NOTES ON KNOT POLYNOMIALS AND HYPERBOLIC VOLUME: ALEXANDER VS JONES POLYNOMIAL

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Abstract. It is known that the volume of an alternating hyperbolic knot is bounded above by a linear function of the L^1-norm of the coefficients of the Alexander polynomial and the L^1-norm of the coefficients of the Jones polynomial. The Alexander polynomial, however, doesn't control the volume of hyperbolic non-alternating knots. For example, there are hyperbolic knots of arbitrarily high volume but trivial Alexander polynomial. On the other hand, experimental evidence suggests that the Jones polynomial should do better than the Alexander polynomial in controlling the volume of arbitrary hyperbolic knots. The data compiled in [CKP] suggests that knots with large volume should have large L^1-norm of the coefficients of the Jones polynomial. In this paper, we provide further evidence is this direction by providing constructions of families of non-alternating hyperbolic knots with arbitrarily large volume that is bounded above by a linear function the L^1-norm of the coefficients of the Jones polynomial. In fact, all the knots in some of these families can be taken to have fixed Alexander polynomial.

1. Introduction

Let $c(K)$ denote the crossing number and let $\Delta_K(t) := \sum_{i=0}^{k} c_i t^i$ and $J_K(t) := \sum_{j=0}^{m} a_j t^j$ denote the Alexander and the Jones polynomial of a knot K, respectively. If K is hyperbolic, let $\mathrm{vol}(S^3 \setminus K)$ denote the volume of its complement. The determinant of K is the quantity

[1]Supported in part by NSF grants DMS-0306995 and NSF/FRG Grant, DMS-0456155.

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$\det(K) := |\Delta_K(-1)| = |J_K(-1)|$. Thus, in general, we have

$$\det(K) \leq ||\Delta_K(t)|| := \sum_{i=0}^{k} |c_i|, \tag{1}$$

and

$$\det(K) \leq ||J_K(t)|| := \sum_{i=0}^{k} |a_i|. \tag{2}$$

It is known ([CR]) that there exists a universal constant $C > 0$ such that if K is a hyperbolic knot with crossing number $c(K)$, then

$$\mathrm{vol}(S^3 \setminus K) \leq Cc(K). \tag{3}$$

If in addition K is alternating then one also has ([BZ])

$$c(K) \leq \det(K) \quad \text{and} \quad \det(K) = ||\Delta_K(t)||. \tag{4}$$

By a result of Menasco ([Me]), all prime, alternating, non-torus knots are hyperbolic. Combining this with (1)-(4), we derive that there is a universal constant $C > 0$ such that

$$\mathrm{vol}(S^3 \setminus K) \leq C\det(K) \quad \text{and} \quad \mathrm{vol}(S^3 \setminus K) \leq C||\Delta_K(t)||, \tag{5}$$

and

$$\mathrm{vol}(S^3 \setminus K) \leq C||J_K(t)||, \tag{6}$$

for every prime, alternating non-torus knot K. [In fact, A. Stoimenow ([St]) showed that the quantity $\det(K)$ in (5) can be replaced by $\log(\det(K))$.] Thus, the Alexander polynomial (and the Jones polynomial) of prime alternating knots dominates the volume of the knot complement. In fact, using work of Lackenby ([La]), Dasbach and Lin showed ([DL], [DL1]) that the volume of hyperbolic alternating knots is bounded above and below by linear functions of coefficients of the Jones (or the colored Jones) polynomial. It is known, however, that (5) doesn't hold for non-alternating knots. For example, in [Kas] we give a way to construct families of hyperbolic knots with trivial Alexander module but arbitrarily large volume and in [SW], Silver and Whitten show that every Alexander polynomial can be realized by hyperbolic knots of arbitrarily large volume. Thus, in general, the Alexander polynomial is far from controlling the volume of hyperbolic knots. The research in this paper is motivated by the question of the extend to which the Jones polynomial controls the volume of non-alternating hyperbolic knots. More specifically, we ask the following:

Question 1.1. *Is there a family of hyperbolic knots with arbitrarily large volume and bounded $||J_K(t)||$?*

Experimental data indicates that the Jones polynomial does a better job in controlling the volume of hyperbolic non-alternating knots. Perhaps the most striking data is this documented by Champanerkar, Kofman and Patterson in [CKP] where the census of the "geometrically simple" knots is completed and the tables of their volumes and their Jones polynomials are presented. The data of [CKP] shows that (6) above holds for all of the geometrically simple knots; it should be noted some of these knots have quite very large crossing numbers. In this note, we construct infinite families of non-alternating hyperbolic knots for which (6) is satisfied. For some of these families, the Alexander polynomial can be taken to remain fixed.

Theorem 1.2. *There exists a family of hyperbolic knots $\{K_n\}_{n\in\mathbf{N}}$ with fixed Alexander polynomial and such that*

$$20v_3||J_K(t)|| \geq \mathrm{vol}(S^3 \setminus K) > n,$$

where $v_3(\approx 1.01494)$ is the volume of regular hyperbolic ideal tetrahedron.

In general, it is hard to obtain closed formulae for the Jones polynomial of families of knots. In this article we examine knots that are obtained by summing up "alternating tangles". This is a broad class of knots that includes alternating knots and many arborescent knots. Such knots are known to be torus knots or hyperbolic. Using work of Stoimenow [St]) and Dasbach-Lin [DL]) one can estimate $||J_K(t)||$. Thus for example, restricting to the class of *semi-alternating* knots defined by Lickorish and Thistlethwaite [LiT]) we obtain the following:

Theorem 1.3. *There exists a constant $C > 0$ such that we have*

$$10v_3||J_K(t)|| \geq \mathrm{vol}(S^3 \setminus K),$$

for every hyperbolic semi-alternating knot.

2. Volume and Jones polynomial of adequate knots

2.1. **Jones polynomial of adequate knots.** Given a knot diagram D let s_+D (resp. s_-D) denote the diagram obtained by applying the 0-resolution (resp. 1-resolution) to all the crossings of D (see Figure 1). Let $|s_+D|$ (resp. $|s_-D|$) denote the number of components of s_+D (resp. s_-D). The following is shown in [Li].

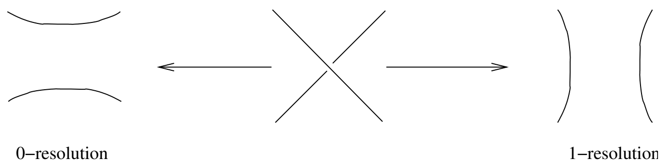

Figure 1

Lemma 2.1. *If D is a knot diagram with n crossings we have $|s_+D| + |s_-D| \leq n+2$ and equality holds if and only D is alternating.*

Definition 2.2. *A knot diagram D is called $+$-adequate (resp. $-$-adequate) if for each crossing in D the two arcs of 0-resolution (resp. 1-resolution) belong on different components in s_+D (resp. s_-D). The diagram D is called adequate if it is $+$-adequate and $-$-adequate. A knot is called adequate if it admits an adequate diagram.*

Let D be an $+$-adequate (resp. $-$-adequate) knot diagram. Following the exposition in [DL1] define the graph G_0 (resp. G_1) as follows: The vertices of G_0 (resp. G_1) are in one-to-one correspondence with the components of s_+D (resp. s_-D) and for every crossing in D we add an edge in G_0 (resp. G_1) connecting the two vertices that correspond to the components of s_+D (resp. s_-D) containing the arcs of the 0-resolution (resp. 1-resolution) of the crossing. Let G'_0 (resp. G'_1) the graph obtained from G_0 (resp. G_1) by removing multiple edges connected to the same pair of vertices. Let v'_0 , e'_0 (res. v'_1, e'_1) denote the number of vertices and edges of G'_0 (resp. G'_1.)

Proposition 2.3. *For a knot diagram D let*

$$< D >= b_m A^m + b_{m-1} A^{m-4} + \ldots + b_{k-1} A^{k+4} + b_k A^k,$$

denote the Kaufman bracket of D so that m (resp. k) are the highest (resp. lowest) power in A. If D is $+$-adequate (resp. $-$-adequate) then

(a) $|b_m| = 1$ *(resp.* $|b_k| = 1$*)*

(b) $|b_{m-1}| = e'_0 - v'_0 + 1$ *(resp.* $|b_{k-1}| = e'_1 - v'_1 + 1$.

Thus, in particular, if D is adequate then we have $|b_m| = |b_k| = 1$ *and* $|b_{m-1}| + |b_{k-1}| = e'_0 + e'_1 - (v'_0 + v'_1) + 2$.

Proof: Part (a) can be found in [LiT]. A reduced alternating diagram is known to be adequate. For such diagrams part (b) is shown in [DL]. The general case is due to Stoimenow; the detailed calculation can be found in [DL1]. □

2.2. **Sums of alternating tangles.** Following [La] a bigon in a knot diagram is a complementary region to the knot projection having exactly two crossings in its boundary. A twist is either a connected collection of bigon regions arranged in a row, that is maximal in the sense that is not part of larger row of bigon regions, or a singe crossing that is not adjacent to any bigon regions. For a knot projection D let $t(D)$ denote the number of twists of D; this is the twist number of D. Clearly, $t(D) \leq c(D)$.

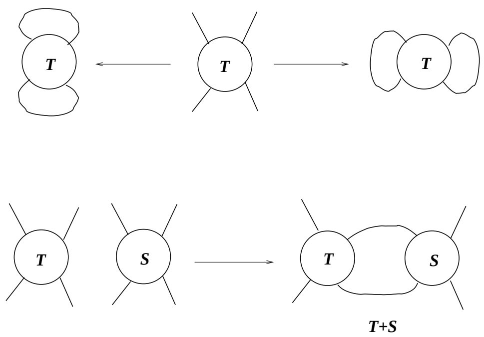

FIGURE 2. The to picture shows the two types of closures for a tangle T: The left (resp. right) hand side closure is $n(T)$ (resp. $d(T)$). The bottom picture shows the operation of summing two tangles.

Next we turn our attention to the class of adequate knots obtained by summing up certain 4-end tangles. For a 4-end tangle T let $n(T)$ and $d(T)$ denote the numerator and denominator closure of T respectively.

Given two 4-end tangles T, S let $T+S$ denote the 4-end tangle obtained by summing T and S; see Figure 2.

Definition 2.4. *A 4-end tangle T will be called alternating (resp. adequate) iff both of the link diagrams $n(T)$ and $d(T)$ are reduced alternating (resp. adequate). A link diagram D is called semi-alternating if $D := n(T)$, where T is a tangle obtained by summing up alternating tangles $T := (\ldots((T_1 + T_2) + T_3) + \ldots) + T_n$.*

Proposition 2.5. *Suppose that D is a semi-alternating link diagram. Let $t(D)$ and $c(D)$ denote the twist and the crossing number of D, respectively, and let denote the Kaufman bracket of D. Then $< D >$ is a Laurent polynomial in A^4 and we have*

$$< D >= b_m A^m + b_{m-1} A^{m-4} + \ldots + b_{k-1} A^{k+4} + b_k A^k,$$

where m (resp. k) are the highest (resp. lowest) power in A and we have

(a) $|b_m| = |b_k| = 1$;

(b) $|b_{m-1}| + |b_{k-1}| = t(D) + c(D) - (|s_+D| + |s_-D|) + 2.$

Proof: By Proposition 4 of [LiT], D is an adequate diagram; thus Proposition 2.3 applies. In particular, part (a) is immediate from Proposition 2.3. For part(b) we have $|b_{m-1}|+|b_{k-1}| = e'_0+e'_1-(v'_0+v'_1)+2$. By definition of G'_0 (resp. G'_1) we have $v'_0 = |s_+D|$ and $v'_1 = |s_-D|$; thus $v'_0 + v'_1 = |s_+D| + |s_-D|$. Now the conclusion will follow from the following:

*Claim:*We have $e'_0 + e'_1 = t(D) + c(D)$.

Proof of Claim: Suppose that D is the closure of just one alternating tangle. Then, by definition, D is a reduced alternating diagram. By the main result of [DL], we have $|b_{m-1}| + |b_{k-1}| = t(D)$ and by Lemma 2.1 we have $v'_0 + v'_1 = |s_+D| + |s_-D| = c(D) + 2$. Thus we have $e'_0 + e'_1 = |b_{m-1}| + |b_{k-1}| + (v'_0 + v'_1) - 2 = t(D) + c(D)$. Now the general case follows from the fact that both the twist and crossing number are additive under tangle addition and by Definition 2.4. □

2.3. **Volume upper bounds.** We need to recall the following:

Theorem 2.6. [Agol-Lackenby-Thurston, [La]] *Let D be a diagram of a hyperbolic knot K with twist number $t(D)$. Then,*

$$\mathrm{vol}(S^3 \setminus K) \leq 10v_3(t(D) - 1),$$

where $v_3(\approx 1.01494)$ is the volume of regular hyperbolic ideal tetrahedron.

Now we have:

Theorem 2.7. *Let K be a hyperbolic knot that is the closure of a sum of alternating tangles. Then we have*

$$\mathrm{vol}(S^3 \setminus K) \leq 10v_3||J_K(t)||,$$

where $v_3(\approx 1.01494)$ is the volume of regular hyperbolic ideal tetrahedron.

Proof: Let D be an adequate diagram for K with twist number $t(D)$ and crossing number $c(D)$. By Theorem 2.6

$$\mathrm{vol}(S^3 \setminus K) \leq 10v_3(t(D) - 1). \tag{7}$$

By the well known relation of the Jones polynomial $J_K(t)$ to the Kaufman bracket $< D >$ we have

$$||J_K(t)|| = || < D > ||, \tag{8}$$

where $|| < D > ||$ denotes the sum of the absolute values of the coefficients of $< D >$. By Proposition 2.5

$$|b_{m-1}| + |b_{k-1}| + |b_m| + |b_k| = t(D) + c(D) - (|s_+D| + |s_-D|) + 4. \tag{9}$$

By Lemma 2.1

$$|s_+D| + |s_-D| \leq c(D) + 2 = . \tag{10}$$

Now (9), (10) implies $||J_K(t)|| = || < D > || \geq t(D) + 2$ which in turn combined with (6) gives the desired result. $\square$

Example 2.8. The 11-crossing Kinoshita-Terasaka knot, which is the first example on the knot table that has trivial Alexander polynomial, is the closure of sum of two alternating knots. We have $J_K(t) = -t^4 + 2t^3 - 2t^2 + 2t + t^{-2} - 2t^{-3} + 2t^{-4} - 2t^{-5} + t^{-6}$, and $\mathrm{vol}(S^3 \setminus K) =$

11.21911773. Thus, $|b_{m-1}|+|b_{k-1}|=4$ and $\mathrm{vol}(S^3\setminus K)\leq 10v_3(|b_{m-1}|+|b_{k-1}|-1)$.

As a Corollary of Proposition 2.3 we have the following.

Corollary 2.9. *With the setting and notation of Proposition 2.5 we have*

$$\min\{|b_{m-1}|,|b_{k-1}|\}-1\leq t(D)\leq |b_{m-1}|+|b_{k-1}|.$$

Remark 2.10. Menasco's theorem characterizing hyperbolic alternating links has been generalized by Futer and Purcell in [FP]: They show that knots that admit diagrams with twist number at least two and such that in each twist region there are at least six crossings are hyperbolic. This criterion provides a systematic ways to construct large families of hyperbolic links. If in addition, the links obtained are adequate and satisfy the condition $e_0'+e_1'=t(D)+c(D)$ then they will also satify the conclusion of Theorem 2.7 and thus (6).

Theorem 2.11. *Let D be a diagram of a knot K such that D is the closure of alternating tangles. Let $<D>=b_mA^m+b_{m-1}A^{m-4}+\ldots+b_{k-1}A^{k+4}+b_kA^k$ denote the Kaufmann bracket of D. Suppose that $t(D)>2$ and that each twist region of D has at least six crossings. Then,*

$$\mathrm{vol}(S^3\setminus K)\leq 10v_3(|b_{m-1}|+|b_{k-1}|-1),$$

where $v_3(\approx 1.01494)$ is the volume of regular hyperbolic ideal tetrahedron.

2.4. **Lower bounds for complicated knots.** For alternating knots, Lackenby shows that the volume also is bounded from below in terms of the twist number. This, in turn, is used by Dasbach and Lin to obtain to obtain obtain upper and lower bounds of the volume of an alternating knot in terms of certain coefficients of the Jones polynomial. A generalization of Lackenby's result is obtained by Purcell ([P]) who shows that the twist number provides a lower bound for the volume provided that each twist region is sufficiently complicated:

Theorem 2.12. [Purcell, [P]] *Let D diagram of a knot K with twist number $t(D)$. Suppose that D is twist-reduced, we have $t(D) > 2$ and each twist region contains at least 113 crossings. Then*

$$\mathrm{vol}(S^3 \setminus K) \geq ct(D),$$

where $c = 2.83565$.

Now combining this result with Corollary 2.9 we have the following result which is can be viewed as a generalization of the Volumish theorem of [DL] for adequate knots.

Corollary 2.13. *Let K be a semialternating knot with Jones polynomial $J_K(t) = a_m t^m + a_{m-1} t^{m-1} + \ldots + a_{k-1} t^{k+1} + a_k t^k$, where m and k are the highest and lowest degree terms, respectively. Suppose that K is the closure of a sum of alternating tangles such each of which contains at least 133 crossings in each twist region. Then we have*

$$c(\min\{|a_{m-1}|, |a_{k-1}|\} - 1) \leq \mathrm{vol}(S^3 \setminus K) \leq 10v_3(|a_{m-1}| + |a_{k-1}| - 1),$$

where $c := 2.83565$ and $v_3(\approx 1.01494)$ is the volume of regular hyperbolic ideal tetrahedron.

Proof: The left hand sided inequality follows immediately from Theorem 2.12 and Corollary 2.9. The right hand side inequality follows from Corollary 2.9 and Theorem 2.6. □

3. Volume and Jones vs Alexander polynomial

In this section we construct families of non-alternating knots with arbitrary large volume that is however linearly bounded from above in terms of the Jones polynomial. We have:

3.1. **Pretzel knots.** Consider the pretzel link $K(a_1, \ldots, a_s, b_1, \ldots, b_r)$ shown below, where a_i, b_i are positive integers denoting half twists in the direction shown below. It is known that after mutation every pretzel link can be put in this form. Note that there are many choices of for the a_i, b_i's for which we obtain families of knots; for example $K(a_1, \ldots, a_s, b_1, \ldots, b_r)$ is a knot if $b_i, a_i > 0$ are odd integers and $s + r$ is even.

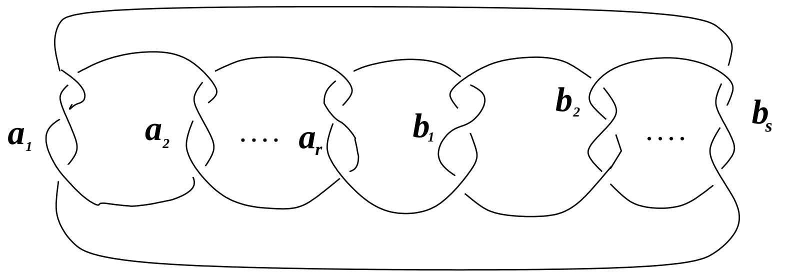

FIGURE 3. The pretzel knot $K(a_1, \ldots, a_s, b_1, \ldots, b_r)$.

In [LiT] it is shown that for $a_i, b_i > 2$ and $r, s > 2$, thee diagram of $K(a_1, \ldots, a_s, b_1, \ldots, b_r)$ shown in Figure 2 is adequate. On the other hand, by Bonahon-Siebenmann ([BoS]) these pretzel knots are hyperbolic.(The result also follows from [F]). Combining these results with Theorem 2.7 above we obtain the following.

Corollary 3.1. *Suppose $r, s > 2$, with $r + s$ even, and a_i, b_i are odd integers for every $1 \leq i \leq s$ and $1 \leq j \leq r$. Then, $K := K(a_1, \ldots, a_s, b_1, \ldots, b_r)$ is a hyperbolic knot such that $10v_3||J_K(t)|| \geq \mathrm{vol}(S^3 \setminus K)$, where $v_3(\approx 1.01494)$ is the volume of regular hyperbolic ideal tetrahedron.*

Before we proceed to produce families of hyperbolic knots with arbitrarily large volume for which the inequality (6) is satisfied we need some preparation. A crossing disc for a knot K is a disc d that intersects K only in its interior exactly twice with zero algebraic intersection number; the boundary $C := \partial d$ is called a crossing circle. Given $D(K)$ a knot diagram with t twist regions, consider the link $L := K \cup C_1, \ldots, C_t$ where C_i is a crossing circle for the i-th twist region. Now let J denote the link obtained from L by removing all the full twists from each twist region of K. Thus, if a twist region on K contains a_i half twists in J we have no twists or a single half twist corresponding to that twist region, according to whether a_i is even or odd. Now let $M := S^3 \setminus \eta(J)$ where $\eta(J)$ denotes a tubular neighborhood of L. It is clear that the knot exterior $S^3 \setminus \eta(K)$ can be obtained by Dehn filling on M: One needs to fill in the torus components corresponding to the crossing circles; the surgery coefficient for the component corresponding to C_i is going to be $\frac{1}{r_i}$, where $|r_i|$ is the number of full twists removed from the i-th twist region of K during the construction of J. The following theorem is shown in [La](where also the notion of twist reduced diagrams is defined); the construction is explained in more detail in [FP]:

Theorem 3.2. [Agol-Thurston, [La]] *Suppose that K is a twist reduced diagram with at least two twist regions and let J the augmented link constructed as above. Then, the complement $M := S^3 \setminus \eta(J)$ admits a completed hyperbolic structure.*

Here, we will apply this theorem to the twist reduced diagrams of Figure 2.

Theorem 3.3. *Given $n \in \mathbf{N}$, there exists infinitely many hyperbolic non-alternating knots K for which we have*

$$10v_3||J_K(t)|| \geq \mathrm{vol}(S^3 \setminus K) > n.$$

Proof: . For $r, s > 2$ such that $r + s > n$ consider the pretzel knot $K := K(a_1, \ldots, a_s, b_1, \ldots, b_r)$, where $a_i, b_j > 2$ are odd integers. The diagram D, shown in Figure 2, is twist reduced. Let J be the augmented link constructed from it. By Theorem 3.2, the complement $M := S^3 \setminus \eta(J)$ admits a completed hyperbolic structure. Since ∂M is a collection of tori, M has finite volume. Associated with each component T^2 of ∂M, there is a cusp in int$M$ homeomorphic to $T^2 \times [1, \infty)$. A result of Adams ([Ad]) states that a complete hyperbolic manifold with N cusps must have volume at least $N.\ v_3$. Since ∂M has at least $n+2$ components, and thus $n+2$ cusps, it follows that $\mathrm{vol}(M) > n\ v_3$. Let $m = r + s$ denote the number of cusps of M. Given an m-tuple of integers $\mathbf{r} := (r_1, \ldots, r_m)$, let $M(\mathbf{r})$ denote the 3-manifold obtained from M as follows: For $1 \leq i \leq m$, perform Dehn filling with slope $\frac{1}{r_i}$-surgery along $\partial\eta(C_i)$. Let $\hat{K}(\mathbf{r})$ denote the image of K in $M(\mathbf{r})$. Clearly, $M(\mathbf{r}) := S^3 \setminus \eta(\hat{K}(\mathbf{r}))$. By Thurston's hyperbolic Dehn filling theorem, if $r_i >> 0$ then $M(\mathbf{r})$ admits a complete hyperbolic structure of finite volume; thus $\hat{K}(\mathbf{r})$ is a hyperbolic knot. By the proof of Thurston's theorem, the hyperbolic metric on $M(\mathbf{r})$ can be chosen so that it is arbitrarily close to the metric of M, provided that the numbers r_i are all sufficiently large. Thus by choosing the r_i's large we may ensure that the volume of $M(\mathbf{r})$ is arbitrarily close to that of M. Since $\mathrm{vol}(M) > n\ v_3$, we can choose the r_i's so that we have $\mathrm{vol}(M(\mathbf{r})) > n$. But since $\mathrm{vol}(M(\mathbf{r})) = \mathrm{vol}(S^3 \setminus \hat{K}(\mathbf{r}))$, setting $K := \hat{K}(\mathbf{r})$ we have $\mathrm{vol}(K) > n$. Now let us explain the assertion that we can have infinitely many such knots: By construction, $\hat{K}(\mathbf{r})$ is a pretzel knot of the form $K := K(a_1, \ldots, a_s, b_1, \ldots, b_r)$, where $a_i = r_i + 1$ and $b_j = r_{s+j}$. By [LiT], the projection of the sort shown at Figure 2 is of minimum crossing number. Clearly we can choose $\mathbf{r} >> 0$ so that we get infinitely many crossing numbers and thus infinitely many knots. Finally since, as remarked earlier, these knots are adequate, by Theorem 2.7 we will have $10v_3||J_K(t)|| \geq \mathrm{vol}(S^3 \setminus K) > n$. $\square$

3.2. **Preserving the Alexander polynomial.** Here we show how to create families of adequate knots with the same Alexander polynomial and arbitrarily large twist number. For $p, q, r \in \mathbf{Z}$, with p even and q, r odd, let $T(p, q, r)$ denote the 3-string pretzel tangle oriented as shown in Figure 3. Now start with a knot diagram say D that is representing a

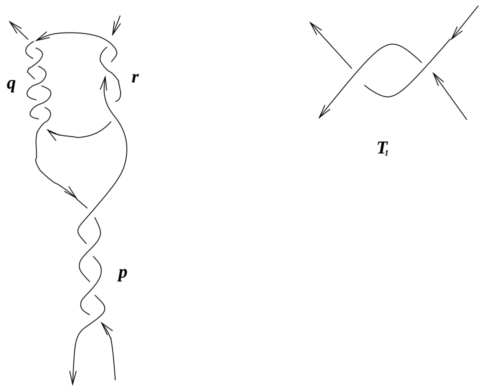

FIGURE 4. The pretzel tangle $T(p, q, r)$ and the clasp tangle T_1. The pretzel tangle shown here is $T(4, 5, -3)$.

knot K. Suppose that the diagram D is projected on the (x, y)-plane so that at each y-level occurs at most one twist region of D and so that the two strings forming the "top"' twist region have opposite orientations. Let T_1 denote the clasp formed by the two leftmost crossings in this top twist region. For example, any of the diagrams of Figure 2 can easily be arranged to fit that description. Now let D_1 denote the knot diagram obtained from D by replacing T_1 with $T(p,q,r)$. We have:

Lemma 3.4. (Stoimenow) *For any $k \in \mathbf{Z}$, set $p := 2k^2$, $q := 1+2k$ and $r := 1-2k$ and let D and D_1 be knot diagrams obtained as above. Then, the knot K_1 represented by D_1 has the same Alexander polynomial as the knot K represented by K.*

Proof: The argument is essentially the same to that of Example 5.3 of [SStW], where the reader is referred for more details and pictures: Let R denote the field of fractions of $\mathbf{Z}[t^{\frac{1}{2}}, t^{\frac{-1}{2}}]$. Let C denote the Conway skein module, over R, generated by oriented tangles modulo the relations $L_+ - L_- - (t^{\frac{1}{2}} - t^{\frac{-1}{2}})L_o$, where L_+, L_-, L_o is any Conway skein triple of links. set $T := T(p,q,r)$. Consider the map $F : C \longrightarrow \mathbf{Z}[t^{\frac{1}{2}}, t^{\frac{-1}{2}}]$, as defined in [SStW], where for a tangle S the image $F(S)$ is the Alexander polynomial of the knot $D(T.S)$; the denominator closure of the product tangle $T.S$. So, $F(S) = \Delta_{D(T.S)}(t)$. Similarly, one defines $G : C \longrightarrow \mathbf{Z}[t^{\frac{1}{2}}, t^{\frac{-1}{2}}]$, by $G(S) = \Delta_{D(T_1.S)}(t)$. See Figure 4. To

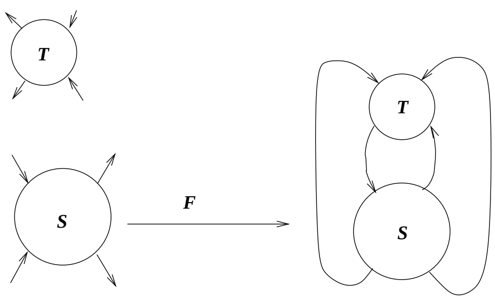

FIGURE 5. The map F.

finish the lemma we must show that for the values of p, q, r as specified above we have $F(S) = G(S)$ for every $S \in C$. Since elementary tangles S_+, S_- shown in Figure 5 form a basis for C it is enough to check that $F(S_\pm) = G(S_\pm)$. Now, the closures $D(T_1.S_-)$ and $D(T_1.S_+)$ are the trefoil and unknot respectively we have $G(S_-) = t^2 - t + 1$ and $G(S_+) = 1$. On the other hand $D(T_1.S_-)$ and $D(T_1.S_+)$ are the pretzel knots $K(p+1,q,r)$ and $K(p-1,q,r)$, respectively. Since we assumed that $p := 2k^2 + 1$, $q := 1 + 2k$ and $r := 1 - 2k$, we have $(p+1)q +$

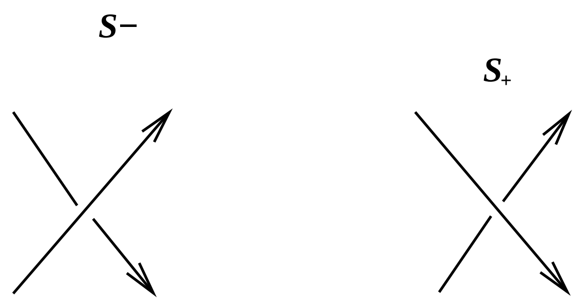

FIGURE 6. The elementary tangles S_-, S_+.

$(p+1)r + qr = 3$ and $(p-1)q + (p-1)r + qr = -1$. Thus, using the formula of page 57 of [Li], we have $F(S_+) = t^2 - t + 1 = G(S_+)$ and $F(S_-) = 1 = G(S_-)$. ☐

We are now ready to give the proof of Theorem 3.5

Theorem 3.5. *There exists a family of hyperbolic knots $\{K_n\}_{n\in\mathbf{N}}$ with fixed Alexander polynomial and such that*

$$20v_3||J_{K_n}(t)|| \geq \mathrm{vol}(S^3 \setminus K_n) > n,$$

where $v_3(\approx 1.01494)$ is the volume of regular hyperbolic ideal tetrahedron. Furthermore, K_n satisfies the inequality of Corollary 2.13

Proof: For $r > 2$ such that $r+2$ is odd consider the pretzel diagram $D := K(a_1, \ldots, a_s, b_1, b_2)$, where $a_i, b_j > 133$ are integers so that the diagram represents a knot. Let $T := T(p, q, r)$, where $|p|, |q|, |r| > 133$ and they are all chosen to satisfy the hypotheses of Lemma 3.4. Now we apply the process described before the statement of Lemma 3.4 to replace the clasp T_1 consisting of the top bigon of the twist region corresponding to a_1; let K_1 the resulting knot. We will furthermore assume that $p, q > 0$ and $r < 0$ (a typical example is that of Figure 5). The knot K_1 is +-adequate. In Figure 6 it is indicated how the resolution all 0-resolution changes if we replace T_1 by T.

The bottom picture of Figure 6 indicates the affect of the replacement of T_1 by T on the reduced 0-graph: We observe that after the replacement of T_1 by T the difference between the edge and the vertices of the reduced 0-graph is increased by one. Now we can repeat the process with the tangle T_1 formed by the two highest crossings in

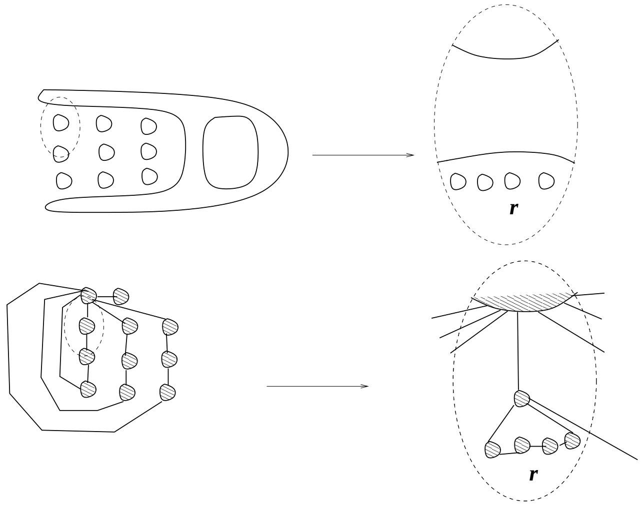

FIGURE 7

the q-string of D_1. Using Lemma 3.4 gives we find that repeating the process n-times increases the second highest coefficient of the Kaufman bracket by n. On the other hand the twist number increases by $2n$. Thus, if we let D_n denote the knot diagram obtained after repeating the process n-times we have $|| < D_n > || \geq s+n+1$ and $t(D_n) = s+2n$. Clearly, $|| < D_n > | \geq \frac{1}{2}t(D_n)$. $\square$

It should also be noted that many known families of knots with bounded $||J_K(t)||$ (most notably the constructions of Kanenobou [Kan]) can be shown to have bounded volume. This together with the results of the current paper prompt the question of whether (6) holds for all hyperbolic knots.

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