $\mathbf{f}_I$ and the framings are defined by parallel curves on S, are isotopic in S^3. In fact, each of these links is isotopic to the sublink of L obtained by disregarding the components whose indices belong in I.

To continue with our notation, let $(L,\ \mathbf{f})$ be a nice link associated to $(S, \mathbf{f}, \mathbf{x})$, with $L = L_1 \cup \ldots \cup L_m$, and let $\mathcal{R} = \{R_1, \ldots, R_h\}$ be a division of L. Set $\mathcal{J} := \{1, \ldots, h\}$ and $\mathcal{I} := \{2, \ldots, m\}$. For a subset $J \subset \mathcal{J}$ recall the sublink $L_J \subset L$ introduced before the statement of

Theorem 3.7. A subset $J \subset \mathcal{J}$ corresponds to a subset of $I_J \subset \mathcal{I}$ by letting $I_J := \{k \in \mathcal{I}, \mid L_k \subset L_J\}$. For every $I \subset \mathcal{I}$, the curves $\mathbf{f}_I$ can be put on a standard handlebody $A_{g_I} \subset S^3$ of genus $g_I := m - \#I$, obtained from A_m by "pinching-off" the 2-handles corresponding to the curves $\{f_i \mid i \in I\}$.

Proposition 7.5. *Let $M \in \mathcal{HM}$ and let $(L,\ \mathbf{f})$ be an adequate nice link with $M = \chi(S^3,\ L)$. Let $(S, \mathbf{f}, \mathbf{x})$ be an adequate H-diagram of M corresponding to $(L,\ \mathbf{f})$ in the sense of Theorem A.5. Then, there exists a division $\mathcal{R} = \{R_1, \dots R_h\}$ of L such that: For every $J \subset \mathcal{J}$, there exists a collection of curves $\mathbf{f}_{I_J} \subset S$ as in Definition 7.3, and an embedding*

$$\phi_{I_J} : A_{g_{I_J}} \to S^3,$$

such that the curves $\phi_{I_J}(\mathbf{f}_{I_J}) \subset \phi_I(A_{g_{I_J}})$ bound disjointedly embedded, oriented surfaces in $\overline{S^3 \setminus \phi_I(A_{g_{I_J}})}$.

In the next subsection we give the proof of Theorem 3.7 assuming Proposition 7.5. The proofs of Proposition 7.5 will occupy the remaining subsections of Section 7.

7.2. **From H-diagrams to surgery presentations.**

The purpose of this subsection is to complete the proof of Theorem 3.7 assuming Proposition 7.5.

Proof: [of Theorem 3.7] By Lemma 3.10 it is enough to prove the theorem for adequate nice links. To that end, let $(L,\ \mathbf{f})$ be an adequate nice link with $M = \chi(S^3,\ L)$. Let $(S, \mathbf{f}, \mathbf{x})$ be an adequate H-diagram of M corresponding to $(L,\ \mathbf{f})$ and let $\mathcal{R} = \{R_1, \dots R_h\}$ be a division of L as in the statement of Proposition 7.5. To make the idea of the proof clearer first we treat a special case in which the notation involved is simpler:

Case 1. Suppose that for every $R_i \in \mathcal{R}$ we have $\#R_i = 1$. In this case the statement of Proposition 7.5 becomes as follows: For every $I \subset \mathcal{I}$, there exists a collection of curves $\mathbf{f}_I \subset S$ as in Definition 7.3, and an embedding

$$\phi_I : A_{g_I} \to S^3,$$

such that the curves $\phi_I(\mathbf{f}_I) \subset \phi_I(A_{g_I})$ bound disjointedly embedded, oriented surfaces in $\overline{S^3 \setminus \phi_I(A_{g_I})}$.

We will show that for every sublink $L' \subset L$, $M_{L'} \in \mathcal{HM}$. By Lemma 3.4 we have that $M_{L'} \in \mathcal{M}$. Notice that if $L^{'}$ doesn't contain the *special* component $L_1 \subset L$, then $M_{L'} = S^3$. Thus, in this case, $M_{L'} \in \mathcal{HM}$. It remains to show that $M_{L'}$ is a homotopy 3-sphere for every sublink $L^{'}$ that contains L_1. Let $L^{'}$ be such a sublink and let $I \subset \mathcal{I}$ be the set of indices corresponding to the components of $L \setminus L^{'}$. Also, let $\mathbf{f}_I$ be a collection of simple curves as in Definition 7.3 that satisfies the conclusion of Proposition 7.5. Recall that L is framed; the components of L are the curves in $\mathbf{f}$ and the framings are determined by parallel curves on S. Thus $L^{'}$ is framed with framing $\mathbf{f}^{'}$ obtained by restricting $\mathbf{f}$ to $L^{'}$. Let $\mathbf{f}^*$ denote the complete system of curves obtained from $\mathbf{f}$ by replacing f_i with y_i (the longitude of $\hat{h}^i$) for all $i \in I$. Let $\mathbf{x}_I = \{x_1\} \cup \{x_i \mid i \in \bar{I}\}$.

By Definition 7.3, the complete systems $\mathbf{f}_I \cup \{y_i \mid i \in I\}$ and $\mathbf{f}^*$ are related by elementary slides and thus the H-diagrams

$$A_1 := (S,\, \mathbf{f}^*,\, \mathbf{x}) \;\text{ and }\; A_2 := (S,\; \mathbf{f}_I \cup \{y_i \,|\, i \in I\},\; \mathbf{x}),$$

define the same 3-manifold, say M^*. Let S_{g_I} denote the boundary of the handlebody A_{g_I} obtained from A_m by pinching off the 2-handles corresponding to $\{(x_i, y_i) \,|\, i \in I\}$. Since A_2 and $(S_{g_I}, \mathbf{f}_I, \mathbf{x}_I)$ are related by a series of distabilizations they define homeomorphic 3-manifolds. That is, we have

$$M^* = H(\mathbf{f}_I) \cup_{S_{g_I}} H(\mathbf{x}_I).$$

By Theorem 2.2 it follows that M^* is a homotopy 3-sphere. Let $(\hat{L}, \mathbf{f}_I)$ denote the framed link whose components are the curves $\mathbf{f}_I$ and with the framing determined by parallel curves on S_{g_I}. The framed links $(\hat{L}, \mathbf{f}_I)$ and $(L^{'}, \mathbf{f}^{'})$ are isotopic (Remark 7.4). Thus, $M_{L'} = \chi(S^3, \hat{L})$ and to finish the proof it is enough to show that $\chi(S^3, \hat{L})$ is a homotopy 3-sphere. By Lemma 2 of [R],

$$\chi(S^3, \hat{L}) = \big(H(\mathbf{f}_I) \cup_{S_{g_I}} H(\mathbf{x}_I)\big) \,\#\, \big(H(\mathbf{f}_I) \cup_{S_{g_I}} H(\mathbf{y}_I)\big)\,,$$

where $\mathbf{y}_I = \{y_1\} \cup \{y_i \,|\, i \in \bar{I}\}$. Now the conclusion, in this case, follows since the first summand is M^* and the second is S^3 (recall that $|f_i \cap y_j| = \delta_{ij}$).

Case 2. Suppose that there are $R_i \in \mathcal{R}$ such that $\#R_i > 1$. We will use the notation of the statement of Theorem 3.7. Fix any $J_1 \subset \mathcal{J}$. Since, by definition, L_{J_1} doesn't contain the special component of L we have $M_{J_1} = S^3$; thus $M_{J_1} \in \mathcal{HM}$. So to finish the proof of the theorem we must show that $M_{\hat{J}_1} \in \mathcal{HM}$. Let J denote the complement of J_1 in $\mathcal{J}$ and let I_J the subset of B corresponding to J as before the statement of Proposition 7.5. By definition, I_J is the set of indices of the components of L that do not belong in $L_{\hat{I}_1}$. By assumption, there exists a collection of simple closed curves $\mathbf{f}_{I_J} \subset S$ and an embedding

$$\phi_{I_J} : A_{g_{I_J}} \to S^3,$$

such that the curves $\phi_{I_J}(\mathbf{f}_{I_J}) \subset \phi_I(A_{g_{I_J}})$ bound disjointedly embedded, oriented surfaces in $\overline{S^3 \setminus \phi_I(A_{g_{I_J}})}$. Now the conclusion will follow applying the argument of Case 1 for $L' := L_{\hat{I}_1}$, $I := I_J$ and $\mathbf{f}_I := \mathbf{f}_{I_J}$. $\square$

7.3. **Nice H-diagrams that are even.**

Let $M \in \mathcal{HM}$ and let $(S, \mathbf{f}, \mathbf{x})$ be an adequate nice H-diagram associated to M. By Theorem 2.2, there exists an embedding $\phi(A_m) \subset S^3$ such that the curves $\phi(\mathbf{f}) \subset \phi(A_m)$ bound disjointedly embedded oriented surfaces, say $S_1, \dots, S_m$, in $\overline{S^3 \setminus \phi(A_m)}$. For the remaining of the section we will fix such an embedding once and for all. By the proof of Theorem 1 of [RR], one can start with a degree 1 map

$$F : S^3 \to M = A_m \cup_S H(\mathbf{f}),$$

and take $\phi(A_m) = F^{-1}(A_m)$. The map F can be taken to be the identity on B^3 and thus we can assume that $\phi(B^3) = B^3$. We often use B^3 to denote $\phi(B^3)$.

For $1 \leq j \leq m$, let $D_j \subset \partial B^3$ be a disc with $\partial D_j = C_j$ and $(D_j^1 \cup D_j^2) \subset \mathrm{int}(D_j)$. Set $\Phi := \cup_{j=1}^m \phi(f_j)$. It will be convenient for us to think that the curves in $\phi(\mathbf{f})$ are colored, in one-to-one fashion, by the elements of $\{1, \dots, m\}$. Then the points in $C_j \cap \Phi$ are colored by the colors of the curves they belong on.

To continue we need some notation and terminology: Let $\{I_1, \dots, I_h\}$ be a partition of $\mathcal{I} := \{2, \dots, m\}$ into disjoint sets. For a subset $J \subset \mathcal{J} := \{1, \dots, h\}$ set $I_J := \cup_{j \in J} I_j$. The following definitions will be helpful in simplifying our exposition:

Definition 7.6. *Given $I \subset \mathcal{I}$ and $i \in I$ we will say that $\Phi^I := \cup_{j \notin I} \phi(f_j)$ is even with respect to C_i iff the points of $C_i \cap \Phi^I$ that lie in each component of $C_i \setminus \{r_1^i, r_2^i\}$ can be paired by arcs that are disjointly embedded in D_i so that each arc connects points of the same color. We will say that Φ^I is even iff it is even with respect to C_i, for every $i \in I$. Finally, we will say that $\phi(\mathbf{f})$ is even iff there exists a partition of $\mathcal{I}$ such that: i) The corresponding partition of $\{L_2, \dots, L_m\}$ is a division of L; and ii) with the notation as above, Φ^{I_J} is even for every $\emptyset \neq J \subset \mathcal{J}$.*

Remark 7.7. Note that if $I, J \subset \mathcal{I}$ are disjoint and both of Φ^I, Φ^J are even the $\Phi^{I \cup J}$ is even. Thus in the definition of $\phi(\mathbf{f})$ being even we may replace condition (ii) by the following weaker condition: Φ^{I_i} is even for every $1 \leq i \leq h$.

Definition 7.8. *Let $1 \leq j \leq m$ and let the notation be as in $(b_1)-(b_2)$. A pair of dual points will be any of the following (unordered) pairs:*

$$(p_k^{ij},\ q_k^{ij}), \quad 1 \leq i, j \leq m\ , \quad 1 \leq k \leq m_{ij},$$

$$(p_k^{j},\ q_k^{j}), \quad 1 \leq j \leq m\ ,\ 1 \leq k \leq m_j,$$

$$(r_1^j\ , r_2^j), \quad 1 \leq j \leq m.$$

Given a subset $P \subset (C_j \cap \Phi)$, the dual set of P is the subset of $C_j \cap f$ consisting of all the points dual to these in P.

Next we will prove Proposition 7.5 in the special case that the embedding $\phi(A_m) \subset S^3$ fixed in the beginning of this subsection has the property that $\phi(\mathbf{f})$ is even. As we will see in the subsequent subsections, under the hypothesis that a nice link corresponding to $(S, \mathbf{f}, \mathbf{x})$ doesn't contain any parallel components the proof of Proposition 7.5 reduces, in fact, to this special case. See Proposition 7.21.

Proposition 7.9. *Let $(S, \mathbf{f}, \mathbf{x})$ be an adequate nice H-diagram representing a homotopy 3-sphere and let $\phi(\mathbf{f}) \subset \phi(A_m)$ be an embedding as fixed in the beginning of the subsection. If $\phi(\mathbf{f})$ is even then the conclusion of Proposition 7.5 is true.*

Proof: Let $\{I_1, \dots, I_h\}$ be a partition of $\mathcal{I}$ that satisfies Definition 7.6. For simplicity of notation, we will treat the case that, for $1 \leq k \leq h$, $\#I_k = 1$ in detail. The general case is completely analogous (compare also with the proof of Theorem 3.7). Now we have $\mathcal{I} = \mathcal{J}$. We must show the following: For every $I \subset \mathcal{I}$, there exists a collection of curves $\mathbf{f}_I \subset S$ as in Definition 7.3, and an embedding

$$\phi_I : A_{g_I} \to S^3,$$

such that the curves $\phi_I(\mathbf{f}_I) \subset \phi_I(A_{g_I})$ bound disjointedly embedded, oriented surfaces in $\overline{S^3 \setminus \phi_I(A_{g_I})}$.

Clearly, the conclusion is true for the empty subset $I \subset \{2, \dots, m\}$ and for all $m \in \mathbf{N}$. Suppose, inductively, that the conclusion has been proven for all $k \leq m$ and all subsets $I \subset \{2, \dots, k\}$ with $\#I < l \leq m$. Let $(S, \mathbf{f}, \mathbf{x})$ be as in the statement of the proposition and let $J \subset \{2, \dots, m\}$ be a subset with $\#J = l$.

Fix $j \in J$. To simplify our notation we will, without loss of generality, assume that $\phi(f_i) \cap \phi(\hat{h}^j) \neq \emptyset$, for all $i \neq j$. Set

$$\bar{\mathcal{A}}_j := \cup_{i \neq j} \phi(\mathcal{A}_{ij}) \text{ and } \mathcal{S}_j := \{S_i \mid i \neq j\}.$$

Here the sets $\mathcal{A}_{ij}$ are as in condition (a_2). The assumption that $\phi(\mathbf{f})$ is even implies that we can find two arcs $\alpha_1, \alpha_2 \in \bar{\mathcal{A}}_j$ of the same color, that are innermost. That is, for $r = 1, 2$, there is an arc $\gamma_r \subset \partial(D_j^r)$ that connects one endpoint of α_1 to one of α_2 and such that $\mathrm{int}(\gamma_r) \cap \Phi^j = \emptyset$. Let $S_k \in \mathcal{S}_j$ such that, for $r = 1, 2$, $\alpha_r \subset \partial(S_k)$ and let $N_r \subset S$ denote a regular neighborhood of γ_r on S. Consider the surface $\hat{S}_k := S_k \cup N_1 \cup N_2$ and let f_{11}, f_{12}, f_{13} denote the components of $\partial(\hat{S}_k)$. By construction, one of them, say f_{13}, lies entirely on $\phi(\hat{h}^j)$ and it bounds an embedded disc $D \subset \phi(\hat{h}^j)$. Let $\bar{S}_k$ denote the surface obtained by attaching D to $\hat{S}_k$ and then isotopying away from $\phi(\hat{h}^j)$ in the complement of $\phi(A_m)$. Now $\partial(\bar{S}_k)$ has two components. Now we can repeat this process with the next pair of innermost arcs in $\bar{\mathcal{A}}_j$ and we can continue till all the arcs in $\bar{\mathcal{A}}_j$ are eliminated. At the end, each surface $S_i \in \mathcal{S}_j$ has given rise to a connected oriented surface Σ_i, with many boundary components. Let

$$\bar{\mathcal{S}}_j := \{\Sigma_i \mid i \neq j\}.$$

Next we attach bands to every $\Sigma_i \in \bar{\mathcal{S}}_j$ to obtain a surface $\bar{\Sigma}_i$, with connected boundary. This should be done so that:

(1) The boundary $\partial(\bar{\Sigma}_i)$ is isotopic to a curve $\phi(f^i_{\{j\}})$ on S. Here the notation is as in Definition 7.3.

(2) The surfaces $\{\bar{\Sigma}_i\}_{i\neq j}$ are mutually disjoint, disjoint from the interior of $\phi(A_m)$ and disjoint from $\phi(\hat{h}^j)$.

To describe how to achieve this, let γ, γ' denote the components of $C_j \setminus \{r^j_1\ , r^j_2\}$. After isotopy we can arrange so that the surfaces in $\bar{\mathcal{S}}_j$ are disjoint from $\phi(\hat{h}^j)$. Furthermore, the intersection of $\Delta := D_j \setminus (D^1_j \cup D^2_j)$ and the boundary curves of the surfaces in $\bar{\mathcal{S}}_j$ consists of a collection of disjoint arcs, properly embedded in Δ. These arcs are divided into two sets , say $\mathcal{A}$, $\mathcal{A}'$, so that each arc in $\mathcal{A}$ (resp. $\mathcal{A}'$) connects two points in γ (resp. γ'). See Figure 9.

Define a function $F : \mathcal{A} \longrightarrow \mathcal{A}'$ by the rule that the image of an arc $\alpha \in \mathcal{A}$ is the arc in $\mathcal{A}'$ that connects the points in γ' that are dual to the points in $\partial(\alpha)$. Clearly the arcs $\alpha, F(\alpha)$ are of the same color and the pairs $(\alpha, F(\alpha))$ are in one to one correspondence with the pairs of innermost arcs $(\alpha_1, \alpha_2) \in \bar{\mathcal{A}}_j \times \bar{\mathcal{A}}_j$ used during the construction of $\bar{\mathcal{S}}_j$. Now, starting from the arcs that are outermost in Δ, for each pair $(\alpha, F(\alpha)) \in \mathcal{A}\times\mathcal{A}'$ attach a band, say $b_\alpha \cong [0,\ 1]\times[0,\ 1]$, to the surface in $\Sigma \in \bar{\mathcal{S}}_j$ whose boundary contains $\alpha, F(\alpha)$. This should be done so that: i) b_α lies in Δ and it is disjoint from all the surfaces in $\bar{\mathcal{S}}_j \setminus \{\Sigma\}$; ii) b_α is disjoint from the interior of Σ; iii) we have $[0,\ 1]\times\{0\} \subset \alpha$ and $[0,\ 1] \times \{1\} \subset F(\alpha)$. Notice that the bands corresponding to different pairs may intersect as one band may cross the interior of another (see Figure 9). For each band b_α we may isotope the part corresponding to $[0,\ 1] \times (0,\ 1)$ off Δ in the complement of $\phi(A_m)$. These isotopies should be done so that the bands become mutually disjoint. At the end we arrive at a collection of surfaces $\{\bar{\Sigma}_i\}_{i\neq j}$ that satisfy conditions (1),

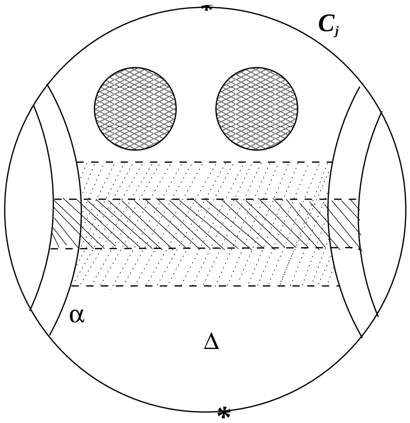

FIGURE 9. Arcs in the sets $\mathcal{A}$, $\mathcal{A}'$: The points $r_j^{1,2}$ are marked by $*$ and the discs $D_j^{1,2}$ are drawn filled in pattern. The bands b_α connecting pairs $(\alpha, F(\alpha))$ are drawn in dashed line and filled in pattern. During the process of obtaining the surfaces $\{\bar{\Sigma}_i\}_{i\neq j}$ the entire b_α except of the two arcs of $\partial(b_\alpha)$ drawn in dashed line, is isotoped off Δ in the complement of $\phi(A_m)$.

(2) above. The conclusion follows by the induction hypothesis applied to the subset $I := J \setminus \{j\}$ of $\{2, \ldots, m\}$. $\square$

Remark 7.10. For the arguments in this subsection and in the next subsection (whose puprose is to introduce notation and terminology) the restriction to adequate nice H-diagrams is not nessecary. In particular, Definitions 7.6, 7.8 and the proof of Proposition 7.9 go through for nice H-diagrams in general.

7.4. **Planar graphs and intersections of surfaces and discs.**

In order to be able to analyze and understand the complexity of the embedding $\phi(A_m)$ and the surfaces $S_1, \ldots, S_m$ we define the notion of a *containing collection*:

Definition 7.11. *For $i = 1, \ldots, m$, let $B_i^3 \subset S^3$ be an embedded 3-ball such that:*

$$B_i^3 \cap B^3 = \partial B_i^3 \cap \partial B^3 = D_i, \quad \phi(\hat{h}^i) \subset B_i^3, \quad \text{and}$$

$$\phi(\hat{h}^i) \cap \partial B_i^3 = \phi(\hat{h}^i) \cap \partial B^3 = D_i^1 \cup D_i^2.$$

Let $\bar{D}_i$ denote the complement of $\mathrm{int}(D_i)$ *in ∂B_i^3. The set of 3-balls $\mathcal{B} := \{B_1^3, \dots, B_m^3\}$ will be called a containing collection for $\phi(A_m)$ and $\mathcal{D} := \{\bar{D}_1, \dots, \bar{D}_m\}$ will be called the set of discs associated to $\mathcal{B}$.*

A containing collection $\mathcal{B}$ will be called simple iff, for every $i, j = 1, \dots, m$, the intersection $\mathrm{int}(B_i^3) \cap \phi(\hat{h}^j)$ *has at most one component.*

The next lemma allows us to restrict ourselves to simple containing collections.

Lemma 7.12. *There exists a simple containing collection $\mathcal{B}$ for $\phi(A_m)$.*

Proof: Let $\mathcal{B}' = \{B_1', \dots, B_m'\}$ be any containing collection for $\phi(A_m)$, and let $\mathcal{D}'$ be the set of discs associated to it. By enlarging B_1' if necessary, we may assume that, for every $j \neq 1$, the set $\phi(\hat{h}^j) \cap \mathrm{int}(B_1')$ is non-empty and connected. Furthermore, we may assume that all of $\phi(\hat{h}^j)$, but two small, unknotted and unlinked tubes, T_1, T_2, near the 3-ball part of $\phi(A_m)$, is contained in B_1'. Now construct a containing collection $\mathcal{B}$ as follows: Let $B_1^3 = B_1'$. For $j = 2, \dots m$, let B_j' denote a 3-ball obtained as a product $I \times D_j$, where D_j is the disc containing the feet of $\phi(\hat{h}^j)$ fixed earlier, such that T_1, T_2 are properly embedded in B_j'. Now let B_j^3 be a 3-ball obtained by first isotopying B_1' so that $B^3 \cap B_1' = \emptyset$, and then replacing B_1' with $B_1' \cup B_j'$. The collection $\mathcal{B} := \{B_j^3 \mid 1 \leq j \leq m\}$ has the desired properties. $\square$

In the view of Lemma 7.12, for the remaining of the section we will restrict ourselves to simple containing collections. Next, we introduce terminology related to the intersections of the surfaces $S_1, \dots, S_m$ and the 3-balls in a containing collection for $\phi(A_m)$. To begin, set

$$\mathcal{S} := \cup_{i=1}^m S_i \quad \text{and} \quad \mathcal{H} := \cup_{i=1}^m \phi(\hat{h}^i),$$

and let $\mathcal{B}$ be a containing collection for $\phi(A_m)$, with associated set of discs $\mathcal{D}$. By general position, we may assume that each component of $\mathcal{H} \cap \mathrm{int}(\bar{D}_j)$ is a meridian disc of some 2-handle in $\mathcal{H}$. The arc components of the intersections $\mathcal{S} \cap \bar{D}_j$ are partitioned into types as follows (see Figure 10):

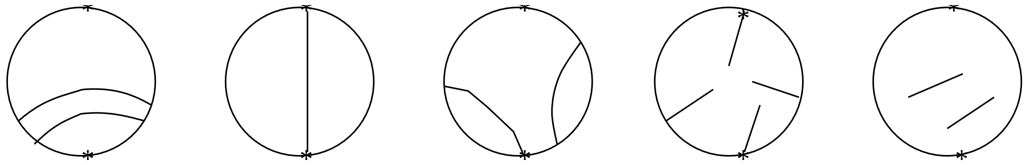

FIGURE 10. Types of arcs of $\mathcal{S} \cap \bar{D}_j$. From left to right: Arcs of type I_1, I_2, II, III_1 and III_2. The points marked by "$*$" indicate $r^j_{1,2}$.

- Type I_1: An arc of $S_i \cap \bar{D}_j$ that is properly embedded on both S_i and $\bar{D}_j$ and with its boundary on different components of $C_j \setminus \{r^j_1, r^j_2\}$.
- Type I_2: An arc of $S_j \cap \bar{D}_j$ that is properly embedded on both S_j and $\bar{D}_j$ and with boundary $\{r^j_1, r^j_2\}$.
- Type II: An arc of $\alpha \subset S_i \cap \bar{D}_j$ that is properly embedded on both S_i and $\bar{D}_j$ and such that either both the points of $\partial(\alpha)$ lie on the same component of $C_j \setminus \{r^j_1, r^j_2\}$ or one of them belongs in $\{r^j_1, r^j_2\}$.
- Type III_1: An arc of $S_i \cap \bar{D}_j$, with one endpoint on $\partial(\bar{D}_j)$ and the other in $\mathrm{int}(\bar{D}_j)$.
- Type III_2: An arc of $S_i \cap \bar{D}_j$, with both its endpoints in $\mathrm{int}(\bar{D}_j)$.

Let $\mathcal{B}$ be a simple containing collection for $\phi(A_m)$ with associated set of discs $\mathcal{D}$. For $k \neq j$ set

$$Z_{kj} := \mathrm{int}(\bar{D}_j) \cap \phi(\hat{h}^k).$$

Then, either $Z_{kj} = \emptyset$ or it consists of two meridian discs of $\phi(\hat{h}^k)$. In order to study the properties of the intersections $\mathcal{S} \cap \bar{D}_j$, it will be convenient to record the sets

$$\mathcal{H}_j := (\mathcal{S} \cup \mathcal{H}) \cap \bar{D}_j,$$

by graphs in $\bar{D}_j$. To a subset $X \subset \mathcal{H}_j$, such that $X \cap \mathcal{H} \neq \emptyset$ or $X \cap \partial(\bar{D}_j) \neq \emptyset$, we assign a graph G_X as follows: Every component of $\mathcal{H} \cap \mathrm{int}(\bar{D}_j) \cap X$ or of $\mathcal{S} \cap \partial(\bar{D}_j) \cap X$ is a vertex of G_X. Also, every arc component of $\mathcal{S} \cap \bar{D}_j \cap X$ is an edge of G_X. The vertices of G_X corresponding to components of $\mathcal{H} \cap \mathrm{int}(\bar{D}_j) \cap X$ (resp. of $\mathcal{S} \cap \partial(\bar{D}_j) \cap X$)

will be called *internal* (resp. *external*. Since the curves of $\phi(\mathbf{f})$ are colored in one-to-one fashion by the elements of $\{1,\dots,m\}$ so are the surfaces $S_1,\dots,S_m$. For $1 \le j \le m$, we can think of $\phi(\hat{h}^j)$ as colored by the color j. Thus the vertices and edges of the graphs $\{G_{\mathcal{H}_j}\}_{j=1}^m$ associated to the sets $\{\mathcal{H}_j\}_{j=1}^m$ are also colored. Since by Lemma 7.2 the curve C_j is chosen so that $\phi(\mathbf{f})$ and C_j have no excess intersections and $S_1,\dots,S_m$ are disjoint and disjoint from the interior of $\phi(A_m)$, $G_{\mathcal{H}_j}$ must be embedded in $\bar{D}_j$ (it is a planar graph). This observation will be useful to us in what follows.

7.5. **Geometric combinatorics of the graphs $G_{\mathcal{H}_j}$.**

Recall that we are working with a nice H-diagram $(S,\mathbf{f},\mathbf{x})$ of a 3-manifold $M \in \mathcal{HM}$ associated to an adequate nice link $(L,\mathbf{f})$ as in Theorem A.5. The link $(L,\mathbf{f})$ is obtained from a projection $P(K)$ of a unit-framed knot $(K,\pm 1)$, with $M = \chi(S^3,\ K)$, through the process given in the proof of Lemma 3.1. Recall also that we have fixed an embedding $\phi(A_m) \subset S^3$ such that the curves $\phi(\mathbf{f}) \subset \phi(A_m)$ bound disjointedly embedded oriented surfaces in $\overline{S^3 \setminus \phi(A_m)}$. In this section we study the properties of the planar graphs $\{G_{\mathcal{H}_j}\}_{j=1}^m$ introduced above, and prove some preparatory lemmas that we need for the proof of Proposition 7.5.

Lemma 7.13. *There exists a simple containing collection $\mathcal{B}$ of $\phi(A_m)$ such that for every $1 \le j,k \le m$, the graph $G_{\mathcal{H}_j}$ has the following properties:*

a) An edge of color k that connects two k-colored vertices corresponds to a type I_2 or type I_1 component of $S_k \cap \bar{D}_k$.

b) An edge of color k that connects two vertices of color t, for some $t \ne j,k$ corresponds to a type I_1 component of $S_k \cap \bar{D}_t$.

c) An edge of color k that connects a vertex of color t to one of color s, for some $t \ne s$, corresponds to a

Proof: Take $\mathcal{B}$ to be a simple collection constructed as in the proof of Lemma 7.12. Properties (a)-(c) follow immediately from the definitions and the constructions of $\mathcal{B}$ and $G_{\mathcal{H}_j}$. $\square$

The proof of the next lemma is an easy exercise left to the reader:

Lemma 7.14. *Let G be a connected planar graph. The number of odd valence vertices of G is even.*

Let $\mathcal{B}$ be a simple containing collection for $\phi(A_m)$ that satisfies the properties of Lemma 7.13, and let $\mathcal{D}$ be the set of discs associated to it. For $1 \leq j \leq m$, let γ_j, γ'_j denote the components of $C_j \setminus \{r^j_1, r^j_2\}$.

Lemma 7.15. *Let $1 \leq j \neq k \leq m$. The graph $G_{\mathcal{H}_j}$ cannot contain a k-colored edge that connects an external vertex that lies on γ_j to one that lies on γ'_j.*

Proof: Suppose, for some $k \neq j$, there exists a k-colored edge $e \subset G_{\mathcal{H}_j}$ that connects a (k-colored) vertex $p \in \gamma_j$ to a (k-colored) vertex $q \in \gamma'_j$. From the construction of $G_{\mathcal{H}_j}$ it follows that p and q must be dual points. Let γ, γ' denote the components of $C_j \setminus \{p, q\}$. By (b_1), (b_2) it follows that each of γ, γ' contains an odd number j-colored vertices of $G_{\mathcal{H}_j}$. We conclude that there is a component $G \subset G_{\mathcal{H}_j}$ such that G has an odd number of vertices on γ. By Lemma 7.14, G must contain an even number of vertices of odd valence. By (a_2), each internal vertex of G has even valence. Thus, G must also have vertices on γ'. But then we have $G \cap e \neq \emptyset$, which contradicts the fact that $G_{\mathcal{H}_j}$ is embedded in $\bar{D}_j$. $\square$

Lemma 7.15 allows us to eliminate components of type I_1 from the intersection $\mathcal{S} \cap \mathcal{D}$, for any collection of discs $\mathcal{D}$ associated to a containing collection of $\phi(A_m)$. Next we turn our attention to components of type III_1. The main technical result about such components is the following lemma:

Basic Lemma 7.16. *Let $\mathcal{B}$ be a simple containing collection for $\phi(A_m)$, with associated set of discs $\mathcal{D}$, with the properties of Lemma 7.13. Let $i \neq j$ with $i, j \neq 1$. Suppose that there exists $k \neq i, j$ such that $S_k \cap \bar{D}_j$ contains a component of type III_1 that ends on Z_{ij}. Suppose, moreover, that L doesn't contain any components that are parallel. Then we have:*

(1) For every $s \in \{i, j\}$ and every $k \neq i, j$, all the type III_1 components of $S_k \cap \bar{D}_s$ end either on Z_{ts} or on Z_{ks}. Here, $\{t\}$ denotes the complement of $\{s\}$ in $\{i, j\}$.

(2) There is $s \in \{i,\ j\}$ so that $\Phi^{\{i,j\}} := \cup_{t \neq i,j} \phi(f_t)$ is even with respect to C_s.

(3) There is a tangle $T(a,b)$ of the projection $P(K)$, with $ab = \pm 1$, and such that one of the components L_i, L_j corresponds to $[a]$ and the other to $[b]^{-1}$ as indicated in Figure 6.

A main ingredient of the proof of the Basic Lemma, which is postponed for a later subsection, is Proposition A.1. In the rest of this subsection we prove two auxiliary lemmas needed for the proof of Proposition 7.5. The proofs of these lemmas do not assume the Basic Lemma.

Lemma 7.17. *Let $\mathcal{B}$ be a simple containing collection for $\phi(A_m)$ with the properties of Lemma 7.13. Let $1 \leq j \leq m$ and let γ, γ' denote the components of $C_j \setminus \{r_j^1,\ r_j^2\}$. Suppose that for every $s \neq j$, an s-colored edge of $G_{\mathcal{H}_j}$ that emanates from C_j and ends on an internal vertex can only end on an s-colored vertex. For $1 \leq s \leq 4$, let $p_s \in \gamma$ be an external vertex of $G_{\mathcal{H}_j}$ such that: i) p_s is connected to an internal vertex by an edge of $G_{\mathcal{H}_j}$; ii) we have $p_2 \in \operatorname{int}(\gamma_1)$ and $p_4 \in \operatorname{int}(\gamma_1')$, where γ_1 denotes the subarc of γ that connects p_1, p_3 and $\gamma_1' := \gamma \setminus \gamma_1$; and iii) there exist $k, i \neq j$ with $k \neq i$, such that p_1, p_3 (resp. p_2, p_4) are i-colored (resp. k-colored). Then, there exists $1 \leq s \leq 4$ and an external vertex $q \in \gamma'$, such that p_s' and q are connected by an edge of $G_{\mathcal{H}_j}$, where p_s' is the dual point of p_s.*

Proof: We will assume that there is no $1 \leq s \leq 4$ such that p_s' is connected to an external vertex on γ' and arrive to a contradiction. By our earlier assumptions and Lemma 7.15 we have:

(a) p_1, p_3 (resp. p_2, p_4) are connected to i-colored (resp. k-colored) internal vertices v_1, v_3 (resp. v_2, v_4) by edges e_1, e_3 (resp. e_2, e_4).

(b) Each of p_1', p_3' (resp. p_2', p_4') is connected to one of v_1, v_3 (resp. v_2, v_4) by edges e_1', e_3' (resp. e_2', e_4').

We claim that we must have $v_1 \neq v_3$ or $v_2 \neq v_4$. For, assume that $v := v_1 = v_3$ and $v' := v_2 = v_4$ and set $G := e_1 \cup v \cup e_3$ and $G' := e_2 \cup v' \cup e_4$. Clearly we must have $G \cap G' \neq \emptyset$, which contradicts the fact that $G_{\mathcal{H}_i}$ is embedded in $\bar{D}_i$.

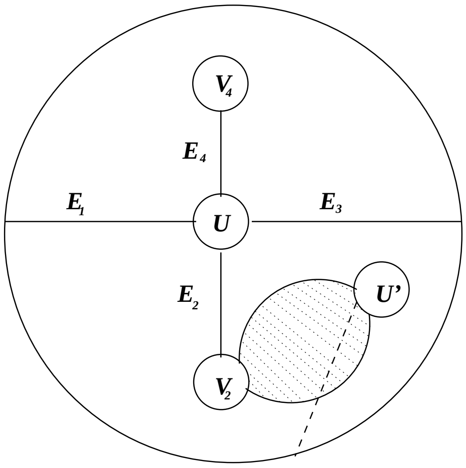

FIGURE 11. Part of the graph $G_{\mathcal{H}_i} \subset \bar{D}_i$: The disc filled in pattern is the disc D defined by the k-colored edges E_2', E_4'. The edge drawn in dashed line illustrates one of the i-colored edges E_3', E_1' that connect U' to an external vertex of $G_{\mathcal{H}_i}$.

Assume, with out loss of generality, that $v_1 \neq v_3$ and let us examine the graph $G_{\mathcal{H}_i}$. By Lemma 7.13 and the construction of $G_{\mathcal{H}_i}$ we have the following: There exist i-colored edges $E_1, E_3 \subset G_{\mathcal{H}_i}$, corresponding to e_1, e_3 respectively, that emanate from distinct components of $C_i \setminus \{r_i^1,\ r_i^2\}$ and end on an internal vertex $U \in G_{\mathcal{H}_i}$. The vertex U is j-colored and corresponds to the component γ of $C_j \setminus \{r_j^1,\ r_j^2\}$. Also, let E_2, E_4 denote the edges of $G_{\mathcal{H}_i}$ corresponding to e_2, e_4 respectively, and let V_2, V_4 denote the vertices corresponding to v_2, v_4. The graph $G := E_1 \cup U \cup E_3$ divides $\bar{D}_j$ into two regions. By assumption ii) of the statement of the lemma, V_2, V_4 must lie in different regions (see Figure 11). Thus, in particular, $V_2 \neq V_4$. Let U' denote the second j-colored vertex of $G_{\mathcal{H}_i}$ and assume without loss of generality that U' lies in the same region with V_2.

For $1 \leq s \leq 4$, let E_s' denote the edge of $G_{\mathcal{H}_i}$ that emanates from U' and corresponds to $e_s' \subset G_{\mathcal{H}_j}$. By assumption E_2', E_4' are k-colored while E_1', E_3' are i-colored. By (b), each of E_2', E_4' ends either on V_2 or on V_1. But since $G_{\mathcal{H}_i}$ is embedded in $\bar{D}_i$, we conclude that both of E_2', E_4' must end on V_2.

Now E_2', E_4' define a disc $D \subset \bar{D}_j$; this disc is shown filled in pattern in Figure 11. By assumption ii) of the statement of the lemma, one of E_3', E_1', say E_1', must emanate from inside D. Since by (b) E_1' ends on C_j we have $E_1' \cap \partial(D) \neq \emptyset$, contradicting the fact that $G_{\mathcal{H}_i}$ is embedded in $\bar{D}_i$. $\square$

Lemma 7.18. *Let $\mathcal{B}$ be a simple containing collection for $\phi(A_m)$, with associated set of discs $\mathcal{D}$, with the properties of Lemma 7.13. Let $1 \leq j \leq m$ and let γ, γ' denote the components of $C_j \setminus \{r_j^1,\ r_j^2\}$. Suppose that for every $s \neq j$, an s-colored edge of $G_{\mathcal{H}_j}$ that emanates from C_j and ends on an internal vertex can only end on an s-colored vertex. For $1 \leq s \leq 4$, let $p_s \in \gamma$ be an external vertex of $G_{\mathcal{H}_j}$ with the following properties: i) One of the pairs $(p_1,\ p_3)$, $(p_2,\ p_4)$, say $(p_2,\ p_4)$, is connected by an edge $e \subset G_{\mathcal{H}_j}$; ii) we have $p_2 \in \mathrm{int}(\gamma_1)$ and $p_4 \in \mathrm{int}(\gamma_1')$, where γ_1 denotes the subarc of γ that connects p_1, p_3 and $\gamma_1' := \gamma \setminus \gamma_1$; and iii) there exist $k, i \neq j$, with $k \neq i$, such that p_1, p_3 (resp. p_2, p_4) are i-colored (resp. k-colored). Then, there is $s \in \{1, 3\}$ and an external vertex $q \in \gamma$ such that p_s and q are connected by an edge of $G_{\mathcal{H}_j}$.*

Proof: We have the following possibilities:

(a) the vertices p_1, p_3 are connected to internal vertices $v_1, v_3 \subset G_{\mathcal{H}_j}$ by edges e_1, e_3 respectively.

(b) at least one of p_1, p_3 is connected to an external vertex of $G_{\mathcal{H}_j}$.

First we show that possibility (a) doesn't occur: As in the proof of Lemma 7.17, there exist i-colored edges $E_1, E_3 \subset G_{\mathcal{H}_i}$, corresponding to e_1, e_3 respectively, that emanate from distinct components of $C_i \setminus \{r_i^1,\ r_i^2\}$ and end on an internal vertex U. The vertex U is j-colored and corresponds to the component γ of $C_j \setminus \{r_j^1,\ r_j^2\}$. The graph $G := E_1 \cup U \cup E_3$ divides $\bar{D}_i$ into two regions ;say R_1, R_2. The edge e of $G_{\mathcal{H}_j}$ gives rise to an edge $E \subset G_{\mathcal{H}_i}$ that emanates from U and ends on U. By assumption ii) of the statement of the lemma E emanates from inside R_1 and ends inside R_2. But then $E \cap G \neq \emptyset$, contradicting the fact that $G_{\mathcal{H}_i}$ is embedded in $\bar{D}_i$. Thus, (a) doesn't occur and we conclude that (b) must hold.

Case 1. Suppose that p_1 is connected to an external vertex $q \in G_{\mathcal{H}_j}$ by an edge e'. Then, we must have $q \in \gamma_1$ since otherwise we would have $e \cap e' \neq \emptyset$. Thus, in this case, the conclusion of the lemma follows.

Case 2. Suppose that p_1 is not connected to an external vertex of $G_{\mathcal{H}_j}$. Then, p_3 must connected to an external vertex $q \in G_{\mathcal{H}_j}$. By Lemma 7.15, $q \notin \gamma'$ and thus $q \in \gamma$ as desired. □

7.6. **The case of nice links without parallel components.**

In this subsection we continue studying the planar graphs $\{G_{\mathcal{H}_i}\}_{j=1}^m$ and investigating how various of their properties reflect on the structure of our fixed embedding $\phi(A_m)$. First, we have the following:

Lemma 7.19. *Let $j \neq 1$. Suppose that for every $k \neq j$ all the type* III_1 *components of $S_k \cap \bar{D}_j$ end on Z_{kj}. Then, $\Phi^{\{j\}}$ is even.*

Proof: Let γ, γ' denote the components of $C_j \setminus \{r_j^1,\ r_j^2\}$. We need the following:

Claim 1: There exists $i \neq j$ and i-colored vertices $p_1, p_3 \in G_{\mathcal{H}_j}$, lying on the same component, say γ, of $C_j \setminus \{r_j^1,\ r_j^2\}$, such that the following is true: For every $1 \leq k \neq j \leq m$, γ_1 contains an even number of k-colored vertices of $G_{\mathcal{H}_j}$. Here, γ_1 denotes the subarc of γ with endpoints p_1, p_3.

Let us first finish the proof of the lemma assuming Claim 1: The proof is a simple induction on the number $T_j := |\gamma \cap \Phi^{\{j\}}|$. For $T_j = 0$ the conclusion is trivially true. Assume $T_j > 0$ and let p_1, p_3 be points that satisfy the conclusion of the claim. Also let $D_j \subset \partial(B^3)$ be a disc as in Definition 7.8. By induction the points in each of γ_1 and $\gamma \setminus \gamma_1$ can be paired by arcs that are disjointly embedded in D_j so that each arc connects points of the same color. But then so can all the points in $\gamma \cap \Phi^{\{j\}}$.

Proof of Claim 1: First we show the following:

Claim 2: Either the conclusion of Claim 1 is true or there exists $i \neq j$ and an i-colored edge $e \subset G_{\mathcal{H}_j}$ that connects two external vertices that lie on the same component of $C_j \setminus \{r_j^1,\ r_j^2\}$.

Proof of Claim 2: Let $i \neq j$, let $p_1, p_3 \in \gamma$ be any pair of i-colored vertices and let γ_1 denote the subarc of γ with endpoints p_1, p_3. We may assume that there is $1 \leq k \neq j \leq m$ and k-colored vertices $p_3, p_4 \in G_{\mathcal{H}_j}$ with $p_3 \in \gamma_1$ and $p_4 \in (\gamma \setminus \gamma_1)$. For, otherwise the conclusion of Claim 1 is true. Suppose, for a moment, that there is $1 \leq s \leq 4$, so that p_s is connected to an external vertex q. By Lemma 7.15, $q \in \gamma$ and Claim 1 is true. Suppose now that, for $1 \leq s \leq 4$, p_s is connected to an internal vertex and let p'_s denote the dual of p_s. By Lemma 7.17, there is $1 \leq s \leq 4$ so that $p's$ is connected to an external vertex on γ'. Thus Claim 2 has been proven.

Now we finish the proof of Claim 1: By Claim 2 we may assume that there exists $i \neq j$ and an i-colored edge $e \subset G_{\mathcal{H}_j}$ that connects two external vertices say p_1, p_3 that lie on the same component of $C_j \setminus \{r_j^1,\ r_j^2\}$. Assume, without loss of generality, that $p_1, p_3 \in \gamma$ and let γ_1 denote the subarc of γ with endpoints p_1, p_3. Suppose, for a moment, that there is $1 \leq k \neq j \leq m$, so that γ_1 contains an odd number of k-colored vertices of $G_{\mathcal{H}_j}$. By (b_1), $\gamma \setminus \gamma_1$ must also contain an odd number of such vertices. It follows that there must be k-colored vertices $p_2 \in \gamma_1$ and $p_4 \in (\gamma \setminus \gamma_1)$ that are both connected to internal vertices of $G_{\mathcal{H}_j}$. But this contradicts Lemma 7.18. Thus, p_1, p_3 must satisfy the conclusion of Claim 1. □

Next, assuming the Basic Lemma, we will prove Proposition 7.5 under the additional hypothesis that the link L contains no parallel components (see Proposition 7.21). For that we need the following lemma the proof of which is similar to these of Lemmas 7.17, 7.18 and 7.19. It should be noted that neither Lemma 7.20 nor Proposition 7.21 is used in the rest of the paper. Thus the reader eager to get to the proof of Proposition 7.5 may skip the remaining subsection without any loss of continuity. On the other hand, reading the proof of Proposition 7.21 might make easier understanding the proof of Proposition 7.5.

Lemma 7.20. *Let $\mathcal{B}$ be a simple containing collection for $\phi(A_m)$ with associated set of discs $\mathcal{D}$. Assume that $\mathcal{D}$ has the properties of Lemma 7.13 and let $j \neq 1$. Suppose that there exists $i \neq j$ such for every*

$s \in \{i,j\}$ and every $k \neq i,j$, all the type III_1 *components of $S_k \cap \bar{D}_s$ end either on Z_{ts} or on Z_{ks}. Here, $\{t\}$ denotes the complement of $\{s\}$ in $\{i,j\}$. Suppose moreover that $\Phi^{\{i,j\}}$ is even with respect to C_i. Then $\Phi^{\{i,j\}}$ is even with respect to C_j. Thus, in particular, $\Phi^{\{i,j\}}$ is even.*

Proposition 7.21. (A special case of Proposition 7.5) *Suppose that $(L,\ \mathbf{f})$ is an adequate nice link such that $M := \chi(S^3,\ L) \in \mathcal{HM}$. Let $(S, \mathbf{f}, \mathbf{x})$ be an adequate H-diagram of M corresponding to $(L,\ \mathbf{f})$. Suppose, moreover, that L doesn't contain any components that are parallel. Then the conclusion of the statement of Proposition 7.5 is true for L.*

Proof: [Assuming the Basic Lemma.] We will show that $\phi(\mathbf{f})$ is even and thus the desired conclusion follows immediately from Proposition 7.9. Define a partition $\mathcal{P}$ of the set $\mathcal{I} := \{2, \dots, m\}$ as follows: Let $j \in \mathcal{I}$. If there exist $i \neq j$ and $k \neq i,j$ such that $S_k \cap \bar{D}_j$ contains components of type III_1 that end on Z_{ij} add the set $\{i,\ j\}$ in $\mathcal{P}$. Otherwise, add the set $\{j\}$. By part (1) of the Basic Lemma it follows that this process gives a partition $\mathcal{P} := \{I_1, \dots, I_h\}$ of $\mathcal{I}$. By construction we have $\#I_i = 1$ or $\#I_i = 2$, for every $1 \leq i \leq h$. Part (3) of the Basic Lemma implies that the partition of $\{L_2, \dots, L_m\}$ corresponding to $\mathcal{P}$ is actually division of L. Let us denote it by $\mathcal{R}$. As remarked after Definition 7.6, it is enough to show that Φ^{I_i} is even for every $1 \leq i \leq h$. To that end fix such an i. If $\#I_i = 1$, then Lemma 7.19 applies to conclude that Φ^{I_i} is even. Suppose that $\#I_i = 2$. Then, $\#I_i = \{i,\ j\}$ where i,j are as in the Basic Lemma. By parts (1) and (2) of the Basic Lemma, Lemma 7.20 applies to conclude that $\Phi^{\{i,j\}}$ is even. $\square$

7.7. **Curve adjacency and a generalized Basic Lemma.**

For the proof of Proposition 7.5, and thus the proof of Theorem 3.7 we need a generalized version of the Basic Lemma. The generalized version, which is stated below, will allow us to treat adequate H-diagrams corresponding to nice links with parallel components. Before we can state this generalized version we remind the reader that given a nice H-diagram $(S, \mathbf{f}, \mathbf{x})$, in the Appendix of the paper (Definition A.9) we defined the notion of adjacency and strong adjacency of two curves

$x_i, x_j \in \mathbf{x}$ with respect to $\mathbf{f}$. In the statement of the lemma we use the notation and terminology introduced in the Appendix and we let the setting be as in the statement of the Basic Lemma.

Lemma 7.22. (A generalization of the Basic Lemma.) *Let $\mathcal{B}$ be a simple containing collection for $\phi(A_m)$, with associated set of discs $\mathcal{D}$, with the properties of Lemma 7.13. Let $i \neq j$ with $i, j \neq 1$. Suppose that there exists $k \neq i, j$ such that $S_k \cap \bar{D}_j$ contains a component of type* III_1 *with one of its boundary points on Z_{ij}. Then there is $s \in \{i, j\}$ so that if we let $\{t\} := \{i, j\} \setminus \{s\}$, the following is true: There exist curves $x_{i_1}, \ldots, x_{i_{r+1}} \in \mathbf{x} \setminus \{x_1\}$ such that one of the following is true:*

(1) We have

$$x_s = x_{i_1} \overset{\mathbf{f}}{\longleftrightarrow} x_{i_2} \overset{\mathbf{f}}{\longleftrightarrow} \ldots \overset{\mathbf{f}}{\longleftrightarrow} x_{i_r} \overset{\mathbf{f}}{\longleftrightarrow} x_{i_{r+1}} = x_t,$$

that is, for $l = 1, \ldots, r+1$, $x_{i_l}, x_{i_{l+1}}$ are strongly adjacent with respect to $\mathbf{f}$.

(2) We have $x_s \in \{x_{i_1}, \ldots, x_{i_r}\}$ and

$$x_{i_1} \overset{\mathbf{f}}{\longleftrightarrow} x_{i_2} \overset{\mathbf{f}}{\longleftrightarrow} \ldots \overset{\mathbf{f}}{\longleftrightarrow} x_{i_r} \overset{\mathbf{f}}{\longrightarrow} x_{i_{r+1}} = x_t,$$

where x_{i_r}, x_{i_t} are adjacent but not necessarily strongly adjacent with respect to $\mathbf{f}$. Moreover, if x_{i_r}, x_{i_t} are not strongly adjacent, then:

(i) Φ^I is even, with respect to C_x for every $x \in \{i_1, \ldots, i_r\}$.

(ii) There exists a tangle of the form $T(\pm 1, \pm r)$ of the projection $P(K)$ such that the components $L_{i_1}, \ldots, L_{i_r}$ (resp. L_t) of L are obtained from the rational tangle $[r]^{-1}$(resp. $[\pm 1]$) of $T(\pm 1, \pm r)$ via the process shown in Figure 6.

Let us now explain how Basic Lemma 7.16 follows from Lemma 7.22: Suppose that the link L doesn't contain parallel components and let i, j and k be indices as in the statement of Basic Lemma 7.16. By Lemma 7.22 and Corollary A.11 we have that $x_i \overset{\mathbf{f}}{\longrightarrow} x_j$ but x_i, x_j are not strongly adjacent with respect to $\mathbf{f}$. Let $s \in \{i, j\}$ and let $\{t\}$ denote the complement of $\{s\}$ in $\{i, j\}$. Also let $1 \leq r \leq m$ such that x_s, x_r are adjacent. By Proposition A.10, we have that $r = t$. By Lemma 7.22 again we have: For every $s \in \{i, j\}$ and every $k \neq i, j$, all the type III_1 components of $S_k \cap \bar{D}_s$ end either on Z_{ts} or on Z_{ks}. Thus part (1) of Basic Lemma 7.16 follows. Parts (2) and (3) follow from (i) and (ii) above, respectively.

7.8. **Type III_1 components lead to bigons between curves.**

The purpose of this subsection is to prove Lemma 7.22. Before we can go on with the proof, which will be broken down into several sublemmas, we need some preparation: Let the notation be as in subsection §7.1 and recall that $\Phi := \cup_{k=1}^{m}\phi(f_k)$. For $1 \leq j \leq m$, consider the following subsets of $C_j \cap \Phi$:

$$P_j := \{p_t^{ij}, p_s^j, r_j^1 \ \mid\ 1 \leq i \leq m,\ 1 \leq t \leq m_{ij},\ 1 \leq s \leq m_j\}$$

$$Q_j := \{q_t^{ij}, q_s^j, r_j^2 \ \mid\ 1 \leq i \leq m,\ 1 \leq t \leq m_{ij},\ 1 \leq s \leq m_j\}$$

In the terminology of Definition 7.8 P_j, Q_j are dual of each other. For a point $p \in (C_j \cap \Phi)$, let p_1, p_2 denote points of $C_j \setminus (C_j \cap \Phi)$ such that: i) p_1, p_2 lie on different sides of p; and ii) the subarc of C_j that has endpoints p_1, p_2 and whose interior contains p does not contain any other points of $C_j \cap \Phi$.

Sublemma 7.23. *Given $i \neq j$ and a point $p \in (C_j \cap \Phi)$, let p_1, p_2 be as above. Suppose that for some $1 \leq k \leq m$ there is a subarc $\alpha \subset \phi(f_k)$ such that $\partial(\alpha)$ consists of p and a point of $C_i \cap \Phi$ and $\mathrm{int}(\alpha) \cap (C_i \cup C_j) = \emptyset$. Then, for $s = 1, 2$, there exists a point $q_s \in C_i \setminus (C_i \cap \Phi)$ and an embedded arc $\gamma_s \subset S$ such that $\partial(\gamma_s) = \{p_s, q_s\}$ and $\mathrm{int}(\gamma_s) \cap (C_i \cup C_j \cup \Phi) = \emptyset$.*

Proof: For $s = 1, 2$, we can take γ_s to start at q_s and run parallel to α. $\square$

Given $i \neq j$ and an embedded arc $\gamma \subset \phi(S)$ constructed as in Sublemma 7.23, let $C_{ij}^{\gamma} := C_i \#_{\gamma} C_j$. The curve C_{ij}^{γ} is simple and bounds a disc $D_{ij}^{\gamma} \subset \phi(S)$; D_{ij}^{γ} can be obtained by band-connecting D_i and D_j along γ. Consider an embedded 3-ball $B_1 := B_{ij}^{\gamma} \subset S^3$ such that:

$$D_{ij}^{\gamma} \subset \partial B_1, \quad B_1 \cap B^3 = \partial B_1 \cap \partial B^3 = \partial B_1 \cap D_{ij}^{\gamma},$$

$$\phi(\hat{h}^i), \phi(\hat{h}^j) \subset B_1, \quad \text{and} \quad (\phi(\hat{h}^i) \cup \phi(\hat{h}^j)) \cap \partial B_1 = (\phi(\hat{h}^i) \cup \phi(\hat{h}^j)) \cap \partial B^3,$$

where B^3 denotes the 3-ball part of $\phi(A_m)$. The complement of D_{ij}^{γ} in $\partial(B_1)$, denoted by $\bar{D}_{ij}^{\gamma}$, will be referred to as *a disc associated to* (C_i, C_j, γ).

Definition 7.24. *Let Φ, C_{ij}^{γ} and D_{ij}^{γ} be as above and set $S_{ij} := S \setminus \mathrm{int}(D_{ij}^{\gamma})$. We will say that the points $p, p' \in (C_{ij}^{\gamma} \cap \Phi)$ are absorbed by a bigon between C_{ij}^{γ} and Φ, iff there exist arcs $\beta^{'} \subset (\Phi \cap S_{ij})$ and $\beta \subset C_{ij}^{\gamma}$ such that $\partial(\beta) = \partial(\beta^{'}) = \{p, p'\}$ and $\beta^{'} \cup \beta$ bounds an embedded disc (a bigon) in S_{ij}. We will often say that p (resp. p') is absorbed by a bigon that ends at p' (resp. p).*

An important step in the proof of Lemma 7.22 is provided by the following Key Sublemma the proof of which uses Proposition A.1:

Key Sublemma 7.25. *Let $\mathcal{B}$ be a simple containing collection for $\phi(A_m)$ with associated set of discs $\mathcal{D}$ and let $i \neq j$. Suppose that there exists $k \neq i, j$ such that $S_k \cap \bar{D}_j$ contains a type III_1 component, say β, that ends on Z_{ij}. Suppose, moreover, that the dual point of $\partial(\beta) \cap \partial(\bar{D}_j)$ is also absorbed by a type III_1 component that ends at the dual point of $\partial(\beta) \cap Z_{ij}$. Then, we can construct a curve C_{ij}^{γ} for which at least one of the following is true:*

(1) For some $s \in \{i, j\}$ there exist dual points $x \in P_s \cap \phi(f_k)$ and $x' \in Q_s \cap \phi(f_k)$ and a set $U \in \{P_t, Q_t\}$ such that x, x' are absorbed by bigons between C_{ij}^{γ} and Φ that end on U. Here, $\{t\}$ denotes the complement of $\{s\}$ in $\{i, j\}$.

(2) For some $s \in \{i, j\}$ there exists a set $R \in \{P_s, Q_s\}$ such that all the points in R are absorbed by bigons between C_{ij}^{γ} and Φ that end on $P_t \cup Q_t$. Here, $\{t\}$ denotes the complement of $\{s\}$ in $\{i, j\}$.

Proof: Since $\phi(f_k)$ intersects both of C_i, C_j, there is a subarc $\alpha \subset \phi(f_k)$ such that $\mathrm{int}(\alpha) \cap (C_i \cup C_j) = \emptyset$ and $\partial(\alpha) := \{y, y'\}$ has exactly one point on each of C_i, C_j. Let γ be an arc corresponding to α in the sense of Sublemma 7.23, let $C := C_{ij}^{\gamma}$ and let $D := D_{ij}^{\gamma}$. Suppose, without loss of generality, that $y \in P_j$ and thus $y' \in T$, for some $T \in \{P_i, Q_i\}$.

By construction, and by conditions $(a_1)-(a_3)$ and $(b_1)-(b_2)$, C has the following properties:

a) There exist disjoint subarcs $\gamma_1, \gamma_2 \subset C$, each of which is parallel on S to γ, such that $\Phi \cap \gamma_{1,2} = \emptyset$. Moreover, the complement $C \setminus (\gamma_1 \cup \gamma_2)$ has two components, say δ_1, δ_2, such that $(P_j \cup Q_j) \subset \delta_1$ while $(P_i \cup Q_i) \subset \delta_2$.

b) There exist subsets $X_j, Y_j \subset P_j$ such that $P_j = X_j \cup Y_j$ and X_j, Y_j lie on opposite sides of Q_j on δ_1. Moreover, we have $r_j^1 \in X_j$, $r_j^2 \in Q_j$ and r_j^1, r_j^2 separate the points in $Q_j \setminus \{r_j^2\}$ from these in $Y_j \cup X_j \setminus \{r_j^1\}$.

c) There exist subsets $X_i, Y_i \subset T$ such that $T = X_i \cup Y_i$ and X_i, Y_i lie on opposite sides of $\bar{T}$ on δ_2. Here, $\{\bar{T}\}$ denotes the complement of $\{T\}$ in $\{P_i, Q_i\}$. Moreover, if we let r_1 (resp. r_2) denote the point in $\{r_i^1, r_i^2\}$ that lies on T (resp. $\bar{T}$) then we have: $r_1 \in X_i$, $r_2 \in \bar{T}$ and r_1, r_2 separate the points in $\bar{T} \setminus \{r_2\}$ from these in $Y_i \cup X_i \setminus \{r_1\}$.

d) The sets X_i, X_j (resp. Y_i, Y_j) lie on the same side of $Q_j \cup \bar{T}$ on C.

e) There exists a set $\mathcal{E}$ of properly

embedded arcs in D such that each arc in $\mathcal{E}$ connects a pair of dual points of $C \cap \Phi$ and for every $\epsilon_1, \epsilon_2 \in \mathcal{E}$ we have the following: Either $\epsilon_1 \cap \epsilon_2 = \emptyset$ or there exists $n \in \{i, j\}$ so that ϵ_1 connects the points r_n^1, r_n^2 and ϵ_2 connects a pair of dual points in $(P_n \cup Q_n) \setminus \{r_n^1, r_n^2\}$. The situation is illustrated in Figure 12.

Let $\bar{D} := \bar{D}_{ij}^{\gamma}$ be a disc associated to (C_i, C_j, γ). By assumption $S_k \cap \bar{D}_j$ contains two type III_1 components that connect a pair of dual points on $\partial(\bar{D}_j)$ to a pair of dual points on Z_{ij}. These components, give rise to properly embedded arcs $(\beta_s, \partial(\beta_s)) \subset (\bar{D}, \partial(\bar{D}))$, where $s = 1, 2$. Let $\partial(\beta^1) := \{p_1, p_2\}$ (resp. $\partial(\beta^2) := \{q_1, q_2\}$) where p_1 and q_1 (resp. p_2 and q_2) are dual.

As in subsection §7.3, out of the set $(\mathcal{S} \cup \mathcal{H}) \cap \bar{D}$ we construct a graph G whose external (resp. internal) vertices are the components of $\mathcal{S} \cap C$ (resp. $\mathcal{H} \cap \mathrm{int}(\bar{D})$ and whose edges are the arc components of $\mathcal{S} \cap \bar{D}$. By properties (a)-(e) above we can see the following (The assumption that $k \neq i, j$ is used here): There exist

$$n \in \{i, j\} \quad \text{and} \quad \{z_1, z_2\} \in \{\{p_1, p_2\},\ \{q_1, q_2\}\},$$

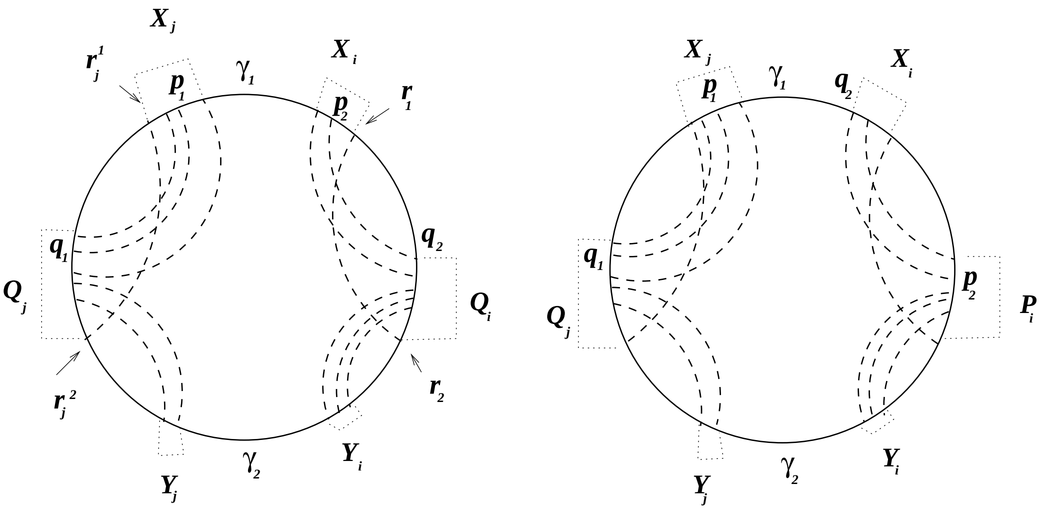

FIGURE 12. The intersection $\Phi \cap C$: In the left (resp. right) hand side picture q_1, q_2 lie on the same (resp. different) component(s) of $C \setminus \{p_1, p_2\}$. In both cases we have $P_j = X_j \cup Y_j$ while in the left (resp. right) hand side picture we have $T = P_i = X_i \cup Y_i$ and $\bar{T} = Q_i$ (resp. $T = Q_i = X_i \cup Y_i$ and $\bar{T} = P_i$). The dashed lines indicate the arcs in $\mathcal{E}$ that connect pairs of dual points of $\Phi \cap C$. Notice that in both cases, there exists $l \in \{i, j\}$ such that r_l^1, r_l^2 belong on different components of $C \setminus \{q_1, q_2\}$.

such each component of $C \setminus \{z_1, z_2\}$ contains an odd number of points of $\phi(f_n) \cap C$ (compare, Figure 12). By lemma 7.14, there must be a component $G_1 \subset G$, that corresponds to a subset of $(S_n \cup \mathcal{H}) \cap \bar{D}$, such that G_1 contains external vertices in both components of $C \setminus \{z_1, z_2\}$. But then $G_1 \cap \beta_z \neq \emptyset$, where $\beta_z \in \{\beta^1, \beta^2\}$ with $\partial(\beta_z) = \{z_1, z_2\}$. Since $S_1, \ldots, S_m$ are disjoint, and disjoint from the complement of $\phi(A_m)$, we conclude that there must be an isotopy of $\Phi \cup C$ on S that reduces $|\Phi \cup C|$. This isotopy must either eliminate a point $x \in \{z_1, z_2\}$, or an external vertex $w_1 \in G_1$ and a point $w_2 \in (\phi(f_n) \cap C)$ such that w_1, w_2 lie on different components of $C \setminus \{z_1, z_2\}$.

Case 1: There is an isotopy of $\Phi \cup C$ that eliminates a point $x \in \{z_1, z_2\}$. By Proposition A.1, and Lemma 7.2, there is a point $z \in (\Phi \cap C)$ such that x and z are absorbed by a bigon, say D_1 between C

and Φ. Without loss of generality we can assume that $x \in (P_j \cup Q_j)$. We will further assume that $x \in X_j$. (The case $y \in Y_j$ is symmetric to that of $x \in X_j$ and the case that $x \in Q_j$ is easily reduced to one of $x \in X_j$, $x \in Y_j$.) By Lemma 7.2, $z \in (P_i \cup Q_i)$. Let β_1 denote the subarc of $\partial(D_1)$ that lies on C. By assumption, $\partial(\beta_1) = \{x, z\}$. Also let β_1' denote the subarc of $\partial(D_1)$ that lies on Φ. Assume for a moment that there exists $s \in \{i, j\}$ and a set $R \in \{P_s, Q_s\}$ such that $R \subset \beta_1$. Then, since Φ is embedded, all the points in R must be absorbed by bigons that end on β_1. By Lemma 7.2, these bigons must end on $P_t \cup Q_t$ and thus (2) of the statement of the sublemma is true in this case. Thus for the remaining of the proof we can assume that we have

$$R \not\subseteq \beta_1 \quad \text{for every} \quad s \in \{i, j\} \ \text{ and } \ R \in \{P_s, Q_s\}. \tag{7.1}$$

By assumption (7.1), Lemma 7.2, and properties (a)-(e) above we conclude $z \in (X_i \cup \bar{T})$ and $((P_j \cup Q_j) \cap \beta_1) \subset X_j$. Furthermore, if $z \in \bar{T}$ then $X_i \subset \beta_1$. Suppose for a moment that $z \in \bar{T}$ and let z' denote the dual of z. By our assumptions we have $X_i \subset \beta_1$ and by properties (a)-(e) we have $z' \in \beta_1$. Thus, z' must be absorbed by a bigon that ends on X_i. But then (1) of the statement of the sublemma is satisfied. Thus we may assume that $z \in X_i$. Using the fact that $S_1, \ldots, S_m$ are disjoint and properties (a)-(e) we can see the following: There exists dual points $a, y \in (\phi(f_k) \cap C)$, with $a \in X_j$, such that a is absorbed by a bigon D_2 that ends at a point w, and b is also absorbed by a bigon. Without loss of generality we assume that $a = x$. Since $x \in X_j$ we have $y \in Q_j$. Let β_2 (resp. β_2') denote the subarc of $\partial(D_2)$ that lies on C (resp. on Φ). By an argument similar to this used for β_1 above, we may assume that we have

$$R \not\subseteq \beta_2 \quad \text{for every} \quad s \in \{i, j\} \ \text{ and } \ R \in \{P_s, Q_s\}. \tag{7.2}$$

We have the following possibilities to consider:

I) $z \in X_i$ and $w \in X_i$
II) $z \in X_i$ and $w \in Y_i$
III) $z \in X_i$ and $w \in \bar{T}$.

In the case of I)-II) the conclusion of the sublemma is true since (1) is satisfied. Let us now treat case III).There two subcases to consider according to whether $\beta_1 \cap \beta_2 = \emptyset$ or $\beta_1 \subset \mathrm{int}(\beta_2)$.

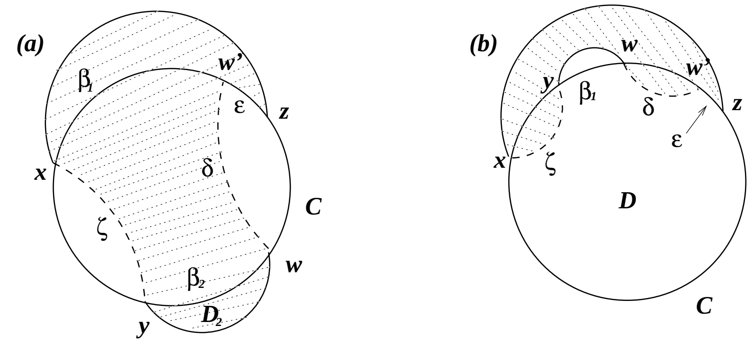

FIGURE 13. Bigons absorbing the pairs x, z and y, w: Part (a) illustrates a situation where $\beta_1 \cap \beta_2 = \emptyset$ and part (b) illustrates a situation where $\beta_1 \subset \mathrm{int}(\beta_2)$. The disc filled with pattern illustrates the disc $\Delta \subset \hat{S}$ bounded by the loop $\theta := \epsilon \cup \beta_1' \cup \zeta \cup \beta_2' \cup \delta$.

Let S' denote the closure of the component of $S \setminus C$ that is disjoint from $\phi(\hat{h}^j) \cup \phi(\hat{h}^i)$. Set $f' := \phi(f_k) \cap S'$ and let $\hat{f}$ denote the closed curve obtained from $\hat{f}$ by joining each pair of dual points of $\partial(\hat{f})$ by an arc in $\mathcal{E}$, where $\mathcal{E}$ is as in property (e) stated above. By (e) and the assumption that $k \neq i, j$, it follows that $\hat{f}$ is embedded in $\hat{S} := S' \cup \mathrm{int}(D)$. Let w' (resp. z') denote the dual of w (resp. z).

Subcase 1. First suppose that $\beta_1 \subset \mathrm{int}(\beta_2)$; now $X_i, X_j \subset \beta_1$. By the properties of C is easy to see that one of z', w' must lie in $\mathrm{int}(\beta_2)$. Suppose, $w' \in \mathrm{int}(\beta_2)$ and let $\zeta \in \mathcal{E}$ (resp. $\delta \in \mathcal{E}$) denote the arcs connecting x, y (resp. w, w'). Let ϵ denote the subarc of β_2 with endpoints w', z. The loop $\theta := \epsilon \cup \beta_1' \cup \zeta \cup \beta_2' \cup \delta$ bounds a disc $\Delta \subset \hat{S}$. The disc is shown filled in pattern in Figure 13(b). Clearly, $\hat{f}$ intersects both of $\mathrm{int}(\Delta)$ and $\mathrm{int}(\hat{S} \setminus \Delta)$. Since $\hat{f}$ is embedded, the only way for $\hat{f}$ to exit Δ is to have $\hat{f} \cap \mathrm{int}(\epsilon) \neq \emptyset$. But this can happen only if there is point $x_1 \in (X_j \cap \phi(f_k))$ such that x_1 and its dual (which lies on $Q_j \cap D$) are absorbed by bigons that end on $X_i \cap \epsilon$. But then the sublemma is proven in this case.

Subcase 2. Suppose that $\beta_1 \cap \beta_2 = \emptyset$. Then $Y_i, Y_j \subset \beta_2$. We can see that we have either $w' \in \mathrm{int}(\beta_1)$ or $z' \in \mathrm{int}(\beta_2)$. If $w' \in \mathrm{int}(\beta_1)$ let ϵ denote the subarc of β_1 with endpoints w', z and let $\Delta \subset \hat{S}$ a disc bounded by the loop $\theta := \epsilon \cup \beta_1' \cup \zeta \cup \beta_2' \cup \delta$. This case is illustrated in Figure 13(a). By analyzing the intersection $\hat{f} \cap D$ as in Subcase 1 we conclude that there must be a point $x_1 \in (Y_j \cap \phi(f_k))$ (or $x_1 \in (Y_i \cap \phi(f_k)))$ that satisfies (1) of the statement of the sublemma. Suppose now that $z' \in \mathrm{int}(\beta_2)$. Then z' is absorbed by a bigon that ends at a point $z'' \in \mathrm{int}(\beta_2)$; z''' denote the dual of z''. We have $z'' \in (Q_j \cap \mathrm{int}(\beta_2))$ and $z''' \in Y_i$ or $z'' \in Y_i$ and $z'' \in (Q_j \cap \mathrm{int}(\beta_2))$. In both cases it can easily be seen that z'', z''' are both absorbed by bigons that end on $\bar{T}$. Thus property (1) of the statement of the sublemma is satisfied.

Case 2. There is an isotopy of $\Phi \cup C$ on S that eliminates an external vertex $w_1 \in G_1$ and a point $w_2 \in (\phi(f_n) \cap C)$ such that w_1, w_2 lie on different components of $C \setminus \{z_1, z_2\}$. By Proposition A.1, we can assume that w_1, w_2 are absorbed by a bigon D between C and Φ. Let β denote the subarc of $\partial(D)$ that lies on C. Clearly, there is $x \in \{z_1, z_2\}$ such that $x \in \mathrm{int}(\beta)$. Thus x is also absorbed by a bigon between Φ and C and the conclusion follows by Case 1. $\square$

Our next sublemma provides the generalization of Key Sublemma 7.25 that is needed to complete the proof of Lemma 7.22.

Sublemma 7.26. *Let $\mathcal{B}$ be a simple containing collection for $\phi(A_m)$ with associated set of discs $\mathcal{D}$ and let $i \neq j$. Suppose that there exists $k \neq i, j$ such that $S_k \cap \bar{D}_j$ contains a type* III_1 *component, say β, that ends on Z_{ij}. Then, we can construct a curve C_{ij}^{γ} for which at least one of the following is true:*

(1) For some $s \in \{i, j\}$ there exist dual points $x \in P_s \cap \phi(f_k)$ and $x' \in Q_s \cap \phi(f_k)$ and a set $U \in \{P_t, Q_t\}$ such that x, x' are absorbed by bigons between C_{ij}^{γ} and Φ that end on U. Here, $\{t\}$ denotes the complement of $\{s\}$ in $\{i, j\}$.

(2) For some $s \in \{i, j\}$ there exists a set $R \in \{P_s, Q_s\}$ such that all the points in R are absorbed by bigons between C^{γ}_{ij} and Φ that end on $P_t \cup Q_t$. Here, $\{t\}$ denotes the complement of $\{s\}$ in $\{i, j\}$.

Proof: [Sketch] Let $C := C^{\gamma}_{ij}$ be a curve constructed out of C_i, C_j as in the beginning of the proof of Key Sublemma 7.25 and let $D := D^{\gamma}_{ij}$. Also let $\bar{D} := \bar{D}^{\gamma}_{ij}$ be a disc associated to (C_i, C_j, γ) and let B_1 be a 3-ball, such that $\partial B_1 = D \cup \bar{D}$, with the properties stated before the statement of Key Sublemma 7.25. By assumption, there is $k \neq i, j$ such that $S_k \cap \bar{D}$ contains a properly embedded arc corresponding to the type III_1 component β above. Let $S'_k \subset (S_k \cap B_1)$ denote the component of $S_k \cap B_1$ whose boundary $\partial S'_k$ contains β. Now $\partial S'_k$ also contains two arcs $\alpha_1, \alpha_2 \subset (f_k \cap B_1)$ such that $\partial \beta := \{p,\ q\}$ with $p \in \partial \alpha_1$ and $q \in \partial \alpha_2$. Let p' (resp. q') denote the second point of $\partial \alpha_1$ (resp. $\partial \alpha_2$). Note that if p', q' are also connected by an arc component of $S_k \cap \bar{D}$ then we are in the situation of Key Sublemma 7.25.

To continue we need to think of B_1 as a cube $[0,\ 1] \times [0,\ 1] \times [0,\ 1]$ where the disc D is identified with $\{0\} \times [0,\ 1] \times [0,\ 1]$ and the curve C with the boundary of the square $\{0\} \times [0,\ 1] \times [0,\ 1]$. We have the following:

I) For $s \in \{i, j\}$, $B_1 \cap \phi(\hat{h}^s)$ is a (possibly) knotted cylinder which is properly embedded in B_1 so that the boundary lies in $\mathrm{int}(D)$.

II) For $s \neq i, j$, one of the following is true: i) $B_1 \cap \phi(\hat{h}^s)$ is empty; ii) $B_1 \cap \phi(\hat{h}^s)$ lies in D and γ intersects x_s; iii) $B_1 \cap \phi(\hat{h}^s)$ is a collection of (possibly) knotted cylinders which are properly embedded in B_1 so that their boundary lies in $\mathrm{int}(\bar{D})$.

For notational simplicity, we will assume that for every $s \neq i, j$ the possibility iii) above occurs. Then, for every $1 \leq s \leq m$, if we let K_s denote the core of the 2-handle $\phi(\hat{h}^s)$, $B_1 \cap K_s$ is a collection of (possibly) knotted arcs which are properly embedded in B_1. Consider $S^3 := \mathbf{R}^3 \cup \{\infty\}$ such that $\mathbf{R}^3$ is parameterized in rectangular coordinates $(x,\ y,\ z)$, so that the disc D lies on the plane $z = 0$. By an isotopy we can arrange so that the z-height function, if restricted on the tangle $\cup_{s=1}^{m} K_s$, is a Morse function. Choose $0 < z_0 < 1$ such that if we let $B := [0,\ z_0] \times [0,\ 1] \times [0,\ 1]$, $B' := [z_0,\ 1] \times [0,\ 1] \times [0,\ 1]$ and $S''_k := B' \cap S'_k$, the following are true:

a) For $s \in \{i,\ j\}$, the intersection $B' \cap K_s$ is a (possibly) knotted arc K'_s that is properly embedded in B' so that its boundary points lie in the interior of the disc $D_1 := \{z_0\} \times [0,\ 1] \times [0,\ 1]$.

b) The intersection $\partial S''_k \cap \partial B' = \partial S''_k \cap D_1$ consists of two arcs each of which has one boundary point of $\partial K'_i$ and the second on $\partial K'_j$.

c) For $s \neq i, j$, the intersection $B' \cap K_s$ is a (possibly) knotted arc K'_s that is properly embedded in B' so that its boundary points lie in the interior of the disc $D_2 := \partial B' \setminus D_1$.

d) For $1 \leq s \leq m$, the intersections of $\phi(f_s)$ and $\phi(y_s)$ (on S) occur inside B'.

e) For $s \in \{i,\ j\}$, the intersection $B \cap K_s$ is two (possibly) knotted and linked arcs that are properly embedded in B such that each arc has one boundary point boundary point in the interior of D and the second in the interior of D_1.

f) For $s \neq i, j$, we have $K_s \cap (D_1 \cup D) = \emptyset$. Furthermore, the intersection $B \cap K_s$ is a collection of (possibly) knotted and linked arcs that are properly embedded in B.

Set $A' := A_m \cup B$ and $S' := \partial A$. By our assumptions above, A' is a handlebody of genus $> m$ such that the 3-ball part of A' is the union $B^3 \cup B$ and 2-handles are in one to one correspondence with the components of

$$(S^3 \setminus B) \cap (\cup_{s=1}^{m} K_s).$$

Let $\mathcal{H}'$ denote the set of 2-handles of A'. We construct a collection of simple closed curves $\mathbf{f}' \subset S'$ from $\mathbf{f}$ as follows: For every curve $f \in \mathbf{f}$ and every 2-handle $h \in \mathcal{H}'$ we can replace each component, say α, of $h \cap f$ with an embedded arc on ∂B that connects the points in $\partial\alpha$. Property (d) above and conditions $(a_1) - (a_3)$ assure that this can be done so that the resulting collection of curves is embedded on S'. Next we can replace the components of $\mathcal{S} \cap B$ by appropriate discs on ∂B. Then we can isotope the interiors of these disc away from ∂B in the complement of B to obtain a collection $\mathcal{S}'$ of oriented, properly embedded in the complement of A', disjoint surfaces bounded by the curves in $\mathbf{f}'$.

The collection $\mathbf{f}'$ is not a complete system of curves for A'. However, by properties (a)-(f) above and our construction of A' it follows that the intersections of $\mathbf{f}'$ with $C' := \partial D_1$ are in one to one correspondence with these of Φ and C. Moreover, if two points in $\mathbf{f}' \cap C'$ are absorbed by a bigon between $\mathbf{f}'$ and C', the corresponding points on $\Phi \cap C$ are also absorbed by a bigon between Φ and C. As in the proof of Key Sublemma 7.25, out of the set $(\mathcal{S}' \cup \mathcal{H}') \cap D_2$ we construct a graph G' whose external (resp. internal) vertices are the components of $\mathcal{S}' \cap C'$ (resp. $\mathcal{H}' \cap \mathrm{int}(D_2)$ and whose edges are the arc components of $\mathcal{S}' \cap D_2$. By properties (a)-(f) above and our earlier discussion, the argument applied to the graph G in the proof of Key Sublemma 7.25 can now be applied to G'. Using this argument and the aforementioned correspondence of bigons between $\mathbf{f}'$ and C' and these between Φ and C we can obtain the desired conclusions. $\square$

We are now ready to complete the proof of Lemma 7.22.

Proof of Lemma 7.22: Let i, j, k be indices as in the statement of Lemma 7.22 and let C_{ij}^{γ} be a curve as in Sublemma 7.26. Recall that the interior of γ is disjoint from $C_i \cup C_j$ and let $\mathbf{x}'$ denote the set of curves in $\mathbf{x}$ that intersect γ. Recall that the H-diagram $(S, \mathbf{f}, \mathbf{x})$ is associated to an adequate nice link $(L, \mathbf{f})$ (Definition 7.1) as in Theorem A.5. Let L' denote the set of components of $P(L)$ corresponding to the curves in $\mathbf{x}'$.

Case 1: Suppose that C_{ij}^{γ} satisfies property (1) of Sublemma 7.26. To simplify our notation we will assume that $s = i$ and thus $t = j$. By Lemma A.13, there are curves $x_{i_1}, \ldots, x_{i_{r+1}}$ such that

$$x_i = x_{i_1} \xleftrightarrow{\mathbf{f}} x_{i_2} \xleftrightarrow{\mathbf{f}} \ldots \xleftrightarrow{\mathbf{f}} x_{i_s} \xrightarrow{\mathbf{f}} x_{i_{r+1}} = x_j,$$

and such that property (ii) of part (2) of Lemma 7.22 is satisfied. The fact that $x_{i_l} \neq x_1$ follows from Proposition A.10. We will show that property (i) of part (2) of Lemma 7.22 latter on.

Case 2: Suppose that C_{ij}^{γ} satisfies property (2) of Sublemma 7.26. Let $s, t \in \{i, j\}$ and $R \in \{P_s, Q_s\}$ as in the statement of that property.

Again, to simplify our notation, we will assume that $s = i$ and thus $t = j$. We distinguish two subcases:

Subcase 1: All the bigons absorbing the points in R end on one of P_j, Q_j. Then by Lemma A.14, there are curves $x_{i_1}, \ldots, x_{i_{r+1}}$ such that satisfy part (1) of the statement of Lemma 7.22. The fact that $x_{i_l} \neq x_1$ follows from Proposition A.10.

Subcase 2: There are bigons absorbing points in R that end on both of P_j, Q_j. Again by Lemma A.14, there are curves $x_{i_1}, \ldots, x_{i_{r+1}}$ such that

$$x_i = x_{i_1} \overset{\mathbf{f}}{\longleftrightarrow} x_{i_2} \overset{\mathbf{f}}{\longleftrightarrow} \ldots \overset{\mathbf{f}}{\longleftrightarrow} x_{i_s} \overset{\mathbf{f}}{\longrightarrow} x_{i_{r+1}} = x_j,$$

and such that property (ii) of part (2) of Lemma 7.22 is satisfied. The fact that $x_{i_l} \neq x_1$ follows from Proposition A.10.

Next we will show that in Case 1 and in Subcase 2, provided that x_{i_r} and x_j are not strongly adjacent, the collection $\mathbf{x}' := \{x_{i_1}, \ldots, x_{i_{r+1}}\}$ can be expanded so that property (i) of part (2) of Lemma 7.22 is also satisfied. This expansion is done as follows: let $\mathbf{z} = \{x_{j_1}, \ldots, x_{j_s}\} \subset \mathbf{x} \setminus (\mathbf{x}' \cup \{x_1\})$ be a collection of curves such that: i) every curve in $\mathbf{z}$ is adjacent to at least one curve in $\mathbf{z} \cup \mathbf{x}'$ and ii) if $w \in \mathbf{x}$ is adjacent to a curve in $\mathbf{z} \cup \mathbf{x}'$ then $w \in (\mathbf{z} \cup \mathbf{x}')$. Among all such collections let $\mathbf{z}$ be the largest one. By Corollary A.12, it follows that the curves in $\mathbf{z}$ can be indexed so that we have $x_{j_1} \overset{\mathbf{f}}{\longleftrightarrow} \ldots \overset{\mathbf{f}}{\longleftrightarrow} x_{j_s} \overset{\mathbf{f}}{\longleftrightarrow} x_{i_1}$.Now replace $\mathbf{x}'$ with $\mathbf{x}' \cup \mathbf{z}$. For simplicity of notation we will assume that $\mathbf{z} = \emptyset$.

Let L' denote the sublink of L whose components are in one to one correspondence with the curves in $\mathbf{x}'$ and let L_j the component corresponding to x_j. Also let $\mathbf{f}'$ denote the collection of curves in $\mathbf{f}$ that are in one to one correspondence with the curves in $\mathbf{x}'$. By Lemma A.13, and its proof,

By the construction of $(S, \mathbf{f}, \mathbf{x})$ at stage (S_5) (see proof of Theorem A.5) there must be a subarc $\alpha \subset P(L_1)$ that intersects each component of $L' \setminus L_j$ exactly once, loops once around L_j and comes back intersecting each component of $L' \setminus L_j$ exactly once again in opposite order and opposite sign from the first time. Now during stages $(S_6) - (S_8)$ of the construction of $(S, \mathbf{f}, \mathbf{x})$ the projection $P(L)$ is altered by finger moves where a small part of certain under crossings is dragged along parts of $P(L_1)$. During this process the intersections of the curves in $\mathbf{f} \setminus (\mathbf{f}' \cup \{f_1\})$ with these in $\mathbf{x}'$ are created by finger moves of the components of $L \setminus (L' \cup \{L_1\})$ along the arc α. Using these observations it is easy to see that Φ^I is even with respect to C_l for every $l \in \{i_l, \dots, i_r\}$. □

7.9. **The general case: Nice links with parallel components.**

In this subsection we complete the proof of Proposition 7.5 and Theorem 3.7. To begin we need some preparation: Recall that $\mathcal{I} := \{2, \dots, m\}$. Fix $x \in \mathcal{I}$ and let $Y_x \subset \mathcal{I}$ denote the set with the following properties:

(1) $x \in Y_x$.

(2) For every $j \in Y_x$ and $k \neq j$ every k-colored type III_1 component of $\mathcal{S} \cap \mathcal{D}$ that emanates from $\partial \bar{D}_j$ ends either on Z_{kj} or on Z_{rj}, for some $r \in Y_x$.

(3) Y_x is the largest set with the properties (1), (2).

Definition 7.27. *Fix $x \in \mathcal{I}$. We will say that a pair of indices $i \neq j \in Y_x$ is admissible iff there exists $k \neq i, j$ and a k-colored type* III_1 *component of $\mathcal{S} \cap \bar{D}_j$ that emanates from $\partial \bar{D}_j$ and ends on Z_{ij}. Note that for such a pair of indices Lemma 7.22 applies. For an admissible pair of indices i, j, let $\mathbf{x}(i,\ j) \subset \mathbf{x}$ be the collection of curves that satisfy the conclusion of Lemma 7.22. Set*

$$J(i,j) := \{y \in \mathcal{I} \mid x_y \in \mathbf{x}(i,\ j)\} \subset \mathcal{I},$$

and let J_x denote the union of the sets $J(i,j)$ over all admissible pairs $i \neq j \in Y_x$.

Next we define a partition $\mathcal{P}$ of $\mathcal{I}$. For $x \in \mathcal{I}$ we proceed as follows:

1. If $\mathcal{S} \cap \bar{D}_x$ does not contain type III_1 components then we add $\{x\}$ to $\mathcal{P}$. Otherwise, we examine the set Y_x.

2. If Y_x doesn't contain any admissible pairs of indices, then we add $\{x\}$ to $\mathcal{P}$.

3. If Y_x contains admissible pairs of indices, then we add J_x to $\mathcal{P}$.

Remark 7.28. Suppose that Y_x doesn't contain any admissible pairs of indices. Then Lemma 7.19 applies to every element $y \in Y_x$ to conclude that $\Phi^{\{y\}}$ is even with respect to C_y. In particular, $\Phi^{\{x\}}$ is even with respect to C_x.

Say $\mathcal{P} = \{I_1, \dots I_h\}$. By Lemma 7.22 it follows the partition of $\{L_2, \dots, L_m\}$ corresponding to $\mathcal{P}$ is a division of height h for L. Let us denote that division by $\mathcal{R} = \{R_1, \dots R_h\}$ where our convention is that R_i is the sublink of L whose components are indexed by the elements of I_i. We will show that $\mathcal{R}$ satisfies the conclusion of Proposition 7.5. By construction of $\mathcal{R}$, Remark 7.28 and Lemma 7.22, each set R_i belongs in one of the following three types:

- Type A: A set R_i for which Φ^{I_i} is even.
- Type B: A set R_i, with $\#R_i > 1$, for which Φ^{I_i} is not even and the curves in $\mathbf{x}$ corresponding to I_i can be indexed in an order say $x_{i_1}, \dots, x_{i_r}$ such that
$$x_{i_1} \overset{\mathbf{f}}{\longleftrightarrow} x_{i_2} \overset{\mathbf{f}}{\longleftrightarrow} \dots \overset{\mathbf{f}}{\longleftrightarrow} x_{i_{r+1}}.$$
- Type C: A set R_i, with $\#R_i > 1$, for which the following are true:
  i) Φ^{I_i} is not even.
  ii) the curves in $\mathbf{x}$ corresponding to I_i can be indexed in an order say $x_{i_1}, \dots, x_{i_r}$ such that
$$x_{i_1} \overset{\mathbf{f}}{\longleftrightarrow} x_{i_2} \overset{\mathbf{f}}{\longleftrightarrow} \dots \overset{\mathbf{f}}{\longrightarrow} x_{i_{r+1}},$$
  where $x_{i_{r+1}}$, x_{i_r} are adjacent but not strongly adjacent.
  iii) Φ^{I_i} is even with respect to C_l for every $l \in \{i_l, \dots, i_r\}$.

We need to study Type B and C sets in more detail. Fix $1 \leq i \leq h$ so that R_i is of type B or C. For $1 \leq j \leq r$, let P_{i_j}, Q_{i_j} the sets of points introduced in the beginning of subsection 7.8. Set $X_j := \Phi^{I_i} \cap P_{i_j}$ and $Y_j := \Phi^{I_i} \cap Q_{i_j}$. By successive band-sums of the curves $C_{i_1}, \dots C_{i_{r+1}}$, as described in Subsection 7.8, we can construct a simple closed curve $\Gamma \subset B^3$ so that after appropriate bigon isotopies of $\Phi \cup \Gamma$ we have: i) For $1 \leq j \leq r+1$, Γ encircles the feet of $\phi(\hat{h}^{i_j})$ and separates them from the feet of $\phi(\hat{h}^k)$, for ever $k \notin I_i$; and ii) $\Phi^{I_i} \cap C = X_1 \cup Y_{r+1}$

(resp. $\Phi^{I_i} \cap C = X_1$) if R_i is of type B (resp. C). Now Γ bounds a disc $D \subset \phi(S)$. Consider an embedded 3-ball $B' \subset S^3$ such that:

$$D \subset \partial B', \quad B' \cap B^3 = \partial B' \cap \partial B^3 = \partial B' \cap D,$$

$$\cup_{j=1}^{r+1} \phi(\hat{h}^{i_j}) \subset B', \quad \text{and} \quad \cup_{j=1}^{r+1} (\phi(\hat{h}^{i_j})) \cap \partial B' = \cup_{j=1}^{r+1} \phi(\hat{h}^{i_j}) \cap \partial B^3,$$

where B^3 denotes the 3-ball part of $\phi(A_m)$. Let D' denote the complement of D in $\partial(B')$. Let $\mathcal{S}' := \cup_{j \notin I_i} S_j$. In a way analogous to that described in Subsection 7.4 we can construct a colored planar graph G_Σ embedded in D', to record the intersections of $\Sigma := \mathcal{S}' \cap D'$ and we can partition the vertices of G_Σ into external and internal. Let $v \in (G_\Sigma \cap X_1)$ be an external vertex. There is a corresponding external vertex v' such that v, v' are the endpoints of a subarc of Φ^{I_i} that is colored by the same color as v, v' and whose interior lies in $\mathrm{int}(B')$. Note that $v' \in (G_\Sigma \cap Y_r)$ or $v' \in (G_\Sigma \cap X_1)$ according to whether R_i is of type B or C. We will say that v' is the *dual* vertex of v. By enlarging B' is necessary we can assume that for every $1 \leq j \leq m$, the intersection $\phi(\hat{h}^j) \cap D'$ is either empty or it consists from two meridian discs of $\phi(\hat{h}^j)$. Note that this assumption, which will remain valid for the remaining of the section, implies that either G_Σ doesn't contain j-colored internal vertices or it contains exactly two of them. In the latter case we will refer to these two vertices as *dual* to each other. We will need the following lemma.

Lemma 7.29. *Suppose that R_i is of type B or C and let the notation be as above. Then, one of the following is true:*

(1) Every external vertex of G_Σ that lies on $X := X_1$ is connected to its dual vertex by an edge of G_Σ.

(2) For every color $k \notin I_i$ there exists at most one k-colored vertex $v \in (G_\Sigma \cap X)$ that is not connected to its dual v' by an edge of G_Σ. Furthermore, we have:

i) v is connected to a k-colored internal vertex u, by an edge $e \subset G_\Sigma$ and v' is connected to the dual vertex of u, say u', by an edge e'.

ii) Every edge $E \neq e$ (resp. $E \neq e'$) that emanates from u (resp. u') ends on u (resp. u').

Proof: To simplify our notation set $G := G_\Sigma$. By the construction of G and the fact that the surfaces in $\mathcal{S}'$ are disjoint we can see the following:

Property 1: Every external vertex $p \in (G \cap X)$ is either connected to its dual p' by an edge or it is connected to an internal vertex q whose color is the same as this of p. Furthermore, in the latter case p' is connected to the dual of q.

Claim 1: Suppose that R_i is of type B. Then, there exists at least one external vertex in X that is connected to its dual by an edge of $e_1 \subset G$.

Proof of Claim 1: Since Φ^{I_i} is not even, there exist colors $k \neq l \notin I_i$ such that we have the following: There exist k-colored vertices $p_1, p_3 \in (G \cap X)$ such that each component of $C \setminus \{p_1, p_3\}$ contains an odd number of l-colored vertices of $G \cap X$. Let $p_2, p_4 \in (G \cap X)$ be l-colored vertices so that p_2 lies between p_1, p_3 while p_4 lies outside. For $1 \leq t \leq 4$, let p'_t denote the dual of p_t. Suppose that both of p_1, p_3 are connected to internal vertices; otherwise the claim is clearly true. By Property 1, both p_1, p_3 are connected to the same k-colored vertex q, by edges e_1, e_3. Moreover, p'_1, p'_3 are connected to the dual of q, say q', by edges e_2, e_4. The graph $e_1 \cup q \cup e_3 \cup e_2 \cup q' \cup e_4$ has two components that separate D' into three regions. By our choice of $p_1, \dots, p_4$ each of these regions contains an odd number of l-colored external vertices. By this observation and Property 1 it follows that there is an l-colored vertex on X that is connected to its dual by an edge. This finishes the proof of the claim.

To continue with the proof of the lemma suppose that there is $k \notin I_i$ and a k-colored vertex v that is connected to a k-colored internal vertex, say u, by an edge $e \subset G_\Sigma$. By Property 1 the dual vertex v' is connected to the dual vertex of u, say u', by an edge e'. We can enlarge B' so that $D_k \subset \partial B'$, where D_k is a disc on B^3 bounded by C_k that contains the feet of $\phi(\hat{h}^k)$. Let D'_k denote the complement of D_k in $\partial B'$. In a way analogous to the one used for G above we can construct a colored planar graph G'' embedded in D'_k, to record the intersections of $\mathcal{S}' \cap D'_k$. As before we partition the vertices of G'' into external and internal.

Suppose, first, that R_i is of type B.

There is a vertex V_X (resp. V_Y) of G'' corresponding to the set X (resp. $Y := Y_r$). Moreover, there are edges E, E', corresponding to e, e' respectively, such that: i) E, E' emanate from dual external vertices $p, p' \in G''$; ii) E ends on V_X and E' ends on V_Y. Set $G_1 := E \cup V_X \cup e_1 \cup V_Y \cup E'$. If R_i is of type C, then there is a vertex V_X of G'' corresponding to the set X . Moreover, there are edges E, E', corresponding to e, e' respectively, emanate from dual external vertices $p, p' \in G''$ and end on V_X. In this case set $G_1 := E \cup V_X \cup E'$. In both cases G_1 separates D'_k into two regions, say R^1, R^2. By property 1 each of R^1, R^2 must contain an even number of k-colored vertices. By conditions $(b_1) - (b_2)$ this possible only when $\{p, p'\} = \{r_1^k,\ r_2^k\}$. Thus the first assertion of part (2) of the statement of the lemma has been proven. It remains to prove part (ii). To that end, it is enough to prove that every external vertex of $v_1 \in G''$, with $v_1 \notin \{p, p'\}$, is connected to another external v_2 vertex by an edge. Note that v_1, v_2 will necessarily lie on the same component of $C_k \setminus \{r_1^k,\ r_2^k\}$ since otherwise the edge connecting them will intersect with G_1. Suppose on the contrary that there is a vertex $v_1 \in G''$, with $v_1 \notin \{p, p'\}$, that is connected to an internal vertex. Then, we can see that there is $s \notin I_i$ and so that the two s-colored vertices of G'', say V, V', lie in different regions R^1, R^2. By conditions $(b_1) - (b_2)$ and Lemma 7.14 there must be a connected s-colored subgraph of G'' that has parts in both of R^1, R^2.

Case 1: Suppose that R_i is of type B. By Property 1 there must be edges of G'' emanating from both of V_X, V_Y and ending on V, V'. Suppose that E_1 is such an edge that emanates from V_X and ends on $V \in R_1$. By Property 1, there is a corresponding edge E'_1 that emanates from V_Y and ends on $V' \in R^2$. By the construction of G'' we can see that this is impossible since E'_2 must emanate from inside R^1. But then E'_2 intersects G_1 which contradicts the fact that the surfaces in $\mathcal{S}'$ are disjoint.

Case 2: Suppose that R_i is of type C. There must be an odd number of s-colored edges emanating from each side of V_X. It is easy to see that condition (iii) in the definition of type C set implies that this is impossible. $\square$

The final piece needed to complete the proof of Proposition 7.5 is the following lemma:

Lemma 7.30. *If $R_i \in \mathcal{R}$ is of type B or C and we set $I := I_i$, there exists a collection of curves $\mathbf{f}_I \subset S$ as in Definition 7.3, and an embedding*

$$\phi_I : A_{g_I} \to S^3,$$

such that the curves $\phi_I(\mathbf{f}_I) \subset \phi_I(A_{g_I})$ bound disjointedly embedded, oriented surfaces in $\overline{S^3 \setminus \phi_I(A_{g_I})}$.

Proof There are four cases to consider:

(I) R_i is of type B and condition (1) of Lemma 7.29 holds.
(II) R_i is of type B and condition (2) of Lemma 7.29 holds.
(III) R_i is of type C and condition (1) of Lemma 7.29 holds.
(IV) R_i is of type C and condition (2) of Lemma 7.29 holds.

We will discuss case (I) in detail. The other cases are similar and are left to the reader. Let D' and B' be a disc and ball respectively corresponding to I as constructed before the statement of Lemma 7.29. Set $\mathcal{S}' := \cup_{\notin I} S_i$. By assumption, $\Phi^I \cap \partial D'$ is a disjoint union of sets X, Y so that every points in X is connected to one in Y by an arc component of $\mathcal{S}' \cap D'$. We label these arc components of $\mathcal{S}' \cap D'$, say $c_1, c_2, \ldots, c_r$, so that c_1 and c_r are the two outermost. Now c_1 cuts off a subdisc $D_1 \subset D'$. Let $a := \partial D_1 \cap \partial D'$. For $i = 2, \ldots, r$, let $D_i \subset D'$ denote the subdisc whose boundary contains $c_{i-1} \cup c_i$. Now $\partial D_i \setminus (c_{i-1} \cup c_i)$ consists of two disjoint arcs a_i, b_i such that $\partial a_i \subset X$ and $\partial b_i \subset Y$. The discs D_i, $1 \leq i \leq m$, might have intersections with $\mathcal{S}'$ but we can attach 2-handles to remove these intersections. This way we can obtain surfaces $\hat{S}_1, \ldots, \hat{S}_r$, corresponding to $D_1, \ldots, D_r$, that are mutually disjoint and disjoint from $\mathcal{S}'$. Let $\mathbf{f}' \subset \phi(\mathbf{f})$ denote the collection of boundary curves of the surfaces in $\mathcal{S}'$. We modify $\mathbf{f}'$ and $\mathcal{S}'$ as follows: Start with the curve $f_1' \in \mathbf{f}'$ that contains the points in ∂c_1 and replace the subarc of f_1' that connects the points in ∂c_1 and lies in B' by a. Let $S_1' \in \mathcal{S}'$ the surface bounded by f_1'. The arc c_1 bisects S_1'

into two pieces one of which say S_1'' lies entirely in B'. Let $S_1''' := S_1' \setminus S_1''$. Now the surface $S_1''' \cup \hat{S}_1$ is bounded by f_1'. To continue, for $i = 2, \ldots, r$, replace the subarc of the curve in $\mathbf{f}'$ that contains ∂c_i and lies in B' by $a_i \cup a'_{i-1} \cup \ldots a' \ldots \cup b'_{i-1} \cup b_i$. Here $a'_{i-1}, \ldots, a', \ldots, b'_{i-1}$ are arcs in $\phi(S) \setminus D$ that are parallel to $a_{i-1}, \ldots, a, \ldots, b_{i-1}$. Next replace the part of the corresponding surface in $\mathcal{S}'$ that doesn't lie entirely in B', by $\hat{T}_1 \cup \ldots \cup \hat{T}_{i-1} \cup \hat{S}_i$. Here $\hat{T}_1, \ldots \cup \hat{T}_{i-1}$ are parallel copies of $\hat{S}_1, \ldots \cup \hat{S}_{i-1}$, respectively, so that $a' \subset \partial \hat{T}_1, \ldots, a'_{i-1} \cup b'_{i-1} \subset \partial \hat{T}_{i-1}$. At the end of the process we have replaced $\mathbf{f}'$ by a collection say $\mathbf{f}''$ and $\mathcal{S}'$ by a collection of surfaces say $\mathcal{S}''$. It is not hard to see that, perhaps after further isotopy, we have $\mathbf{f}'' = \phi(\mathbf{f}_I)$ where $\mathbf{f}_I \subset S$ is as in Definition 7.3. $\square$

We are now ready to finish the proof of Proposition 7.5:

Proof of Proposition 7.5: Let $\mathcal{R}$ be the division of L constructed in the beginning of the subsection. Let β denote the number of the sets in $\mathcal{R}$ that are of type B or C. If $\beta = 0$, then $\phi(f)$ is even and the conclusion follows by Proposition 7.9. Otherwise apply Lemma 7.30 and induct on β. $\square$

Appendix A. Some properties of nice H-diagrams

Our purpose in this Appendix, is to study the Heegaard splittings corresponding, in the sense of Theorem 2.1, to nice links in more detail. We prove some technical results that are used in Section 7 of the paper.

Let $S^3 = A_g \cup_S B_g$ denote a genus g standard Heegaard splitting of S^3 with corresponding H-diagram $(S, \mathbf{x}, \mathbf{y})$. Two simple closed curves $\alpha, \beta \subset S$ are said to have an excess intersection if there are curves α', β' in the isotopy classes of a and b respectively, such that $|\alpha' \cap \beta'| < |\alpha \cap \beta|$. A collection of simple closed curves $\mathbf{a}$ and a simple closed curve β are said to have an excess intersection if there is $\alpha \in \mathbf{a}$ so that α and β have an excess intersection. The curves α and β are said to cobound a *bigon* if there exist arcs $a \subset \alpha$ and $b \subset \beta$ such that i) a, b meet only at their endpoints and do not meet other points of $\alpha \cup \beta$ and ii) the loop $a \cup b$ bounds a disc $D \subset S$. Let α' denote the simple closed curve obtained from α by replacing a with an arc parallel to b that lies outside D. The curve α' is said to be obtained from α by a *bigon move*. Clearly, $|\alpha' \cap \beta| < |\alpha \cap \beta|$. We recall the following:

Proposition A.1. *(*[FLP]*, Proposition 3.10 or* [HS]*, Lemma 3.1) Suppose that $\alpha, \beta \subset S$ are simple closed curves that have an excess intersection. Then, α and β cobound a bigon.*

In what follows we use the notation and terminology introduced before the statement of Theorem 2.1. Consider an H-diagram $(S, \mathbf{f}, \mathbf{x})$, with properties (i)-(iv) of Theorem 2.1. Assume, moreover, that for every $1 \leq i, j \leq m$ we have $f_i \cap x_j^1 = \emptyset$; that is all the intersections between f_i and x_j occur on x_j^2. We will say that the points $p, q \in (f_i \cap x_j^2)$ are successive on f_i if there exists a subarc $a \subset f_i$ with the following properties: i) following the orientation of f_i, the initial (resp. terminal) point of a is p (resp. q); ii) $\mathrm{int}(a) \cap x_j^2 = \emptyset$. Let $b \subset x_j^2$ denote the subarc with $\partial(b) = \{p,\ q\}$.

Definition A.2. *Let $1 \leq i, j \leq m$. We will say that f_i intersects x_j^2 twice in the row if there exist points $p, q \in (f_i \cap x_j^2)$ that are successive on f_i, and subarcs $a \subset f_i$, $b \subset x_j^2$ as above, such that: i) We have $a \cap x_k^2 = \emptyset$, for every $k \neq j$; and ii) $a \cup b$ bounds a disc $\partial D \subset D^3$ so that, for every $k \neq j$, none of the feet of h_k is contained in D.*

We need the following:

Lemma A.3. *Let $(S, \mathbf{f}, \mathbf{x})$ be an H-diagram with the properties (i)-(iv) of Theorem 2.1. Suppose that f_i intersects x_j^2 twice in the row and let p, q, a, b and D be as in the statement of Definition A.2. Then, either*

(1) the curves f_i and x_j have an excess intersection; or

(2) we have $i = j$. Furthermore, there exists a point $t \in (f_i \cap x_i^2)$ that lies in $\mathrm{int}(D)$ and such that f_i passes through t immediately after it runs over h_i or it runs over h_i immediately after it passes through t.

Proof: There are three cases to consider:

Case 1: The disc D contains exactly one foot of h_j; then p, q have the same sign of intersection (see Figure 14(a)). Suppose without loss of generality that f_i goes into $\mathrm{int}(D)$ right after it passes through q.

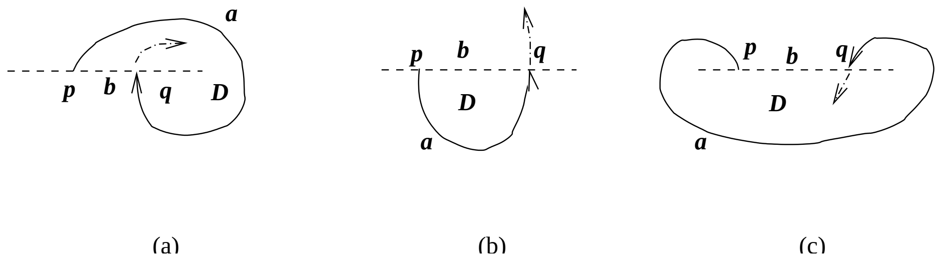

FIGURE 14. The curve f_i intersects x_j^2 twice in the row. In all cases f_i (resp. x_j^2) is drawn by the solid (resp. dashed) line.

Since f_i cannot intersect itself it will either intersect $\text{int}(b)$ with a sign opposite to that of q to exit D, or, perhaps after intersecting x_j^2 repeatedly, it will run over h_j. In the first case it follows that f_i and x_j must have excess intersections. In the second case, let t denote the last intersection of f_i and x_j^2 before f_i runs over h_j. Since $|f_i \cap y_j| = \delta_{ij}$, we conclude that $i = j$.

Case 2: The points p, q have opposite intersection signs and D doesn't contain any of the feet of h_j (Figure 14(b)) . Then, D is a bigon cobounded by f_i and x_j. Hence, f_i and x_j have excess intersections.

Case 3: The points p, q have opposite intersection signs and D contains both of the feet of h_j (Figure 14(c)). In this case, by an argument similar to this of Case 1, we can see that f_i and x_j have excess intersections. □

To continue let $L = L_1 \cup \ldots \cup L_m$ be an m-component nice link where L_1 is the special component. The components $L_2, \ldots, L_m$ bound disjoint discs $D_2, \ldots, D_m$ respectively.

Definition A.4. *Let $i, j > 1$ with $i \neq j$. We will say that L_i is adjacent to L_j if L_1 pierces D_j immediately after it pierces D_i for the first time, and then, immediately, it pierces D_i for the second time.*

In the main body of the paper, we often need to chooce nice Heegaard diagram to satisfy certain admissibility properties. The purpose of the rest of the appendix is to demonstrate that these choices are possible. We begin with the following theorem:

Theorem A.5. *Let $M \in \mathcal{HM}$ and let L be a nice link with $M = \chi(S^3,\ L)$. There exists an m-component nice link $(\bar{L}, \mathbf{f})$ that is obtained from L by admissible moves, a standard Heegaard splitting $A_m \cup_S B_m$ of S^3, with a corresponding H-diagram $(S, \mathbf{x}, \mathbf{y})$, and a complete system of curves $\mathbf{f} := \{f_1, \dots, f_m\}$ on $S := \partial B_m$ such that:*

a) The H-diagram $(S, \mathbf{f}, \mathbf{x})$ has properties (i)-(iv) of Theorem 2.1.

b) For $i = 1, \dots, m$, $f_i \cap h_i$ consists of single arc that intersects each of y_i and x_i^1 exactly once. Furthermore, f_i doesn't intersect x_i^2 immediately after it runs over h_j or right before it runs over h_j.

c) There are no indices i, j so that f_i intersects x_j^2 twice in the row.

Proof: Set $r := \#L$ and let the notation be as introduced in the beginning of subsection 3.1. Let $P(L) \subset R$ be a projection representing L as plat-closure of a braid $p \in P_{2r}$ with respect to the height on the z-axis so that the properties Lemma 3.2 are satisfied. For $1 \leq j \leq r$, let a_j (resp. b_j) denote the part of $P(L_j)$ that lies below (resp. above) the line $z = 1$ (resp. $z = 2$). Let z_1 denote a segment on the line $z = 2$ connecting the endpoints of b_1. For $j > 1$, let z_j denote a segment on the line $z = 2$ connecting the endpoints of a_j. We may suppose the following: i) For $k = 1, \dots, r-1$, the $(2k-1)$-th and $(2k)$-th endpoint of p on the line $z = 1$ (and on $z = 2$) belongs on L_{k+1} while the $(2r-1)$-th and $(2r)$-th endpoint of p belongs on L_1; ii) the discs $D_2, \dots, D_r$, bounded by $L_2, \dots, L_r$, project to embedded discs on R and $P(L_2) \cup \ldots \cup P(L_r)$ is the plat closure of the trivial $(2r-2)$-braid; and iii) $P(L_1)$ is embedded.

In order to obtain a nice link and a Heegaard diagram with the desired properties, we need to perform various modifications to the projection $P(L)$, some of which will change L by admissible moves. These modifications will be divided into stages:

(S_1). We can arrange so that all the elementary braid appearances in p occur at different z-heights. We order these appearances, moving from higher to lower z values. For every $j \neq 1$, we have two elementary

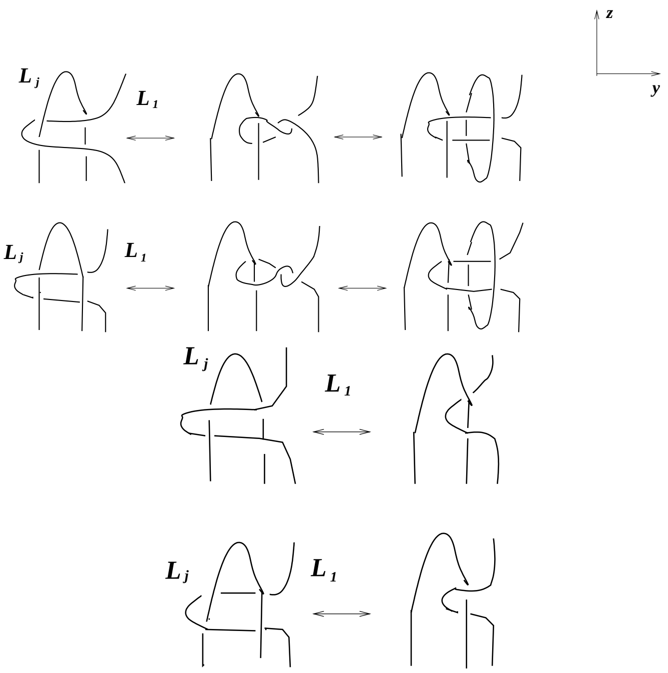

FIGURE 15. Moving the crossings $P(L_j)$ and $P(L_1)$ on the same side of $P(L_j)$.

braids realizing the geometric linking between $P(L_j)$ and $P(L_1)$; let p_j^{lo} (resp. p_j^{hi}) denote the one appearing at a lower (resp. higher) z-height. Let z_j^{lo} (resp. z_j^{hi}) the height at which p_j^{lo} (resp. p_j^{hi}) occurs. We isotope $P(L_j)$ so that there are no intersections with $P(L_1)$ at a height lower than z_j^{lo} or higher than z_j^{hi}. We may arrange this by isotopying the components of $P(L)$ so that the arcs a_j, b_j all lie at different z-heights. During this isotopy we can also move the arcs z_j so that they all lie at different z-heights.

(S_2). A component $P(L_j)$, $j \neq 1$, will be called inadmissible if $P(L_j)$ intersects the strings of $P(L_1)$ that lie in the zone $z_j^{lo} < z < z_j^{hi}$. The crossings of $P(L_j)$ and $P(L_1)$ that occur in the zone $z_j^{lo} < z < z_j^{hi}$ will be called inadmissible. After isotopies that rearrange the components $P(L_j)$, $j \neq 1$, we can assume that in all the inadmissible crossings of $P(L)$, $P(L_1)$ under crosses (some $P(L_j)$, $j \neq 1$).

(S_3). We apply to $P(L)$ the moves shown in Figure 15 (and their mirror images with respect to the y-axis) so that, for $j = 2, \dots, r$, the

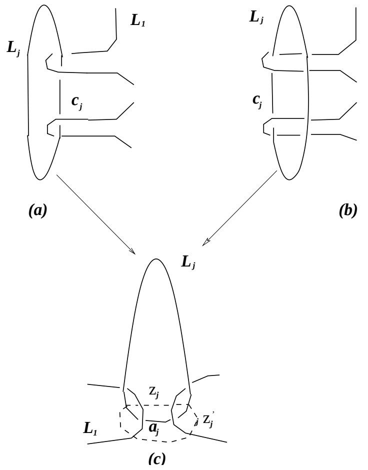

FIGURE 16. Crossings between $P(L_j)$, $j \neq 1$, and $P(L_1)$.

crossings between $P(L_j)$ and $P(L_1)$, corresponding to p_j^{hi}, p_j^{lo} occur on the same string of p, as shown in parts (a) and (b) of Figure 16.

(S_4). From now we will assume that among all projections of L on R, that satisfy the properties establishes in stages $(S_1) - (S_3)$, $P(L)$ has the minimum number of inadmissible crossings. If $P(L)$ does not contain any inadmissible crossings, then this stage of modifications is ommited. Otherwise, first order the inadmissible components of $P(L)$ according to the y-height they appear at, from left to right. Then, on each inadmissible component, order the double points of $P(L)$ corresponding to inadmissible crossings according to the z-height they appear at, from top to bottom. We will refer to these double points as inadmissible points. This way we can obtain a complexity that orders lexicographically the set of inadmissible points of $P(L)$. Let C_1 denote the inadmissible point of smallest complexity; suppose that C_1 occurs on $P(L_j)$ for some $2 < j \leq r$. Then, by $(S_1) - (S_3)$ and the assumption that the number of inadmissible points of $P(L)$ is minimum, we see that there exists an inadmissible point C_2 on $P(L_j)$, with $C_2 < C_1$, and $2 \leq k < j$ such that there is a subarc $\delta \subset P(L_1)$ with the following properties: i) we have $\partial\delta = \{C_1, C_2\}$; and ii) the interior of δ pieces D_k exactly once. The point $\delta \cap D_k$ may correspond to p_k^{hi} or to p_k^{lo}. We will explain how to proceed in the case that it corresponds to p_k^{hi}; the case of p_j^{lo} is completely analogous. Now both of C_1, C_2 appear in the zone $z_j^{lo} < z < z_j^{hi}$ and we have $z_k^{lo} < z_j^{lo} < z_k^{hi} < z_j^{hi}$; the situation is illustrated in Figure 17. Note that δ decomposes into two subarcs, say δ_1, δ_2; one with endpoints $\{C_1, p_k^{hi}\}$ and the other with endpoints $\{C_2, p_k^{hi}\}$. We will say that the pair C_1, C_2 of inadmissible points is *proper* if there is a component L_s ($s \neq 1$) such that the disc $D_s \subset R$ bounded by $P(L_s)$ is pierced by exactly one of the arcs δ_1, δ_2. Otherwise, we will say that C_1, C_2 is an *improper* pair.

(S_4^1). If C_1, C_2 is a proper pair then move to an inadmissible point of $P(L)$, say C_3, whose complexity is less than these of C_1, C_2. Otherwise, continue as follows:

(S_4^2). If there are components of L that are adjacent to L_k first isotope these components and their parallel ones near p_k^{hi} and p_k^{lo}. Then slide the ones that are near p_k^{lo} over $P(L_k)$ so that they intersect $P(L_1)$ near p_k^{hi} and arrange them so that they all appear at different y-values. Let V denote the component that appears at the largest y-value. Let $\gamma \subset L_1$ be the subarc whose endpoints correspond to z_k^{lo} and z_j^{lo}, whose interior is disjoint from $L_j \cup L_k$ and it doesn't contain any of the arcs b_1, a_1. Between V and L_j add a pair of parallel, oppositely unit-framed components J_1, J_2 so that each of them bounds a disc that is pierced exactly once by each of γ, δ. These components should be added in such a fashion so that, they lie near V, and δ (resp. γ) pierces the discs bounded by J_1, J_2 after (resp. before) it pierces D_k at p_k^{hi} (resp. at p_k^{lo}). See Figure 17.

(S_4^3). Proceed to repeat the process of $(S_4^1) - (S_4^2)$ to an inadmissible point of $P(L)$, say C_3, whose complexity is less than these of C_1, C_2 and continue till all the inadmissible points have been axhausted.

(S_5). The next modifications also consern inadmissible points of $P(L)$ and should be ommitted if there are no such points. Let $j > 1$ such that $P(L_j)$ contains inadmissible points. Then, there is a subarc $\theta \subset P(L_1)$ such that i) one endpoint of θ is p_j^{hi} and the second endpoint is an inadmissible point of $P(L_j)$; ii) the interior of θ is disjoint from $P(L_j)$ and it doesn't contain any of a_1, b_1. See Figure 17. We will say that $P(L_j)$ is a *bad* component if the arc θ doesn't intersect any of component

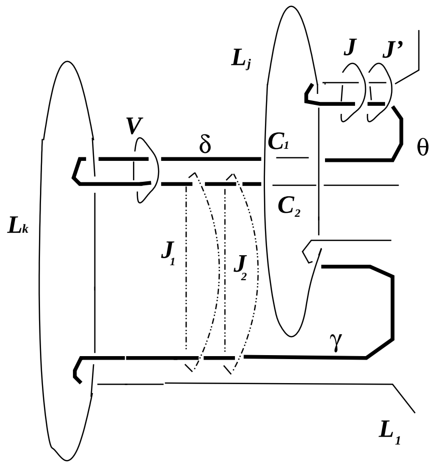

FIGURE 17. Modifying the link L: The framings of the inserted components J_1, J_2 must be of opposite sign. Note that $\chi(S^3,\ L) = \chi(S^3,\ L \cup J_1 \cup J_2) = \chi(S^3,\ L \cup J_1 \cup J_2 \cup J \cup J')$.

$P(L_k)$, $k \neq 1, j$. The modification to be done at this state is to insert a pair of parallel, oppositely unit-framed components J, J' near p_j^{hi} as shown in Figure 17. After stages (S_4^2), (S_5) have been completed, we have an m-component $(m > \#r)$ nice link that is obtained from L by a number of admissible moves. To simplify our notation we will continue denoting the new link by L.

(S_6). For $j > 1$, let z'_j be an arc on R that connects the points in ∂a_j and whose interior lies close to a_j on the side opposite to that in which z_j lies.

Isotope $P(L_1)$ as shown in part (c) of Figure 16, so that for $j \neq 1$ and for each p_j^{hi}, p_j^{lo}, $P(L_1)$ intersects each of z_j, z'_j once. Furthermore, for each of these intersections $P(L_1)$ should cross once over the arc $a_j \subset P(L_j)$.

(S_7). After stage (S_6), the only intersections of $P(L_j)$, $j \neq 1$, and $P(L_1)$ occur on a_j. However, stage (S_5) might introduce self crossings of $P(L_1)$. Indeed if $P(L_j)$ is an inadmissible component, during the isotopy of p_j^{hi} on a_j, $P(L_1)$ overlaps itself in the zone $z_j^{lo} < z < z_j^{hi}$. See Figure 18. We can order these overlaps according to the ordering given on the set $\{p_j^{hi}\}_{j \neq 1}$ at stage (S_1). Then we can start with the

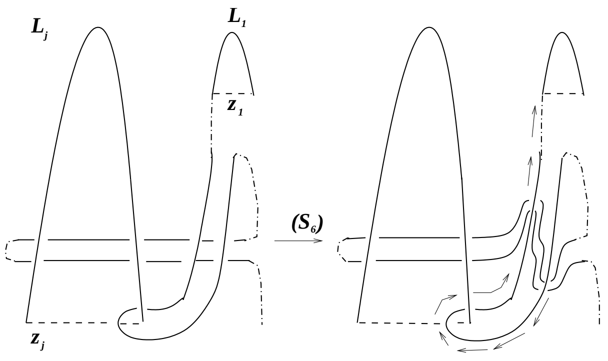

FIGURE 18. Pushing the self crossings of $P(L_1)$ on z_1.

first overlap and proceed as follows: For every self crossing of $P(L_1)$ in that overlap, we create a finger move on a small subarc $\alpha \subset P(L_1)$ corresponding to the "under" part of the crossing. We slide these fingers along $P(L_1)$ till α intersects z_1 once. This process is indicated in Figure 18 where the arrows drawn indicate the direction in which the fingers moves should be performed.

(S_8). At stage (S_6), for every $j \neq 1$, $P(L_j)$ crosses under $P(L_1)$ twice. Let $\mathcal{C}$ denote the set of crossings of $X := \cup_{j\neq 1} P(L_j)$ under $P(L_1)$ ordered as in (S_1). After (S_6) some crossings in $\mathcal{C}$ have been replaced by a set of crossings. Let $\mathcal{CT}$ denote the set obtained from $\mathcal{C}$ by replacing these crossings with the corresponding set of crossings; $\mathcal{CT}$ inherits an ordering from $\mathcal{C}$. Next we will push the under crossings of X with $P(L_1)$ on $\cup_{i=1}^m z_i$:

(S_8^1). Set $x_1^2 := z_1$.

(S_8^2). Let $j_1 \in \{2, \dots, m\}$ such that the first element, say C_1, of $\mathcal{CT}$ corresponds to crossings of $P(L_{j_1})$ under $P(L_1)$. For every crossing in C_1, create a finger move on a small subarc $\alpha \subset P(L_j)$ and corresponding to the "under" part of the crossing. slide these fingers along $P(L_1)$ till α intersects z_1 once (see Figure 19). The fingers corresponding to different crossings should remain disjoint during the entire isotopy. Let p_{j_1}, q_{j_1} denote the endpoints of a_j. If during these isotopies $P(L_{j_1})$ doesn't intersect z_{j_1} immediately after (or right before) it passes through $\partial(a_{j_1})$, set $x_{j_1}^2 =: z_{j_1}$. Otherwise, set $x_{j_1}^2 =: z'_{j_1}$.

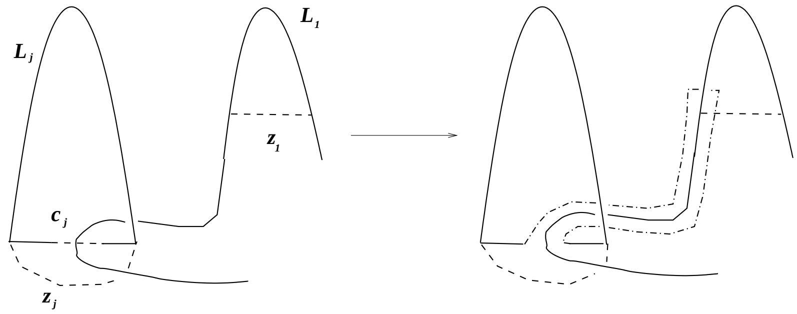

FIGURE 19. Pushing the under crossings of $P(L_j)$ with $P(L_1)$ on z_1.

(S_8^3). During the isotopies described above $P(L_{j_1})$ might intersect some arcs z_k, $k \neq 1, j_1$. These will add more crossings into the elements of $\mathcal{CT} \setminus \{C_1\}$; we will proceed with these enlarged sets.

(S_8^4). Suppose that the process has been carried out for all elements $C_t \in \mathcal{CT}$ with $t < k$. Suppose that the under part of each crossing in C_k belongs on $P(L_l)$. We create a finger move for every crossing in C_k as follows: For a crossing of $P(L_l)$ under $P(L_s)$, $s < k$, the "under arc" of $P(L_l)$ is dragged along $P(L_s)$ until it intersects z_s once. If C_k is the first element of $\mathcal{CT}$ corresponding to $P(L_l)$, we choose x_l^2 as indicated earlier so that $P(L_l)$ doesn't intersect x_l^2 immediately after (or right before) it passes through $\partial(a_l)$. If C_k is the second element of $\mathcal{CT}$ corresponding to $P(L_l)$, x_l^2 was chosen when the process was applied to the first element of $\mathcal{CT}$ corresponding to $P(L_l)$. Now we need to make sure that the two subarcs of $P(L_l)$ corresponding to the under part of the crossings that are closer to $\partial(a_l)$ are pushed in the same side of x_l^2.

We are now ready to construct an H-diagram having the properties of the statement of the theorem. Let $\hat{a}_j := P(L_j) \setminus \mathrm{int}(a_j)$ for $j \neq 1$ and let $\hat{a}_1 := b_1$. Lift the arcs $\hat{a}_1, \dots, \hat{a}_m$ slightly in $x > 0$ so that on R there remains a collection $\mathcal{A}$ of m-disjoint arcs. Let D be a disc neighborhood of $\mathcal{A}$ on R. We construct a standard handlebody B_m as follows: Let D^3 be a 3-ball in $\{(x, y, z) \mid x \leq 0\}$, such that the intersection of D^3 with the plane $x = 0$ is D. For $j = 1, \dots, m$, attach a 2-handle, say h_i, to $\partial(D^3)$ with core $\hat{c}_j$. Choose y_i to be the boundary of a meridional disc of h_i on $\partial(B_m)$. Choose x_i to be a standard longitude of h_i so that the part of x_i on $\partial(D^3)$ is x_i^2. Let $x_i^1 \subset x_i$ denote the part of x_i on h_i. We isotope the curves $L_1, \ldots, L_m$ on $\partial(B_m)$ and then we isotope further on so that $|L \cap (\cup_{j=1}^m x_j)|$ is minimal. By construction, the framing on L defined by parallel curves on $\partial(B_m)$ is zero. For $i = 1, \ldots, m$, choose a simple closed curve $f_i \subset \partial(B_m)$ so that: i) The part of f_i on $\partial(D^3)$ is the corresponding part of L_i; ii) the part of f_i on h_i is an arc that intersects each of y_i and x_i^1 exactly once; and iii) the framing defined by a curve parallel to f_i on $\partial(B_m)$ is the framing of L_i in $(L, \mathbf{f})$. Let $S := \partial(B_m)$ and $A_m := \overline{S^3 \setminus B_m}$. The H-diagram $(S, \mathbf{f}, \mathbf{x})$, where $\mathbf{x} := \{x_1, \ldots, x_n\}$, $\mathbf{f} := \{f_1, \ldots, f_n\}$ has the desired properties: Properties (i)-(iv) of Theorem 2.1 are checked as in the proof of Theorems 4.1 of [BP]. The first property stated in part b) follows from the construction of $\mathbf{f}$ and the second property follows from the constructions at stage $(S_5) - (S_8)$. Part c) follows from the assumption that $|L \cap (\cup_{j=1}^m x_j)|$ is minimal, Lemma A.3 and part b). $\square$

Remark A.6. The modifications performed on the link at stage (S_4) of the proof of Theorem A.5, are done in order to assure that $P(L)$ dosen't contain any improper pairs of inadmissible points. This property is not needed in order to assure that the H-diagram associated to L has properties (a)-(c) of Theorem A.5. However, the H-diagrams obtained from the construction in the proof of Theorem A.5 have additional properties that are needed in the main body of the paper. In the rest of this appendix we will discuss these properties.

Definition A.7. *We will say that a projection $P(L) \subset R$ of a nice link $(L, \mathbf{f})$ is good iff: i) P(L) has the properties described at stages $(S_1) - (S_3)$ in the proof of Theorem A.5; ii) it contains no improper pairs of inadmissible points; iii) it contains no bad inadmissible component (see (S_5)); and iv) there is no $i \neq 1$ such that $P(L_i)$ can be isotoped on R so that it bounds disc whose interior is disjoint from $P(L)$.*

Remark A.8. The proof of Theorem A.5 shows that after a number of admissible moves (of type (2) of Definition 3.9) every nice link admits a projection that satisfies the properties (i)-(iii) of Definition

A.7. By applying some admissible moves of type (1) we can assure that property (iv) is also satisfied. Thus, every nice link admits a good projection after a number of admissible moves. In particular, given a 3-manifold $M \in \mathcal{HM}$, we can find a nice H-diagram associated to a good projection of a nice link $(L, \mathbf{f})$ that represents M by surgery.

Let $(S, \mathbf{x}, \mathbf{f})$ be an H-diagram associated, in the sense of Theorem A.5, to a good projection $P(L)$ of a nice link $(L, \mathbf{f})$. For $i = 1, \dots, m$, let $n(x_i) \subset S$ denote a regular neighborhood of x_i on S and let x_{i1}, x_{i2} denote the components of $\partial(n(x_i))$. Set $N := \cup_{i=1}^{m} n(x_i)$ and $\bar{S} := \overline{S \setminus N}$. Let $\bar{\mathbf{f}}$ denote the image of $\mathbf{f}$ on $\bar{S}$. Clearly, $\bar{\mathbf{f}}$ is a collection of properly embedded, disjoint arcs.

Definition A.9. *Let $x_i, x_j \in \mathbf{x}$, where $i \neq j$. The curve x_i will be called adjacent to x_j with respect to $\mathbf{f}$, iff there is a component $c \subset \partial(n(x_i))$, such that: All the components in $\bar{\mathbf{f}}$ that have one boundary point on c have their second boundary point on $\partial(n(x_j))$. We will write $x_i \xrightarrow{\mathbf{f}} x_j$. The curves x_i and x_j will be called adjacent with respect to $\mathbf{f}$ iff we have $x_i \xrightarrow{\mathbf{f}} x_j$ or $x_j \xrightarrow{\mathbf{f}} x_i$. We will say that x_i is strongly adjacent to x_j with respect to $\mathbf{f}$, we will write $x_i \xleftrightarrow{\mathbf{f}} x_j$, iff all the components in $\bar{\mathbf{f}}$ that have one boundary point on c have their second boundary point on the same component of $\partial(n(x_j))$. The curves x_i and x_j will be called strongly adjacent with respect to $\mathbf{f}$ iff we have $x_i \xleftrightarrow{\mathbf{f}} x_j$ or $x_j \xleftrightarrow{\mathbf{f}} x_i$.*

By definition, for every $x_i \in \mathbf{x}$ there can be at most two curves in $\mathbf{x}$ that are adjacent to x_i. The next proposition provides a stronger statement. To state it, recall that the components $L_i \neq L_j \subset L$ are called parallel iff they cobound an embedded annulus in the complement of $L \setminus (L_i \cup L_j)$.

Proposition A.10. *Let $(S, \mathbf{x}, \mathbf{f})$ be an H-diagram associated to a good projection $P(L)$ of a nice link $(L, \mathbf{f})$. Then the following are true:*

a) There is no $i \neq 1$ such that x_i, x_1 are adjacent with respect to $\mathbf{f}$.

b) Suppose that $\bar{L}$ does not contain parallel components. Then, for every $i = 1, \dots, m$, there can be at most one $j \neq i$, such that x_j, x_j are adjacent with respect to $\mathbf{f}$.

Proof : Let $P(L)$ denote the projection of L on the plane R used in the proof of Theorem A.5. For $i = 2, \ldots, m$, let N_i denote a neighborhood of a_i on R. For $6 \leq k \leq 8$ and $i = 1, \ldots, m$, let $P_k(L_i)$ denote the image of $P(L_i)$ at stage (S_k) of the construction of $(S, \mathbf{x}, \mathbf{f})$. Set

$$\Lambda_k := \overline{P_k(L_1) \setminus (P_k(L_1) \cap (\cup_{i=1}^m N_i))}.$$

a) Suppose that there exists $i \neq 1$ such that x_i, x_1 are adjacent with respect to $\mathbf{f}$. By construction of $\mathbf{f}$ in the proof of Theorem A.5, it follows that there must be a component $a \subset \Lambda_6$ with both its boundary points on the same component of $\partial(N_i)$ and such that $b_1 \subset a$. See Figure 18. But then $P(L_i)$ can be isotoped to bound a disc disjoint from $P(L)$ contrary to our assumption.

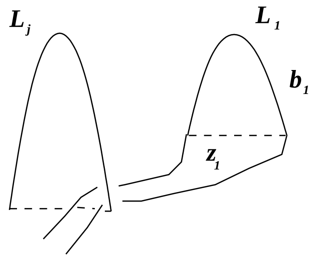

FIGURE 20

b) Suppose that $x_i, x_j \in \mathbf{x}$ are adjacent with respect to $\mathbf{f}$. By (a), we have $i, j \neq 1$. Suppose, without loss of generality, that $x_i \xrightarrow{\mathbf{f}} x_j$. With the notation as in Definition A.9, there are two cases to consider:

Case 1: We have $x_i \xleftrightarrow{\mathbf{f}} x_j$. That is, all the components of $\bar{\mathbf{f}}$ that have one boundary point on a component $c \subset \partial(n(x_i))$ have their second boundary point on the same component of $\partial(n(x_j))$. At stage (S_6) of the construction of $\mathbf{f}$ there are two components $\hat{\alpha}, \hat{\beta} \subset \Lambda_6$ each having a boundary point on a component $\hat{c} \subset \partial(N_i)$. We claim claim that each of $\hat{\alpha}, \hat{\beta}$ has thei second boundary point on the same component of $\partial(N_j)$. For, otherwise $\hat{\alpha}, \hat{\beta}$ must contain undercrossings of $P(L_1)$ that are pushed on $\partial(N_j)$ during stage (S_7). However, our modifications of $P(L)$ at stage (S_4) assure that then we cannot have $x_i \xleftrightarrow{\mathbf{f}} x_j$.

Case 2: There are components $a, b \subset \bar{\mathbf{f}}$ such that: i) each of $\partial(a), \partial(b)$ has one point on the component $c \in \{x_{i1},\ x_{i2}\}$; and ii) the second point of $\partial(a)$ (resp. $\partial(b)$) belongs on x_{j1} (resp. x_{j1}). As above it follows that there must be two components $\hat{\alpha}, \hat{\beta} \subset \Lambda_6$ and a component $\hat{c} \subset \partial(N_i)$ such that each of $\partial(\hat{\alpha}), \partial(\hat{\beta})$ have one point on $\hat{c}$ and have their second point on distinct components of $\partial(N_j)$.

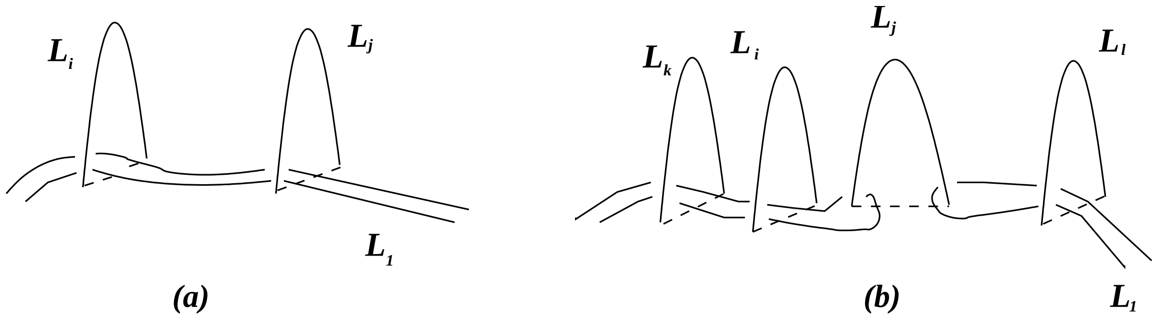

FIGURE 21. Part (a) (resp. Part (b)) corresponds shows part of the projection $P_1(L)$ at stage (S_6) in Case 1 (resp. Case 2).

The situation in both cases discussed above is illustrated in Figure 19. In Case 1, $P_1(L_i)$ and $P_1(L_j)$ can be isotoped so that cobound an annulus on the projection plane R (see Figure 19(a)). But then L_i, L_j must be parallel contrary to our assumption. Thus, under our assumptions, Case 1 cannot occur. Lets us now turn to Case 2 and show that there is no $k \neq i, j$ so that x_k is also adjacent, with respect to $\mathbf{f}$, to x_i or to x_j. To that end, suppose that there exists $k \neq i, j$ so that x_k, x_i are adjacent with respect to $\mathbf{f}$. Then as in Case 1 above, it follows that L_i, L_k must be parallel contrary to our assumption. Now suppose that there is $l \neq i$ so that x_j, x_l are adjacent with respect to $\mathbf{f}$. Clearly, we must have $x_l \xrightarrow{\mathbf{f}} x_j$. There must be two components $\hat{\gamma}, \hat{\delta} \subset \Lambda_1$ and a component $\hat{d} \subset \partial(N_l)$ such that each of $\partial(\hat{\gamma}), \partial(\hat{\delta})$ have one point on $\hat{d}$ and have their second point on the distinct components of $\partial(N_j)$. Again it is seen that this situation cannot occur as it would lead to the conclusion that L_j, L_l are parallel (see Figure 19(b)). $\square$.

We continue with the following corollaries the proofs of which follow from our arguments in the proof of Proposition A.10.

Corollary A.11. *Let $(S, \mathbf{x}, \mathbf{f})$ be an H-diagram associated to a good projection $P(L)$ of a nice link $(L, \mathbf{f})$. Suppose the curves $x_i, x_j \in \mathbf{x}$ are strongly adjacent with respect to $\mathbf{f}$. If and let $i, j \neq 1$, then the components of L correspondng to x_i, x_j are parallel.*

Corollary A.12. *Let $(S, \mathbf{x}, \mathbf{f})$ be an H-diagram associated to a good projection $P(L)$ of a nice link $(L, \mathbf{f})$.*

i) Let $i, j \neq 1$ and let $k \neq j$. If $x_i \xrightarrow{\mathbf{f}} x_j$ and x_i, x_k are adjacent, then either x_i, x_j are strongly adjacent or we have $x_i \xleftrightarrow{\mathbf{f}} x_k$.

ii) Let $x_i, x_j \in \mathbf{x}$. If $x_i \xrightarrow{\mathbf{f}} x_j$, then either x_i, x_j are strongly adjacent or there is no curve $z \in \mathbf{x} \setminus \{x_i\}$ such that x_j, z are adjacent to each other with respect to $\mathbf{f}$.

We conclude this Appendix with two technical lemmas used in the proof of Lemma 7.22. To state them we need some preparation. Let $(S, \mathbf{f}, \mathbf{x})$ be a nice H-diagram for M corresponding to $(L, \mathbf{f})$, as constructed in Theorem A.5. With the notation as in the statement of Theorem A.5, we will view $\mathbf{f}$ and $\mathbf{x}$ on the boundary of the standard handlebody A_m whose meridian discs are bounded by the curves in $\mathbf{x}$ as we've done in throughout the entire Section 7. (See beginning of Subsection 7.1.) The notation and terminology used in the statements of the lemmas below is that used in the statement of Lemma 7.22.

Lemma A.13. *Let $(S, \mathbf{x}, \mathbf{f})$ be an H-diagram associated to a good projection $P(L)$ of a nice link $(L, \mathbf{f})$. Let $i \neq j$ and let C^{γ}_{ij} be a simple closed curve on $S := \partial A_m$ that is constructed as in Sublemma 7.23. Suppose that for some $s \in \{i, j\}$ there exists $k \neq i, j, 1$ and dual points $p \in P_s \cap \phi(f_k)$ and $p' \in Q_s \cap \phi(f_k)$ and a set $U \in \{P_t, Q_t\}$ such that p, p' are absorbed by bigons between C^{γ}_{ij} and f_k that end on U. Here, $\{t\}$ denotes the complement of $\{s\}$ in $\{i, j\}$. Then, there exist curves $x_{i_1}, \ldots, x_{i_{r+1}} \in \mathbf{x} \setminus \{x_1\}$ such that*

$$x_s = x_{i_1} \xleftrightarrow{\mathbf{f}} x_{i_2} \xleftrightarrow{\mathbf{f}} \ldots \xleftrightarrow{\mathbf{f}} x_{i_r} \xrightarrow{\mathbf{f}} x_{i_{r+1}} = x_t,$$

and the components $L_{i_1}, \ldots, L_{i_r}$ of L are obtained from a tangle of the form $T(\pm 1, r)$ of the projection $P(K)$ as shown in Figure 6.

Proof: Recall that by construction (see proof of Sublemma 7.23) C^{γ}_{ij} is a band connect sum of C_i, C_j along the arc γ. Thus, $\partial\gamma$ has one point on each of C_i, C_j and $\mathrm{int}(\gamma) \cap (C_i \cup C_j) = \emptyset$. Let $\mathbf{x}' \subset \mathbf{x}$ denote the collection of curves that are intersected by γ. By assumption there are points $q, q' \in U$ such that p, q (resp. p', q') are absorbed by a bigon D (resp. D') between C^{γ}_{ij} and f_k. That is we have $\partial D = \alpha \cup \beta$ (resp. $\partial D' = \alpha' \cup \beta'$), where $\alpha \subset C^{\gamma}_{ij}$ and $\beta \subset f_k$ (resp. $\alpha' \subset C^{\gamma}_{ij}$ and $\beta' \subset f_k$) and $\alpha \cap \beta = \{p, q\}$ (resp. $\alpha' \cap \beta' = \{p', q'\}$). Since C_i, C_j lie on the 2-sphere part of S, it follows that each of β, β' intersects each curve in $\mathbf{x}'$ in exactly the same way as γ does and it doesn't intersect any of the curves in $\mathbf{x} \setminus \mathbf{x}'$. Let L' denote the sublink of L whose components are in one to one correspondence with the curves in $\mathbf{x}'$ and let L_t the component corresponding to x_t. Also let $\mathbf{f}'$ denote the collection of curves in $\mathbf{f}$ that are in one to one correspondence with the curves in $\mathbf{x}'$. By the construction of $(S, \mathbf{f}, \mathbf{x})$ at stage (S_1) there is a subarc $\theta \subset P(L_1)$ that starts from L_s, intersects each component of $L' \setminus L_t$ exactly one, loops once around L_t and comes back intersecting each component of $L' \setminus L_t$ exactly once again in opposite order and opposite sign from the first time. (Compare, Figure 19(b)). Index the components of L' in the order met by θ as it travels from L_s to L_t. Let $L_{i_1}, \dots, L_{i_{r+1}}$ denote this indexing.

???????? (WORK)

During stage (S_2) of the construction of $(S, \mathbf{f}, \mathbf{x})$ the projection $P(L)$ is altered by finger moves where a small part of certain under crossings is dragged along parts of $P(L_1)$. During this process the intersections of the curves in $\mathbf{f} \setminus (\mathbf{f}' \cup \{f_1\})$ with these in $\mathbf{x}'$ are created by finger moves of the components of $L \setminus (L' \cup \{L_1\})$ along the arc θ. From these observations the desired conclusion follows easily. $\square$

We close the Appendix with the following lemma whose proof is similar to this of Lemma A.13.

Lemma A.14. *Let $i, j \neq 1$ with $i \neq j$ and let C^{γ}_{ij} be a simple closed curve on $S := \partial A_m$ that is constructed as in Sublemma 7.23. Suppose that for some $s \in \{i, j\}$ there exists a set $R \in \{P_s, Q_s\}$ such that all the points in R are absorbed by bigons between C^{γ}_{ij} and Φ that end on*

$P_t \cup Q_t$. Here, $\{t\}$ denotes the complement of $\{s\}$ in $\{i, j\}$. Then, there exist curves $x_{i_1}, \ldots, x_{i_{r+1}} \in \mathbf{x} \setminus \{x_1\}$ that either satisfy the conclusion of Lemma A.13 or we have

$$x_s = x_{i_1} \overset{\mathbf{f}}{\longleftrightarrow} x_{i_2} \overset{\mathbf{f}}{\longleftrightarrow} \ldots \overset{\mathbf{f}}{\longleftrightarrow} x_{i_r} \overset{\mathbf{f}}{\longleftrightarrow} x_{i_{r+1}} = x_t,$$

that is, for $l = 1, \ldots, r+1$, $x_{i_l}, x_{i_{l+1}}$ are strongly adjacent with respect to $\mathbf{f}$.

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