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THE HOMFLY POLYNOMIAL FOR LINKS IN RATIONAL HOMOLOGY 3-SPHERES†

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We use intrinsic 3-manifold topology to construct formal power series invariants for links in a large class of rational homology 3-spheres, which generalizes the 2-variable Jones polynomial (HOMFLY). As a consequence, we show that a certain completion of the HOMFLY skein module of a homotopy 3-sphere is isomorphic to that of the genuine 3-sphere. © 1998 Elsevier Science Ltd. All rights reserved.

1. INTRODUCTION

The theory of quantum groups gives a systematic way of producing families of polynomial invariants, for knots and links in $\mathbb{R}^3$ or S^3 (see for example [18]). In particular, the Jones polynomial [8] and its generalizations [4, 9], can be obtained that way. All these Jones-type invariants are defined as state models on knot diagrams or as traces of braid group representations, and the proofs of their topological invariance offer little insight into the underlying topology. The lack of a topological interpretation of these polynomial invariants always makes it hard, if not impossible, to generalize them for knots and links in other 3-manifolds.

In his study of the topology of the space of knots in S^3 [19] Vassiliev constructed a vast family of knot invariants which are now known as Vassiliev invariants or invariants of finite type. Some of the nice aspects of the theory of invariants of finite type are that they have a simple combinatorial description and that they provide a unifying way to view various knot polynomials [2, 3, 14]. On the other hand, the fact that the theory of finite type invariants rests on topological foundations allows the generalization of many of its aspects to knots in other 3-manifolds [11, 16]. In this paper, using the machinery developed in [11, 16], we will show the existence and uniqueness of formal power series link invariants obeying the HOMFLY skein relation in a large class of rational homology 3-spheres.

In order to state our main results we need to introduce some notation: Suppose that M is an orientable 3-manifold. Let $\pi = \pi_1(M)$ and let $\hat{\pi}$ denote the set of *non-trivial* conjugacy classes of π. Notice that $\hat{\pi}$ can be identified with the set of non-trivial free homotopy classes of oriented loops in M. An n-component *link* is a collection of n unordered oriented circles, tamely and disjointly embedded in M. Hence, n-component links are in 1–1 correspondence with unordered n-tuples of elements in $\hat{\pi} \cup \{1\}$. In every homotopy class of links, we will fix, once and for all, a link CL and call it a *trivial link*. If CL has k components which are homotopically trivial, our choice will be such that $CL = CL^* \coprod U^k$,

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where U^k is the standard unlink with k components in a small ball neighborhood disjoint from CL^*, and U^1 will be abbreviated to U later on. We will denote by $\mathscr{CL}^*$ the set of all *trivial links* with none of their components homotopically trivial. It is obvious that the elements in $\mathscr{CL}^*$ are in 1–1 correspondence with unordered n-tuples of elements in $\hat{\pi}$, for all $n > 0$.

Every link L is homotopic to a certain $CL^* \coprod U^k$ for some $CL^* \in \mathscr{CL}^*$, possibly empty. But the aim of link theory in the 3-manifold M is to understand how two links can differ up to a (tame) isotopy if they are homotopic. Let $\mathscr{L}$ be the set of isotopy classes of links in M and let $R = \mathbb{C}[v^{\pm 1}, z^{\pm 1}]$ be the ring of Laurent polynomials in v and z. A map $\mathscr{L} \to R$ will be called a *link polynomial.*

Now let $z = t^{1/2} - t^{-1/2}$ and let $\mathscr{I}$ be the ideal of $R[t]$ generated by $v - v^{-1}$ and t. Let $\hat{R}$ be the pro-$\mathscr{I}$ completion of $R[t]$, i.e. the inverse limit of

$$\cdots \to R[t]/\mathscr{I}^n \to R[t]/\mathscr{I}^{n-1} \to \cdots.$$

THEOREM A. *Let M be a rational homology 3-sphere which is either atoroidal or a Seifert fibered space with orientable orbit space. Then, there is a unique map $J_M\colon \mathscr{L} \to \hat{R}$ satisfying the HOMFLY skein relation*

$$v^{-1} J_M(L_+) - v J_M(L_-) = z J_M(L_0)$$

and with given values $J_M(U)$ and $J_M(CL^*)$ for every $CL^* \in \mathscr{CL}^*$.

Here, the three links L_+, L_- and L_0 appearing in the HOMFLY skein relation differ only in a small ball neighborhood in M where, as usually, they intersect at a positive crossing, a negative crossing, and a smoothing of a crossing, respectively.

It seems to be worthwhile to comment on the relationship between Theorem A and the study of the HOMFLY skein modules of 3-manifolds (see [6] for a survey). With the notation as above, the HOMFLY skein module $\mathscr{S}_3(M)$ is defined to be the R-module spanned by $\mathscr{L}$, and subject to the HOMFLY skein relation

$$v^{-1} L_+ - v L_- = z L_0.$$

Let $S(R\hat{\pi})$ be the symmetric tensor algebra of the free R-module $R\hat{\pi}$, generated by $\hat{\pi}$. J. Przytycki proposed the following conjecture.

CONJECTURE. *If M is compact and contains no closed non-separating surfaces, then*

$$\mathscr{S}_3(M) \cong S(R\hat{\pi})$$

as R-modules.

In the special case of $M = X \times [0, 1]$ where X is a compact surface, it was proved by Przytycki (see [6] and references therein) that $\mathscr{S}_3(M) \cong S(R\hat{\pi})$ as R-algebras. Many other partial verifications of the conjecture are known. In the presence of non-separating closed surfaces in M, one can construct examples in which $\mathscr{S}_3(M)$ is not torsion free.

Notice that $\mathscr{CL}^* \cup \{U\}$ is in one-one correspondence with a basis of $S(R\hat{\pi})$. So Theorem A provides supporting evidence to (the dual version) of Przytycki's conjecture. We denote $\mathscr{S}_3^* = \mathrm{Hom}(\mathscr{S}_3(M), \hat{R})$. In the special case when $\pi_1(M) = 1$, Theorem A is of particular interest because of the Poincare conjecture. In this special case, since $\mathscr{CL}^* = \emptyset$ when $\pi_1(M) = 1$, we may rephrase Theorem A in the language of skein modules, as follows:

THEOREM B. *If M is a homotopy 3-sphere, then*

$$\mathscr{S}_3^*(M) \cong \mathscr{S}_3^*(S^3) \cong \hat{R}$$

as R-modules.

Let us now briefly discuss the main ideas of the paper:

For an $L \in \mathscr{L}$ let $\mathscr{M}^L$ denote the space of maps, equipped with the compact-open topology, from a disjoint union of circles to M, which are homotopic to L. Any two links in some $\mathscr{M}^L$ are related by a sequence of "crossing changes". When we make a crossing change from one link to another, we produce a singular link with one transverse double point as an intermediate step. If we are given a link invariant $F: \mathscr{L} \to \mathscr{R}$, where $\mathscr{R}$ is a ring, we may define an invariant f of singular links with one double point by

$$f(L_\times) = F(L_+) - F(L_-), \tag{1}$$

where $\times$ stands for a double point and $L_\pm$ are links obtained from resolving the double point into a positive or negative crossing, respectively.

A natural question is the following: Starting with a singular link invariant f, we want to find necessary and sufficient conditions that f has to satisfy so that it is derived from a link invariant, via (1). This question was shown in [11, 16] to play an important role in understanding invariants of finite type for links in 3-manifolds. In Section 3 we answer the question for links in rational homology spheres which are either *atoroidal* or *non-special Seifert fibered spaces*. In Section 4, we use the results in Section 3 to prove Theorem A. Theorem B is a direct corollary of Theorem A.

The question of whether f is induced by a link invariant turns out to be a question about the "integrability" and of f along paths in every $\mathscr{M}^L$. Let Φ be a path in $\mathscr{M}^L$ connecting two links. After perturbation, we may assume that there are only finitely many points on Φ where we see singular links with one double point. Moreover, we can assume that when the parameter of the path passes through a point where we see a singular link, the nearby links are changed by a crossing change. The sum of suitably signed values of f on these singular links along the path Φ, denoted here by X_Φ, can be thought of as the integral of f along Φ.

In order for f to be derived from a link invariant, it is necessary and sufficient that X_Φ is independent of Φ relative to the end points, or equivalently that $X_\Phi = 0$, for every loop Φ in $\mathscr{M}^L$. It turns out that the first thing that one has to do is to find a set of finitely many *local integrability conditions* which guarantee $X_\Phi = 0$ for every null-homotopic loop Φ in $\mathscr{M}^L$. Our technical assumptions about M, in the statement of Theorem A, will then imply that these local integrability conditions guarantee $X_\Phi = 0$ for every loop Φ in $\mathscr{M}^L$.

If Φ is null-homotopic in $\mathscr{M}^L$, then we achieve our goal by putting the null-homotopy into *almost general position*. This is done in [16] (see also 3.2). The treatment for the general case is based on the machinery developed in [11]. Here we need to change our point of view and think of Φ as a map from a disjoint union of tori into M. This naturally leads to the study of tori in M and to the use of the results in [7, 17] in order to treat essential Φ's. More precisely, by employing the homotopy classification of essential tori in Seifert fibered spaces, we are able to homotope Φ into a certain nice position so that the local integrable conditions imply $X_\Phi = 0$ (see 3.3 and 3.4).

We organize the paper as follows: In Section 2 we recall the generic picture of a family of maps from a compact 1-polyhedron into a 3-manifold, parameterized by a disc, and we give the preliminaries from the topology of 3-manifolds that we use in subsequent sections. In Section 3 we answer the integrability question addressed above. The main result of this section is Theorem 3.1.2. In Section 4 we use Theorem 3.1.2 to construct formal power series link invariants which satisfy the same crossing change formulae as the HOMFLY polynomial (Theorem 4.2.1). Theorem A, and subsequently Theorem B, will follow from Theorem 4.2.1 immediately.

Remark. (a) We do not know whether $J_M(U)$ and $J_M(CL^*)$ can be appropriately chosen in R so that J_M *is a* link polynomial, i.e. $J_M(L) \in R$ for every $L \in \mathscr{L}$. In [12] the first named author shows that the obstruction to doing so is a particular class of links in M. The *convergence problem* is an important one and we hope to further address it in the future.

(b) A different approach to power series invariants for links in rational homology spheres can be found in [14].

2. PRELIMINARIES

2.1. Almost general position for a disjoint union of circles

In this section we summarize from [16] the results about the generic picture of a family of maps from a disjoint union of circles to a 3-manifold, parameterized by a disc.

Let P be a one-dimensional compact polyhedron. Let M be a 3-manifold and let D^2 denote the 2-disc. A map $\Phi: P \times D^2 \to M$ gives rise to a family of maps $\{\phi_x: P \to M; x \in D^2\}$ where $\phi_x(*) = \Phi(*, x)$ for $x \in D^2$. Suppose that every ϕ_x is a piecewise–linear map and let S_Φ be the closure of the set $\{x \in D^2; \phi_x \text{ is not an embedding}\}$. One can see that S_Φ is a sub-polyhedron of F.

Two maps $\phi_1, \phi_2: P \to M$ are called ambient isotopic if there exists an isotopy $h_t: M \to M$, $t \in [0, 1]$ with $h_0 = id$ and $h_1\phi_1 = \phi_2$.

Let us now introduce some terminology about one-dimensional polyhedra in 3-manifolds.

Let P be a one-dimensional polyhedron. Every point $q \in P$ has a neighborhood homeomorphic to a bouquet of finitely many arcs such that q is the common endpoint of these arcs. The number of arcs in the bouquet is called the *valence* of q. A point $q \in P$ with valence different than 2 is called a *vertex* of P. A component of the complement of vertices is called an *edge* of P.

A double point of a map $\phi: P \to M$ is a point $p \in M$ such that $\phi^{-1}(p)$ consists of two points. A double point of a piecewise linear map ϕ: $P \to M$ is called a transverse double point if there exist two 1–simplexes σ_1, σ_2 contained in the 1–skeleton of P such that

(1) ϕ is linear and non-degenerate on σ_1 and σ_2,
(2) $\phi(\sigma_1) \cap \phi(\sigma_2)$ is the double in question,
(3) $\phi(\sigma_1)$ and $\phi(\sigma_2)$ intersect transversally in their interiors and they do not lie on the same plane.

We call a one-dimensional sub-polyhedron $S \subset D^2$ *neat* if $S \cap \partial D^2$ consists of finitely many points and each of them is a valence 1 vertex of S. We call these vertices *boundary vertices* of S and we call the vertices of S lying in the interior of D^2 *interior vertices* of S.

We suppose now that P is a disjoint union of circles. Then we have

PROPOSITION 2.1.1. (Lin [16]). *A map $\Phi: P \times D^2 \to M$ can be changed by an arbitrary small perturbation so that S_Φ is a neat one-dimensional sub-polyhedron of F. Moreover,* *we have*

(1) If $x, x' \in D^2$ *belong to the same component of* $D^2 \backslash S_\Phi$ *or* $S_\Phi \backslash \{$*interior vertices*$\}$, *then* ϕ_x *and* $\phi_{x'}$ *are ambient isotopic.*
(2) *The interior vertices of* S_Φ *are of valence either four or one.*
(3) *If* $x \in S_\Phi$ *lies on an edge of* S_Φ *or is a boundary vertex, then* ϕ_x *has exactly one transverse double point.*
(4) *If* $x \in S_\Phi$ *is an interior vertex of valence four, then* ϕ_x *has exactly two transverse double points.*
(5) *If* $x \in S_\Phi$ *is an interior vertex of valence one, then* ϕ_x *is an embedding ambient isotopic to the nearby embeddings.*

We say that the resulting map in Proposition 2.1.1 is *in almost general position.* Figure 1 below illustrates $S_\Phi \subset D^2$ for a map Φ in almost general position.

Remark 2.1.2. (a) The proposition above is true for maps $\Phi : P \times X \to M$, where X is a compact surface with $\partial X \neq \emptyset$, as well as for more general types of polyhedra P. It was used extensively, in [11], to define and study invariants of finite type for knots in 3-manifolds.

(b) If $\Phi | P \times \partial D^2$, is in almost general position already, then the perturbation in Proposition 2.1.1 can be fixed on ∂D^2.

2.2. Seifert fibered spaces

In this section we give some terminology from the topology of 3-manifolds and recall some results from [7, 17] that are used in subsequent sections.

Definition 2.2.1. A surface $X \neq S^2$, properly embedded in a 3-manifold M (or embedded in ∂M), is compressible if there exists a disc $D \subset M$ such that $D \cap X = \partial D$ and ∂D is not homotopically trivial in X. Otherwise X is called incompressible in M. A compact, orientable, irreducible 3-manifold is called a Haken (or sufficiently large) manifold, if it contains a two-sided incompressible surface.

Definition 2.2.2. Let M be a closed 3-manifold and X a surface. A map $\phi : X \to M$ is called essential if $\ker\{\phi_*: \pi_1(X) \to \pi_1(M)\} = 1$.

Let (μ, ν) be a pair of relatively prime integers. Let

$$D^2 = \{(r, \theta);\ 0 \leqslant r \leqslant 1,\ 0 \leqslant \theta \leqslant 2\pi\} \subset \mathbb{R}^2$$

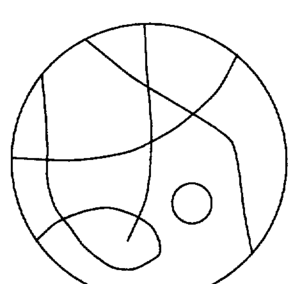

Fig. 1. $S_\Phi \subset D^2$.

A fibered solid torus of type (μ, ν), is the quotient of the cylinder $D^2 \times I$ via the identification $((r, \theta), 1) = ((r, \theta + 2\pi\nu/\mu), 0)$. If $\mu > 1$ the fibered solid torus is said to be *exceptionally fibered* and the core is the *exceptional fiber*. Otherwise the fibered solid torus is *regularly fibered* and each fiber is a *regular fiber*.

Definition 2.2.3. An orientable 3-manifold is said to be a Seifert fibered space, if it is a union of pairwise disjoint simple closed curves, called fibers, such that each one has a closed neighborhood, consisting of a union of fibers, which is homeomorphic to a fibered solid torus via a fiber preserving isomorphism.

In a Seifert fibered space M, a fiber is called *exceptional* if it has a neighborhood homeomorphic to an exceptionally fibered solid torus and the fiber in question corresponds to the exceptional fiber of the solid torus. The orbit space, B, of M is the quotient obtained by identifying every fiber of M to a point. Notice that B is a surface.

Definition 2.2.4. Let M be a Seifert fibered space, with a fixed fibration and let $p : M \to B$ be the fiber projection. Let X be a surface. A map $\Phi : X \to M$ is called vertical or saturated, with respect to p, if $p^{-1}(p\Phi(X)) = \Phi(X)$ and $\Phi(X)$ contains no exceptional fibers.

It is known that if $\pi_1(M)$ is infinite and M is a Seifert fibered space which is not Haken, then it has to fiber over the 2-sphere with three exceptional fibers. Moreover, if N is the subgroup of $\pi_1(S)$ generated by a regular fiber of S, then the quotient $\pi_1(S)/N$ is the triangle group $\Delta(p, q, r)$, where p, q and r are the multiplicities of the exceptional fibers. As shown in [17], an essential map $S^1 \times S^1 \to S$, can always be homotoped to a vertical one if $\Delta(p, q, r)$ is a hyperbolic triangle group. The remaining cases are the Euclidean Seifert surfaces corresponding to the three triangle groups $\Delta(3, 3, 3)$, $\Delta(2, 4, 4)$ and $\Delta(2, 3, 6)$. These together with $S^1 \times S^1 \times S^1$, the orientable S^1-bundle fibering over the Klein bottle and $\mathbb{R}P^2(-1; 2, 2)$ form the entire list of irreducible Seifert fibered manifolds that contain essential tori that can not be homotoped to vertical position with respect to all fibrations. Here $\mathbb{R}P^2(-1; 2, 2)$, is the Seifert space with orbit space $\mathbb{R}P^2$ that has two exceptional fibers each with multiplicity 2, and Euler number -1. We will call these manifolds *special.*

Throughout this paper we are dealing with Seifert fibered spaces that are rational homology 3-spheres (i.e $H_1(M)$ is finite). The only *special* manifold in this class is $\mathbb{R}P^2(-1; 2, 2)$.

Proposition 2.2.5. *Suppose that M is a non-special Seifert fibered rational homology 3-sphere with a fixed fibration. Let $\Phi : T = \coprod T_i \to M$ be an essential map, where each T_i is a torus. Then there exists a homotopy $\Phi_t : T \to S$, $t \in [0, 1]$, such that $\Phi_0 = \Phi$ and Φ_1 is vertical with respect to the given fibration of M.*

The proof of the proposition is given in [7] for the case when M is Haken. For the rest of the cases, see [17].

3. INTEGRATING INVARIANTS OF SINGULAR LINKS

In this section we introduce singular links and study their invariants. Our purpose is to give conditions under which an invariant of singular links gives rise to a link invariant.

3.1. Definitions and the statement of the main result

Let M be an oriented 3-manifold and let P be a disjoint union of oriented circles.

Definition 3.1.1. A singular link of order n is a piecewise-linear map $L : P \to M$ that has exactly n transverse double points. Two singular links L and L' are equivalent if there is an isotopy $h_t : M \to M$, $t \in [0, 1]$ such that $h_0 = id$, $L' = h_1(L)$ and the double points of $h_t(L)$ are transverse for every $t \in [0, 1]$.

We will also use L to denote $L(P)$. A singular link of order 0 is simply a link.

Let $p \in M$ be a transverse double point of a singular link L. Then $L^{-1}(p)$ consists of two points $p_1, p_2 \in L$. There are disjoint 1-simplexes, σ_1 and σ_2, on P with $p_i \in \mathrm{int}(\sigma_i)$, $i = 1, 2$, such that for a small ball neighborhood B of p in M

$$L \cap B = L(\sigma_1) \cup L(\sigma_2).$$

Moreover, there is a proper 2-disc D in B such that $L(\sigma_1)$, $L(\sigma_2) \subset D$ intersect transversally at p, and the isotopy h_t of Definition 3.1.1 carries the ball disc pair (B, D) through for all the double points of L.

We can resolve a transverse double point of a singular knot of order n in different ways. Notice that $L(\partial\sigma_1) \cup L(\partial\sigma_2)$ consists of four points on ∂D. Also, since σ_i inherits an orientation from that of P we can talk about the initial point and terminal point of σ_i and $L(\sigma_i)$.

Now choose arcs a_1, a_2, b_1, b_2 with disjoint interiors such that

(1) a_1 and a_2 go from the initial point of $L(\sigma_1)$ to the terminal point of $L(\sigma_1)$ and lie in distinct components of $\partial B \setminus \partial D$; and

(2) b_1 and b_2 lie on ∂D with β_1 going from the initial point of $L(\sigma_1)$ to the terminal point of $L(\sigma_2)$ and b_2 from the initial point of $L(\sigma_2)$ to the terminal point of $L(\sigma_1)$. See Fig. 2.

Define

$$L_+ = \overline{L \setminus L(\sigma_2)} \cup a_1$$

$$L_- = \overline{L \setminus L(\sigma_2)} \cup a_2$$

$$L_0 = \overline{L \setminus L(\sigma_1 \cup \sigma_2)} \cup (b_1 \cup b_2).$$

Clearly L_+, L_- are well-defined singular links of order $n - 1$. We call L_+ (respectively, L_-) a positive (respectively, a negative) resolution of L.

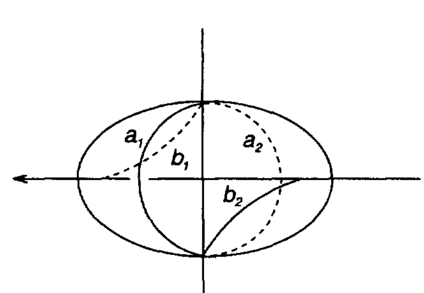

Fig. 2. Resolving a transverse double point.

We will denote by $\mathscr{L}^n$ the set of equivalence classes of singular links of order n in M. Let $\mathscr{R}$ be a ring. A singular link invariant is a map $\mathscr{L}^n \to \mathscr{R}$. Notice that for $n = 0$ we have a link invariant. From a link invariant $F : \mathscr{L} \to \mathscr{R}$ we can always define a singular link invariant $f : \mathscr{L}^1 \to \mathscr{R}$ as follows:

Let $L_\times \in \mathscr{L}^1$ where $\times$ stands for the only double point. Then $L_+, L_- \in \mathscr{L}^0 = \mathscr{L}$. We define $f : \mathscr{L}^1 \to \mathscr{R}$ by

$$f(L_\times) = F(L_+) - F(L_-). \tag{2}$$

In this section we will answer the following question: Suppose that we are given a singular link invariant $f: \mathscr{L}^1 \to \mathscr{R}$. Under what conditions can we find a link invariant $F : \mathscr{L} \to \mathscr{R}$ so that (2) holds for all $L_\times \in \mathscr{L}^1$.

In [2], Bar-Natan thinks of (2) when going from the link invariant F to the singular link invariant f as the "first derivative" of F. In this spirit the question above concerns the "integrability" of a singular link invariant.

For the rest of the paper we will assume, unless otherwise stated, that M is a rational homology 3-sphere such that either (i) it has trivial π_2 and there are no essential maps $S^1 \times S^1 \to M$ or (ii) it is a Seifert fibered rational homology 3-sphere which is not *special.* If M is as in (i) we will say that it is *atoroidal.* Notice that if M is as in (ii) then it is *irreducible* and hence we have $\pi_2(M) = \{1\}$ by the sphere theorem (see for example [5]). The orbit space of a rational homology sphere is either S^2 or $\mathbb{R}P^2$.

We will also assume that $\mathscr{R}$ is a ring which is torsion free as an abelian group. Our main result in this section is the following theorem, which answers the integrability question for a large class of rational homology 3-spheres.

THEOREM 3.1.2. *Suppose that M and $\mathscr{R}$ are as above, and let $f : \mathscr{L}^1 \to \mathscr{R}$ be a singular link invariant. Moreover assume that if M is a Seifert fibered space, the orbit space of M is S^2. There exists a link invariant $F : \mathscr{L} \to \mathscr{R}$ so that* (2) *holds for all $L_\times \in \mathscr{L}^1$ if an only if f satisfies*

$$f(\infty) = 0 \tag{3}$$

$$f(L_{\times +}) - f(L_{\times -}) = f(L_{+\times}) - f(L_{-\times}). \tag{4}$$

Notation. Before we proceed with the proof of Theorem 3.1.2, let us explain the notation above. In (3) the kink stands for a singular link $L_\times \in \mathscr{L}^1$ where there is an embedded 2-disc $D \subset M$ such that $L_\times \cap D = \partial D$, and the unique double point of $L_\times$ lies on ∂D. In (4) we start with an arbitrary singular link $L_{\times\times} \in \mathscr{L}^2$. The four singular links in $\mathscr{L}^1$ are obtained by resolving one double point of $L_{\times\times}$ at a time. We will call conditions (3) and (4) above the *local integrability conditions.*

Proof of Theorem 3.1.2. One direction of the theorem is clear. That is if a singular link invariant $f: \mathscr{L}^1 \to \mathscr{R}$ is derived from a link invariant $F : \mathscr{L} \to \mathscr{R}$ via (2), then it satisfies (3) and (4). To see that (3) is satisfied observe that the positive and the negative resolution of the double point in the kink are equivalent. For (4) observe that, using (2), both sides of (4) can be expressed as $F(L_{++}) - F(L_{-+}) - F(L_{+-}) + F(L_{--})$.

We now turn into the proof of the other direction. Namely, assuming that a singular link invariant $f: \mathscr{L}^1 \to \mathscr{R}$ is given satisfying (3) and (4), we show that it can be derived from a link invariant $F : \mathscr{L} \to \mathscr{R}$ via (2), provided that M is as in the statement of Theorem 3.1.2.

Let $L \in \mathscr{L}$ be a link in M. We also use L to denote a representative $L : P \to M$, of L. Let $\mathscr{M}^L(P, M)$ denote the space of maps $P \to M$ homotopic to L, equipped with the compact-open topology. For every $L' \in \mathcal{M}^L(P, M)$, we choose a homotopy $\phi_t : P \times [0,1] \to M$ such that $\phi_0 = L'$ and $\phi_1 = L$. After a small perturbation, we can assume that for only finitely many points $0 < t_1 < t_2 < \cdots < t_n < 1$, ϕ_t is not an embedding. Moreover, we can assume that ϕ_{t_i}, for $i = 1, 2, \ldots, n$ are singular links of order 1 (i.e. $\phi_{t_i} \in \mathscr{L}^{(1)}$). For different t's in an interval of $[0,1] \backslash \{t_1, t_2, \ldots, t_n\}$, the corresponding links are equivalent. When t passes through t_i, ϕ_t changes from one resolution of ϕ_{t_i} to another.

We define

$$F(L') = F(L) + \sum_{i=1}^{n} \varepsilon_i f(\phi_{t_i}). \tag{5}$$

Here $\varepsilon_i = \pm 1$ is determined as follows: If $\phi_{t_i+\delta}$, for $\delta > 0$ sufficiently small, is a positive resolution of ϕ_{t_i} then $\varepsilon_i = 1$. Otherwise $\varepsilon_i = -1$.

To prove that F is well defined we have to show that modulo "the integration constant" $F(L)$, the definition of $F(L')$ by (5) is independent of the choice of the homotopy. For this we consider a closed homotopy Φ: $P \times S^1 \to M$. After a small perturbation, we can assume that there are only finitely many points $x_1, x_2, \ldots, x_n \in S^1$, ordered cyclicly according to the orientation of S^1, so that $\phi_{x_i} \in \mathscr{L}^1$ and ϕ_x is equivalent to ϕ_y for all $x_i < x, y < x_{i+1}$. To prove that F is well defined we need to show that

$$X_\Phi := \sum_{i=1}^{n} \varepsilon_i f(\phi_{t_i}) = 0 \tag{6}$$

where $\varepsilon_i = \pm 1$ is determined by the same rule as above. We will call (6) the *global integrability condition* around Φ.

The proof of (6), which will be broken into many steps, occupies most of the rest of Section 3. At the end of Section 3 we will briefly discuss Seifert fibered rational homology spheres fibering over $\mathbb{R}P^2$.

3.2. The proof of the global integrability condition in some special cases

Assume that M is an oriented 3-manifold, with $\pi_2(M) = \{1\}$, and that $f: \mathscr{L}^1 \to \mathscr{R}$ is a singular link invariant. Let $L: P \to M$ be a link, and recall that $\mathcal{M}^L(P, M)$ denotes the space of maps $P \to M$ homotopic to L, equipped with the compact-open topology. A closed homotopy $\Phi: P \times S^1 \to M$ from L to itself, can be viewed as a loop in $\mathcal{M}^L(P, M)$.

LEMMA 3.2.1. *Let M, P, and Φ be as above. Moreover, suppose that Φ can be extended to a map $\hat{\Phi}: P \times D^2 \to M$, where D^2 is a 2-disc with $\partial D^2 = \{*\} \times S^1$. Then, Φ satisfies the global integrability condition, i.e. $X_\Phi = 0$.*

Proof. We perturb $\hat{\Phi}$ to an almost general position map as in Proposition 2.1.1. Then each edge of the set of singularities $S_{\hat{\Phi}}$, corresponds to a singular link of order 1. So by using the invariant f we can assign an element of $\mathscr{R}$ to every edge of $S_{\hat{\Phi}}$. We can reduce the global integrability condition around $\hat{\Phi}$, to local integrability conditions around each interior vertex in $S_{\hat{\Phi}}$.

More precisely, for every interior vertex of $S_{\hat{\Phi}}$ draw a small circle C around it, so that the number of points in $C \cap S_{\hat{\Phi}}$ is equal to the valence of the vertex. For a picture see Fig. 3. It suffices to show that

$$\sum_{x \in C \cap S_{\hat{\Phi}}} \pm f(\hat{\phi}_x) = 0 \tag{7}$$

for every interior vertex of $S_{\hat{\Phi}}$. Here $\hat{\phi}_x(S^1) = \hat{\Phi}(P \times \{x\})$.

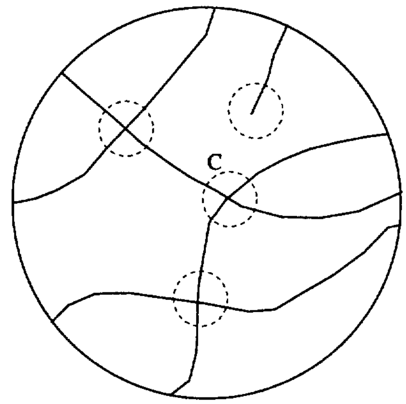

Fig. 3. From global to local integrability conditions.

Case 1. The valence of the interior vertex is one: in this case it is easy to see that for $x \in S_{\hat{\Phi}}$, near that vertex, the unique double point of $\hat{\phi}_x$ is at a kink. So (7) is implied by the local integrability condition (3)

Case 2. The valence of the interior vertex is four: in this case the four points in $C \cap S_{\hat{\Phi}}$ correspond to the four singular knots appearing in the local integrability condition (4) and one can show that (7) is guaranteed by it. □

LEMMA 3.2.2. *Let $f: \mathscr{L}^1 \to \mathscr{R}$ be a singular link invariant and let $\Phi: S^1 \to \mathscr{M}^L(P, M)$ be a loop. Then X_Φ only depends on the free homotopy class of Φ in $\mathscr{M}^L(P, M)$.*

Proof. Let Φ' be another closed homotopy in almost general position such that Φ, $\Phi': S^1 \to \mathscr{M}^L(P, M)$ are freely homotopic loops in $\mathscr{M}^L(P, M)$. Then there exists a homotopy $\Phi_t: P \to \mathscr{M}^L(P, M)$ with $t \in [0, 1]$, such that $\Phi_0 = \Phi$ and $\Phi_1 = \Phi'$.

Let γ be the path in $\mathscr{M}^L(P, M)$ defined by $\gamma(t) = \Phi_t(L)$. After putting γ in almost general position we have

$$X_{\gamma\Phi\gamma^{-1}} = X_\gamma + X_{\Phi'} - X_\gamma = X_{\Phi'}.$$

Hence, we can assume that both X_Φ and $X_{\Phi'}$ are based at L and the homotopy Φ_t is taken relatively L. The homotopy Φ_t gives rise to a map $\mathscr{H}: P \times S^1 \times I \to M$. We cut the annulus $S^1 \times I$ into a disc D along a proper arc $\alpha \subset S^1 \times I$. Then, we have

$$X_{\partial D} = \pm (X_\Phi - X_{\Phi'} - X_\alpha + X_\alpha).$$

By Lemma 3.2.1 we obtain $X_{\partial D} = 0$, and hence $X_\Phi = X_{\Phi'}$. □

To continue, we first need to introduce some notation. Suppose that P has m components; that is

$$P = \coprod_{i=1}^{m} P_i$$

where each P_i is an oriented circle. Let $L: P \to M$ be a link. Pick a base point $p_i \in P_i$ and let a_i denote the homotopy class of $L(P_i)$ in $\pi_1(M, L(p_i))$. Finally, we denote by $Z(a_i)$ the centralizer of a_i in $\pi_1(M, L(p_i))$.

LEMMA 3.2.3. *Assume that* M, P, *and* f *are as in the statement of Lemma* 3.2.2. *Let* $L: P \to M$ *be a link such that the abelianization of* $Z(a_i)$ *is finite, for every* $i = 1, \ldots, m$. *Then* $X_\Phi = 0$, *for every closed homotopy* $\Phi: P \times S^1 \to M$ *from* L *to itself.*

Proof. We denote by $\mathcal{M} = \mathcal{M}^L(P, M)$ the space of maps $P \to M$, which are homotopic to L, equipped with the compact-open topology.

Let

$$\pi = \pi_1(\mathcal{M}^L(P, M), L).$$

Since $\pi_2(M) = \{1\}$ one can see that π is isomorphic to the direct product of the centralizers $\{Z(a_i)\}_{i=1,\ldots,m}$.

By Proposition 3.3 of [16], the assignment $\Phi \to \chi(\Phi)$ is a group homomorphism $\chi: \pi \to \mathcal{R}$.

Since $\mathcal{R}$ is abelian, χ must factor through the abelianization of π which is finite by assumption. Now, since $\mathcal{R}$ is torsion free, we must have $\chi = 0$ and thus

$$\chi(\Phi) = X_\Phi = 0$$

which is the desired conclusion. □

COROLLARY 3.2.4. *Assume that* M *is a rational homology sphere. Let* $L: P \to M$ *be a link and let* $\Phi: P \times S^1 \to M$ *be a closed homotopy from* L *to itself.*

(1) *If all components of* L *are homotopically trivial in* M, *then* $X_\Phi = 0$.
(2) *If* $\pi_1(M)$ *is finite, then* $X_\Phi = 0$.
(3) *Assume that* M *is a Seifert fibered space, and that each component of* L *is either homotopic to a regular fiber of the fibration, or it is homotopically trivial. Then* $X_\Phi = 0$.

Proof. (1) and (2) follow immediately from Lemma 3.2.3.

(3). By Lemma 32.8 of [7], we know that the centralizer of a regular fiber is $\pi_1(M)$. Hence the result follows from Lemma 3.2.3. □

LEMMA 3.2.5. *Let* $L: P \to M$ *be a link and let* $\Phi: P \times S^1 \to M$ *be a closed homotopy from* L *to itself. Assume that for some* $i = 1, \ldots, m$ *the abelianization of* $Z(a_i)$ *is finite. Let* $P' = P \backslash P_i$, *and let* $\Phi' = \Phi | (P' \times S^1)$. *If* $X_{\Phi'} = 0$, *then* $X_\Phi = 0$.

Proof. Without loss of generality we may assume that the abelianization of $Z(a_1)$ is finite. Let L_1 denote the restriction of L on P_1 and let Φ_1 denote the restriction of Φ on $P_1 \times S^1$. We denote by $\mathcal{M}_1 = \mathcal{M}^{L_1}(P_1, M)$ the space of maps $P_1 \to M$, which are homotopic to L_1, equipped with the compact-open topology. Let

$$\pi = \pi_1(\mathcal{M}^{L_1}(P_1, M), L_1).$$

We have that $\pi = Z(a_1)$. Clearly, Φ_1 represents an element in π.

Let $\Psi: P_1 \times S^1 \to M$ be a loop in $\mathcal{M}_1$ based at L_1. We define $\tilde{\Psi}: P \times S^1 \to M$ by

$$\tilde{\Psi} | P_1 \times S^1 = \Psi$$

$$\tilde{\Psi}(P' \times S^1) = \Phi(P' \times S^1)$$

where $P' = P \backslash P_1$. Then $\tilde{\Psi}$ is the closed homotopy from L to itself.

Define

$$\chi(\Psi) := X_{\tilde{\Psi}}$$

CLAIM. *The assignment $\Psi \to \chi(\Psi)$ is a group homomorphism $\chi : \pi \to \mathcal{R}$.*

Proof. It is enough to show that $\chi(\Psi)$ is independent of the choice of the representative of $[\Psi] \in \pi$. Let $\Psi_1 : P_1 \times S^1 \to M$ be another loop in almost general position, which is homotopic to Ψ. Then clearly $\tilde{\Psi}$ and $\tilde{\Psi}_1$ are homotopic loops in $\mathcal{M}^L(P, M)$, and the claim follows from Lemma 3.2.2.

Since $\mathcal{R}$ is abelian, χ must factor through a finite abelian group. Thus, we must have $\chi = 0$ since $\mathcal{R}$ is torsion free. In particular,

$$\chi(\Phi_1) = X_\Phi = 0$$

as desired. □

3.3. Closed homotopies of links and essential tori

The purpose of this paragraph is to study closed homotopies of links thought of as singular tori in 3-manifolds. Since we are mainly interested in the global integrability condition, and in view of Corollary 3.2.4, we may (and will) consider only 3-manifolds with infinite π_1.

Assume that M is a Seifert fibered space with orbit surface B and fiber projection $p : M \to B$. Let $\Phi : T = S^1 \times S^1 \to M$ be a closed homotopy of the knot $K = \Phi(S^1 \times \{*\})$, such that $\Phi(T)$ is vertical with respect to the given fibration. Let $Q = S^1 \times \{*\}$, let $H = \{*\} \times S^1$ denote the parameter space of the homotopy and let $\alpha = p(\Phi(T))$. Then α is an orientation preserving, immersed closed curve on B (without triple points). Let $B_1 \subset B$ denote a regular neighborhood of α. Then a regular neighborhood, S, of $\Phi(T)$ in M, is an orientable S^1 bundle over B_1. Now, let

$$\tilde{M} \to M$$

be the covering space corresponding to the cyclic normal subgroup generated by a regular fiber. We say that $K = \Phi(S^1 \times \{*\})$ *does not wrap around the fibers of* M if the following are true: (i) $\Phi | H$ lifts to $\tilde{M}$ and (ii) Q is a section of the S^1 bundle obtained as the pull-back of the S^1 bundle over B_1 to S^1, via the map $S^1 \to B_1$ defined by α. In particular, $p(K) = \alpha$.

LEMMA 3.3.1. *Let M, B and Φ be as above. Moreover assume that $\Phi(S^1 \times \{*\})$ does not wrap around the fibers of M and the curve $\alpha = p(\Phi(T))$ does not contain any orientation reversing sub-loops. Then the closed homotopy Φ is homotopic to another closed homotopy Φ' with the following property: For every $x_1, x_2 \in S^1$, there is a homeomorphism $h^{12} : M \to M$ such that*

(1) *$h^{12} = id$ outside of a regular neighborhood of $\Phi'(T)$ in M,*
(2) *$h^{12}(\phi_{x_1}) = \phi_{x_2}$, where $\phi_x = \Phi' | S^1 \times \{x\}$,*
(3) *h^{12} is isotopic to the identity map $id : M \to M$.*

See Lemma 3.12 in [11].

LEMMA 3.3.2. *Let M be a non-special Seifert fibered rational homology sphere, with fiber projection $p : M \to B$. Let $\Phi : T = S^1 \times S^1 \to M$ be an essential map. Then, there exists a map $\Phi_1 : T \to M$ which is homotopic to Φ, and a finite covering $\tau : S^1 \times S^1 \to S^1 \times S^1$ such that the map $\Phi_1 \circ \tau : S^1 \times S^1 \to M$ can be extended to a map $\hat{\Phi} : S^1 \times X \to M$. Here X is a surface with $\partial X = \{*\} \times S^1$.*

Proof. By Proposition 2.2.5, Φ is homotopic to a map $\Phi_1 : T \to M$ which is vertical with respect to the fibration of M. Then, there exists a decomposition $T = S^1 \times S^1$ such that

(a) $\Phi_1(S^1 \times \{*\})$ covers a regular fiber h, of M.
(b) We have $p(\Phi_1(\{*\} \times S^1)) = p(T)$.

Let H (respectively, Q) denote the curve $S^1 \times \{*\}$ (respectively, $\{*\} \times S^1$ on T), and let N be the cyclic normal subgroup $\pi_1(M)$ generated by the regular fiber h. Since M is a rational homology 3-sphere, the abelianization of the *Fuchsian* group $\Delta = \pi_1(M)/N$, is finite. Let d be its order, and consider the d-fold covering $\tau : \hat{T} :\to T = H \times Q$, corresponding to the subgroup $\mathbb{Z} \oplus d\mathbb{Z}$ of $\pi_1(T) = \mathbb{Z} \oplus \mathbb{Z}$. Let $\hat{H}$ and $\hat{Q}$ denote the liftings, on $\hat{T}$, of H and Q respectively. Then, the map $\hat{Q} \to B$ induced by $\Phi_1 \circ \tau$ extends to a map $X \to B$ for some compact surface X with boundary $\hat{Q}$. This gives us a map $\pi_1(X) \to \Delta$, which in turn lifts to a map $\hat{\Phi} : S^1 \times X \to M$ (recall that $\pi_1(M)$ is an extension of Δ by $\mathbb{Z} \cong N$), with $\hat{\Phi} | S^1 \times \partial X = \Phi_1 \circ \tau$. □

Let P be a disjoint union of oriented circles and let $\Phi : P \times S^1 \to M$ be a closed homotopy from a link $L : P \to M$ to itself. Let $f : \mathscr{L} \to \mathscr{R}$ be a singular link invariant and let X_Φ be the quantity defined in (6). Suppose that P has m components, that is

$$P = \coprod_{i=1}^{m} P_i$$

LEMMA 3.3.3. *Assume that M is a Seifert fibered rational homology 3-sphere with orbit space S^2, and let P, Φ be as above. Moreover, assume that $\Phi | P_i \times S^1$ is an essential map, for every $i = 1, \ldots, m$. Then there exists a map $\tilde{\Psi} : P \times D^2 \to M$ such that*

$$X_{\partial \tilde{\Psi}} = aX_\Phi \tag{8}$$

for some $a \in \mathbb{Z}$. Here, $\partial \tilde{\Psi} = \tilde{\Psi} | P \times \partial D^2$ and D^2 is a 2-disc. In particular, we have $X_\Phi = 0$.

Proof. Let $T_i = P_i \times S^1$ and let $\Phi_i = \Phi | T_i$, for $i = 1, \ldots, m$. Denote by l_i (respectively, m_i) the simple closed curve $P_i \times \{*\}$ (respectively, $\{*\} \times S^1$) on T_i.

By Lemma 3.3.2, and after a homotopy to vertical position, there exist a finite covering $\tau_i : \hat{T}_i \to T_i$, such that $\Phi_i \circ \tau_i$ extends to a map $\hat{\Phi}_i : S^1 \times Y_i \to M$. Here Y_i is a compact surface and $S^1 \times \partial Y_i = \hat{T}_i$. Moreover all the τ_i's can be taken to be of the same degree d.

Case 1. $d = 1$ so that $\hat{T}_i = T_i$. Notice that this is always the case if $H_1(M) = 0$.

Recall that the quantity X_Φ does not change under homotopy (Lemma 3.2.2.), and let H_i (respectively, Q_i) denote $S^1 \times \{*\}$ (respectively, $\{*\} \times \partial Y_i$). Suppose that $l_i = a_i H_i + b_i Q_i$, for some $a_i, b_i \in \mathbb{Z}$. We distinguish two sub-cases:

Sub-case 1. Suppose that $a_i \neq 0$ for every $i = 1, \ldots, m$. Let $q_i : \tilde{T}_i \to T_i$ be the covering of T_i corresponding to the subgroup $a_i\mathbb{Z} \oplus \mathbb{Z}$ of $\pi_1(T_i) = \mathbb{Z} \oplus \mathbb{Z}$. Let $\tilde{l}_i$, $\tilde{Q}_i$, $\tilde{H}_i$ and $\tilde{m}_i$ denote the liftings of l_i, Q_i, H_i and m_i, respectively. We have $\tilde{l}_i = \tilde{H}_i + b_i\tilde{Q}_i$.

Each map q_i extends to an $|a_i|$-fold covering,

$$\tilde{q}_i : \tilde{l}_i \times \tilde{Y}_i \to S^1 \times Y_i$$

where $\tilde{Y}_i$ is a compact surface with $\partial \tilde{F}_i = \tilde{Q}_i$, and $\tilde{l}_i \times \tilde{Q}_i = \tilde{T}_i$.

Let $\tilde{\Phi}_i = \hat{\Phi}_i \circ \tilde{q}_i$ and let

$$\tilde{\Phi} = \coprod_{i=1}^{m} \tilde{\Phi}_i$$

CLAIM. *We have that*

$$X_{\partial\tilde{\Phi}} = aX_{\Phi}$$

where $|a| = \max\{|a_1|, \ldots, |a_m|\}$.

Proof. Let q_{i*} denote the map induced by q_i on the fundamental groups. One can easily see that $q_{i*}(\tilde{m}_i) = a_i m_i$ and

$$\tilde{Q}_i = c_i \tilde{l}_i + \tilde{m}_i \tag{9}$$

for some $c_i \in \mathbb{Z}$. We identify the curves $\tilde{Q}_i$ by a common parameterization, and call the result $\tilde{Q}$. The parameterization should be such that corresponding points on the $\tilde{Q}_i$'s map, under the q_i's, to the same point on the parameter space of Φ. By (9) this induces a common parameterization of the curves $\tilde{m}_i$. Identify them and call the result $\tilde{m}$. Now, $\tilde{\Phi}$ induces a map $\tilde{l} \times \tilde{m} \to M$, where

$$\tilde{l} = \coprod_{i=1}^{m} \tilde{l}_i$$

We continue to denote this map by $\tilde{\Phi}$. Clearly, we have

$$\tilde{\Phi}(\tilde{l} \times \{x\}) = \coprod_{i=1}^{m} \Phi_i(P_i \times \{q_i(x)\})$$

for every x on $\tilde{m}$. Notice that each point on the parameter space of Φ, for which $\Phi(P)$ is not an embedding, corresponds to $|a|$ points $x \in \tilde{m}$ for which $\tilde{\Phi}(\tilde{l} \times \{x\})$ is not an embedding. Now, the claim follows easily. Let us finally observe that, because of (9), the quantity $X_{\partial\tilde{\Phi}}$ does not change if we replace the parameter space $\tilde{m}$, by $\tilde{Q}$.

To continue with the proof of the lemma, we choose a collection of proper arcs

$$\{\alpha_i^j\}_{j=1}^{m_i} \subset \tilde{Y}_i$$

such that: (a) each $\tilde{Y}_i$ if cut along the $\{\alpha_i^j\}$'s becomes a disc (as each $\tilde{Y}_i$ can be chosen to be connected), and (b) the end points of the $\{\alpha_i^j\}$'s avoid the points for which $\tilde{\Phi}(\tilde{l} \times \{*\})$ is not an embedding. Let Γ_i denote the space obtained by cutting $\tilde{l}_i \times \tilde{Y}_i$ along the collection of annuli

$$\{A_i^j\}_{j=1}^{m_i}$$

where $A_i^j = \tilde{l}_i \times \alpha_i^j$. Let us denote by $\tilde{\Psi}_i$ the map induced on Γ_i, by $\hat{\Phi}_i \circ \tilde{q}_i$. Finally, let

$$\Gamma = \coprod_{i=1}^{m} \Gamma_i$$

and let

$$\tilde{\Psi} = \coprod_{i=1}^{m} \tilde{\Psi}_i.$$

The map induced on Γ by $\tilde{\Psi}$, is the desired map.

Sub-case 2. Suppose that $a_i = 0$ for some $i = 1, \ldots, m$.

Suppose for example that $a_1 = 0$. Then $\Phi(l_1)$ does not wrap around the fibers of M. Since $B = S^2$, by Lemma 3.3.1, we may assume that $\Phi_1(P_1 \times \{x_1\})$ and $\Phi_1(P_1 \times \{x_2\})$ are isotopic for every x_1 and $x_2 \in S^1$. Let

$$P' = \coprod_{j \neq 1} P_j$$

and let

$$\Phi' = \coprod_{j \neq 1} \Phi_j.$$

Observe that $\Phi(P \times \{*\})$ is not an embedding if either $\Phi'(P' \times \{*\})$ is not an embedding, or $\Phi_1(P_1 \times \{*\})$ intersects with some $\Phi_j(P_j \times \{*\})$. We may change Φ' by composing it with the inverse of the isotopy of $\Phi_1(P_1 \times \{*\})$. Hence, X_Φ does not change if we assume that

$$\Phi_1(P_1 \times \{x\}) = \Phi_1(P_1 \times \{x_0\})$$

where x_0 is fixed and x runs on $\{*\} \times S^1$. Hence, Φ_1 extends to a map $\hat{\Phi}_1 : S^1 \times D_1 \to M$, where D_1 is a disc. Then we proceed as in Sub-case 1 above.

Case 2. Now assume that $d > 1$, where d is the common degree of the coverings $\tau_i : \hat{T}_i \to T_i$.

In this case we apply Lemma 3.3.2 to a suitable covering of T_i rather that T_i itself. More precisely suppose that T_i is in vertical position, and that $l_i = a_i H_i + b_i Q_i$, for some $a_i, b_i \in \mathbb{Z}$.

If $a_i \neq 0$ for every $i = 1, \ldots, m$, let

$$q_i : \tilde{T}_i \to T_i$$

be the covering of T_i corresponding to the subgroup $a_i\mathbb{Z} \oplus \mathbb{Z}$ of $\pi_1(T_i) = \mathbb{Z} \oplus \mathbb{Z}$. Let $\tilde{l}_i$, $\tilde{Q}_i$, $\tilde{H}_i$ and $\tilde{m}_i$ denote the liftings of l_i, Q_i, H_i and m_i, respectively. We have $\tilde{l}_i = \tilde{H}_i + b_i\tilde{Q}_i$ and we may choose $\tilde{L}_i$ and $\tilde{Q}_i$ as a system of generators of $\pi_1(\tilde{T}_i)$. Let

$$\tilde{\Phi} = \coprod_{i=1}^{m} \Phi_i \circ q_i.$$

In view of the claim inside the proof of Case 1, it is enough to prove the assertion in the statement of the lemma for $\tilde{\Phi}$. Now let $\tilde{\tau}_i : \tilde{T}_i^* \to \tilde{T}_i$ be coverings as in the proof of Lemma 3.3.2, and let d be their common degree. Let $\Phi_i^* = \Phi_i \circ q_i \circ \tilde{\tau}_i$ and let

$$\Phi^* = \coprod_{i=1}^{m} \Phi_i^*.$$

One can see that

$$X_{\Phi^*} = dX_{\tilde{\Phi}}$$

and proceed as in Case 1 above to prove the desired assertion for Φ^*.

If $a_i = 0$ for some $i = 1, \ldots, m$, then we proceed as in Sub-case 2 above. □

LEMMA 3.3.4. *Assume that M is an oriented rational homology* 3-*sphere with* $\pi_2(M) = \{1\}$. *Let L be a link that does not have any homotopically trivial components. Let Φ be a closed homotopy from L to itself, such that*

$$\Phi \mid P_i \times S^1$$

is an inessential map, for every $i = 1, \ldots, m$. Then $X_\Phi = 0$.

Proof. As before, we denote $T_i = P_i \times S^1$, $\Phi_i = \Phi \mid T_i$, $l_i = P_i \times \{*\}$, and $m_i = \{*\} \times S^1$ for $i = 1, \ldots, m$.

We have assumed that $\pi_1(M)$ is infinite. So it is torsion free (Theorem 9.8 of [5]). Hence, Φ_i extends to a map $\hat{\Phi}_i : S^1 \times D_i \to M$, where D_i is a 2-disc and $S^1 \times \partial D_i = T_i$. Let H_i (respectively, Q_i) denote $S^1 \times \{*\}$ (respectively, $\{*\} \times \partial D_i$). Suppose that $l_i = a_i H_i + b_i Q_i$, for some $a_i, b_i \in \mathbb{Z}$.

By our assumption we have that $a_i \neq 0$ for every $i = 1, \ldots, m$, We proceed as in Case 1 of the proof of Lemma 3.3.3 to obtain a map $\tilde{\Psi} : P \times D^2 \to M$ such that

$$X_{\partial \tilde{\Psi}} = a X_\Phi \tag{10}$$

for some $a \in \mathbb{Z}$. The desired conclusion then follows immediately. □

3.4. Completing the proof of Theorem 3.1.2

Recall that M is a rational homology 3-sphere which is either atoroidal or a Seifert fibered space as described in the statement of Theorem 3.1.2. Also, $\pi_1(M)$ is infinite. As before, P is a disjoint union of oriented circles and

$$\Phi : P \times S^1 \to M$$

is a closed homotopy from some link $L : P \to M$ to itself. Let $f : \mathscr{L}^1 \to \mathscr{R}$ be a singular link invariant as in the statement of Theorem 3.1.2. We have to show that

$$X_\Phi = 0 \tag{11}$$

where X_Φ is the signed sum of values of f around Φ defined in (6).

First suppose that M is a Seifert fibered space. Let E (resp. I) denote the set of components of $P \times S^1$ on which Φ is essential (resp. inessential).

(a) Suppose that the link to begin with contains no homotopically trivial components. If E or I is empty the claim follows from Lemma 3.3.3 or 3.3.4. In general we have the following:

Claim. *There exists a map $\tilde{\Psi} : P \times D^2 \to M$ such that*

$$X_{\partial \tilde{\Psi}} = a X_\Phi$$

for some $a \in \mathbb{Z}$. In particular, we have $X_\Phi = 0$.

Proof. The proof is very similar to the proofs of Lemmas 3.3.3 and 3.3.4. The only difference is that, for example, when cutting each surface Y_i to discs (see the proof of Lemma 3.3.3), we have to make sure that the end points of the cutting arcs also avoid singular links where the double points are other than intersections between components of L in E. The details are left to the reader.

(b) Suppose that the link L to begin with, contains homotopically trivial components. Let Φ' denote the restriction of Φ, on the tori corresponding to the non-trivial components of L. By (a) we have that $X_{\Phi'} = 0$. Thus, by Lemma 3.2.5 we obtain $X_\Phi = 0$.

If M is atoroidal, the conclusion follows from Lemmas 3.3.4 and 3.2.5.

This finishes the proof of Theorem 3.1.2. □

The reader might have noticed that the only place that the orientability of the orbit space B is needed is in Lemmas 3.3.1 and 3.3.3 to avoid closed homotopies $\Phi : (\coprod S^1) \times S^1 \to M$ such that (i) the restriction of Φ on some component $T = S^1 \times S^1$ does not wrap around the fibers of M; and (ii) the curve $\alpha \subset B$ contains some orientation reversing sub-loops. We will call such a homotopy *inadmissible*.

Since $\alpha = p(\Phi(S^1 \times \{*\})) \subset B$ is an orientation preserving closed curve whose only singularities are transverse double points, the only way for (ii) above to happen is if $\alpha = p(\Phi(S^1 \times \{*\}))$ contains double points whose two "lobes" are orientation reversing loops on B. Let M be a Seifert fibered space with non-orientable base space, and let $\Phi : T = S^1 \times S^1 \to M$ be an inadmissible closed homotopy of the knot $\Phi | S^1 \times \{*\}$. Then the homotopy $\Phi : T = S^1 \times S^1 \to M$ may not be a self- isotopy of the knot $\Phi | S^1 \times \{*\}$. That is $\Phi | S^1 \times \{t\}$ may be a singular knot for some values of the parameter t. See Remark 3.13 of [11]. Moreover, in some cases, the local integrability conditions may not imply that $X_\Phi = 0$. For examples see [13].

THEOREM 3.4.1. *Suppose that M is a non-special rational homology sphere fibering over $\mathbb{R}P^2$ and let $f : \mathscr{L}^1 \to \mathscr{R}$ be a singular link invariant. There exists a link invariant $F : \mathscr{L} \to \mathscr{R}$ so that* (2) *holds for all $L_\times \in \mathscr{L}^1$ if and only if f satisfies*

$$f(\infty) = 0 \tag{12}$$

$$f(L_{\times +}) - f(L_{\times -}) = f(L_{+\times}) - f(L_{-\times}) \tag{13}$$

$$X_{\Phi_{\text{inadm}}} = 0 \tag{14}$$

for all inadmissible *closed homotopies* Φ_{inadm}.

4. AN INTRINSIC DEFINITION OF THE HOMFLY POWER SERIES

4.1. Preliminaries

It is known [10] that the 2-variable Jones (or HOMFLY) polynomial [4, 9] for links in $\mathbb{R}^3$ or S^3 is equivalent to sequence $\{J_n = J_n(t)\}_{n \in \mathbb{Z}}$ of 1-variable Laurent polynomials. They are completely determined by the following skein relations:

$$J_n(U) = 1 \tag{15}$$

$$t^{(n+1)/2} J_n(L_+) - t^{-(n+1)/2} J_n(L_-) = (t^{1/2} - t^{-1/2}) J_n(L_0) \tag{16}$$

where L_+, L_-, L_0 are the resolutions of a singular link $L_\times \in \mathscr{L}^{(1)}$ described in 3.1. In this context, the original Jones polynomial is J_{-3}.

Notice that the initial value $J_n(U) = 1$ is not essential. Any choice of the initial value together with (16) will determine a unique J_n. Let

$$u_n(t) = \frac{t^{(n+1)/2} - t^{-(n+1)/2}}{t^{1/2} - t^{-1/2}}$$

By (16) one obtains

$$J_n(L \amalg U) = u_n(t) J_n(L) \tag{17}$$

where the link $L \amalg U$ is obtained from L by adding an unknotted and unlinked component U.

The coefficients of the power series $J_n(x)$, obtained from $J_n(t)$ by substituting $t = e^x$, are invariants of finite type [2, 3]. In the theorem below we reverse this procedure, and guided by (16) we will construct inductively power series invariants for links in 3-manifolds generalizing the $J_n(x)$'s.

4.2. The construction of the invariants

Assume that M is an orientable, rational homology 3-sphere which is either atoroidal or Seifert fibered space as in the statement of Theorem 3.1.2. For every $n \in \mathbb{Z}$, we will construct a sequence of knot invariants

$$v_n^0,\ v_n^1,\ \dots, v_n^m,\ \dots$$

such that the formal power series

$$J_{\{M,n\}}(L) = \sum_{m=0}^{\infty} v_n^m(L)x^m$$

satisfies (16), under the change of variable $t = e^x$, for every $L \in \mathscr{L}$.

We will construct our invariants inductively (induction on m) by using Theorem 3.1.2. More precisely, each v_n^m is going to be obtained by integrating a suitable singular link invariant determined by the v_n^j's with $j < m$.

Recall that a link invariant obtained by integrating a singular link invariant is well defined up to a collection of "integral constants" (see the beginning of the proof of Theorem 3.1.2). This means that in order to define $v_n^0, v_n^1, \dots, v_n^m, \dots$ uniquely, we need to make a choice of "initial links".

Let L be an S-component link and recall from Section 3 that $\mathscr{M}^L$ denotes the space of maps $\coprod S^1 \to M$ which are homotopic L. The spaces $\mathscr{M}^L$ corresponding to links with S components are in one to one correspondence with the unordered S-tuples of conjugacy classes in $\pi = \pi_1(M)$. In every such space we will fix, once and for all, a link CL and call it a *trivial link*. If CL has k components which are homotopically trivial, our choice will be such that $CL = CL^* \coprod U^k$, where U^k is the standard unlink with k components in a small ball neighborhood disjoint from CL^*. Let $\mathscr{CL}$ be the set of trivial links and $\mathscr{CL}^*$ be the set of trivial links with homotopically non-trivial components.

Notice that when M is simply connected there is a natural choice of trivial links. Namely, one chooses each CL to be the unlink U^k. Thus $\mathscr{CL}^* = \emptyset$ in this case.

THEOREM 4.2.1. *Let M, $\mathscr{L}$ and $\mathscr{CL}^*$ be as above. There exists a unique sequence of complex valued link invariants $v_n^0, v_n^1, \dots, v_n^m, \dots$, with given values on the links in $\mathscr{CL}^* \cup \{U\}$, such that if we define a formal power series*

$$J_{\{M,n\}}(L) = \sum_{m=0}^{\infty} v_n^m(L)x^m$$

for $L \in \mathscr{L}$ then

$$t^{(n+1)/2}J_{\{M,n\}}(L_+) - t^{-(n+1)/2}J_{\{M,n\}}(L_-) = (t^{1/2} - t^{-1/2})J_{\{M,n\}}(L_0) \tag{18}$$

where $t = e^x = 1 + x + x^2/2 + \cdots$

Notation. To simplify our notation, and throughout this proof, we will write J_n instead of $J_{\{M,n\}}$.

Proof. By our assumption, the values $v_n^0(CL^*), v_n^1(CL^*), \dots, v_n^m(CL^*), \dots$ are given for every $CL^* \in \mathscr{CL}^*$. Hence, we can form the power series $J_n(CL^*)$. Also, we may form $J_n(U)$ using the given values $v_n^m(U)$'s.

Guided by (17) we define

$$J_n(CL \coprod U) = u_n(t) J_n(CL) \tag{19}$$

where

$$u_n(t) = \frac{t^{(n+1)/2} - t^{-(n+1)/2}}{t^{1/2} - t^{-1/2}} \tag{20}$$

Thus, J_n has been defined on all trivial links. We define the link invariant v_n^0 by

$$v_n^0(L) = v_n^0(CL),$$

where CL is the trivial link homotopic to L. Inductively, suppose that the invariants $v_n^0, v_n^1, \ldots, v_n^{m-1}$ have been defined such that if we let

$$J_n^{(m-1)}(L) = \sum_{i=1}^{m-1} v_n^i(L) x^i,$$

then

$$J_n(L \coprod U) = u_n(t) J_n(L) \bmod x^m, \tag{21}$$

$$t^{(n+1)/2} J_n^{(m-1)}(L_+) - t^{-(n+1)/2} J_n^{(m-1)}(L_-) = (t^{1/2} - t^{-1/2}) J_n^{(m-1)}(L_0) \bmod x^m. \tag{22}$$

From (22) we obtain

$$\begin{aligned} J_n^{(m-1)}(L_+) - J_n^{(m-1)}(L_-) &= (t^{-(n+1)} - 1) J_n^{(m-1)}(L_-) \\ &\quad + t^{-(n+1)/2}(t^{1/2} - t^{-1/2}) J_n^{(m-1)}(L_0) \bmod x^m \end{aligned}$$

which leads us to define

$$J_n^{(m)}(L_\times) := (t^{-(n+1)} - 1) J_n^{(m-1)}(L_-) + t^{-(n+1)/2}(t^{1/2} - t^{-1/2}) J_n^{(m-1)}(L_0) \bmod x^{m+1}. \tag{23}$$

This is a polynomial of degree m, with trivial constant coefficient. The coefficients of x^i, $i = 1, 2, \ldots, m-1$, in this polynomial are singular link invariants derived from v_n^i, $i = 1, 2, \ldots, m-1$. However, the coefficient of x^m is a *new* singular link invariant. We are going to prove that it is derived from a knot invariant by using Theorem 3.1.2. For that we need to check that the local integrability conditions (3) and (4) are satisfied. It is enough to check them modulo x^{m+1}. In what follows the symbol $\equiv$ will denote calculation modulo x^{m+1}.

To check (3), suppose we start with a singular link $L^1 \in \mathscr{L}^1$ that contains a kink $= \infty$. Let L be the link obtained by resolving the double point of L^1 and let U be unknot in M. Then we have

$$\begin{aligned} J_n^{(m)}(L^1) &\equiv (t^{-(n+1)} - 1) J_n^{(m-1)}(L) + t^{-(n+1)/2}(t^{1/2} - t^{-1/2}) J_n^{(m-1)}(L \coprod U) \\ &\equiv [(t^{-(n+1)} - 1) + t^{-(n+1)/2}(t^{1/2} - t^{-1/2}) u_n(t)] J_n^{(m-1)}(L) \\ &\equiv 0. \end{aligned}$$

To check (4), we calculate

$$\begin{aligned} &J_n^{(m)}(L_{\times +}) - J_n^{(m)}(L_{\times -}) \\ &\equiv (t^{-(n+1)} - 1) J_n^{(m-1)}(L_{-+}) + t^{-(n+1)/2}(t^{1/2} - t^{-1/2}) J_n^{(m-1)}(L_{0+}) \\ &\quad - (t^{-(n+1)} - 1) J_n^{(m-1)}(L_{--}) - t^{-(n+1)/2}(t^{1/2} - t^{-1/2}) J_n^{(m-1)}(L_{0-}) \end{aligned}$$

$$\equiv (t^{-(n+1)} - 1)[(t^{-(n+1)} - 1)J_n^{(m-1)}(L_{--})$$
$$+ t^{-(n+1)/2}(t^{1/2} - t^{-1/2})J_n^{(m-1)}(L_{-0}) + o(x^m)]$$
$$+ t^{-(n+1)/2}(t^{1/2} - t^{-1/2})[(t^{-(n+1)} - 1)J_n^{(m-1)}(L_{0-})$$
$$+ t^{-(n+1)/2}(t^{1/2} - t^{-1/2})J_n^{(m-1)}(L_{00}) + o(x^m)]$$
$$\equiv (t^{-(n+1)} - 1)^2 J_n^{(m-1)}(L_{--}) + [t^{-(n+1)/2}(t^{1/2} - t^{-1/2})]^2 J_n^{(m-1)}(L_{00})$$
$$+ (t^{-(n+1)} - 1)t^{-(n+1)/2}(t^{1/2} - t^{-1/2})[J_n^{(m-1)}(L_{0-}) + J_n^{(m-1)}(L_{-0})].$$

Since the result is symmetric with respect to the two double points we deduce that

$$J_n^{(m)}(L_{\times +}) - J_n^{(m)}(L_{\times -}) \equiv J_n^{(m)}\ (L_{+\times}) - J_n^{(m)}(L_{-\times})$$

Thus, the singular link invariant defined in (23) is induced by a link invariant. Using the given values $\{v_n^m(CL)\colon CL \in \mathscr{CL}\}$, we can define a link invariant v_n^m, such that if we let

$$J_n^{(m)}(L) = \sum_{i=1}^{m} v_n^m(L)x^i$$

we have

$$J_n^{(m)}(L_+) - J_n^{(m)}(L_-) = J_n^{(m)}(L_\times)$$

for $L \in \mathscr{L}$ and $L_\times \in \mathscr{L}^1$. Therefore the invariant $J_n^{(m)}$ defined in this way satisfies the inductive hypothesis (22).

Now, a straightforward calculation shows that

$$J_n^{(m)}(L_\times \coprod U) \equiv u_n(t)\ J_n^{(m)}\ (L_\times)$$

which together with (19) shows that $J_n^{(m)}$ satisfies the inductive hypothesis (21).

To finish our proof we need to show uniqueness. Inductively, we assume that $v_n^0, v_n^1, \ldots, v_n^{m-1}$ are uniquely determined by (18) and their values on $\mathscr{CL}$, for every $n \in \mathbb{Z}$. Then, the conclusion for v_n^m follows from the fact that

$$v_n^m(L) = v_n^m(CL) + \sum_{i=1}^{r} \pm v_n^m(L_i)$$

where $L_1, \ldots, L_s$ are singular links in $\mathscr{L}^1$, and CL is the representative of L in $\mathscr{CL}$. □

To illustrate how the power series $J_n = J_{\{M,n\}}$ depend on the choice of $\mathscr{CL}$, let us restrict ourselves to the case of knots. Let $\mathscr{C}$ denote the set of homotopy classes of loops in M. Then $\mathscr{CK}$ will contain exactly one knot, say K_C from every $C \in \mathscr{C}$. Moreover, we suppose that the initial values of J_n are all equal to 1. To stress the dependence on K_C, let

$$J_n^{K_C}(K) := J_n(K)$$

for every $K \in \mathscr{M}^{K_c}(S^1, M)$. Let $\{\tilde{K}_C\}_{C\in\mathscr{C}}$ be a different choice of trivial knots. Then one can see that $J_n(K)$ is well defined up to a multiplicative constant in the ring of formal power series with complex coefficients. More precisely,

PROPOSITION 4.2.2. *We have*

$$J_n^{\tilde{K}_C}(K) = J_n^{\tilde{K}_C}(K_c)J_n^{K_C}(K)$$

for every $K \in \mathscr{M}^{K_c}(S^1, M)$ *and* $c \in \mathscr{C}$.

Proof. Follows immediately from the definition. □

Remark 4.2.3. Recall that an invariant f, is called of finite type if there exists an integer m, such that the singular link invariant derived from f is zero on all singular links with more than m double points. One can see that the invariants $v_n^0, v_n^1, \ldots v_n^m, \ldots$ constructed above are of finite type. It would be interesting to find a direct relation of the invariants constructed here with these coming from the $SU(N)$-perturbative Chern–Simons theory [1, 20].

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