SURGERY n-TRIVIALITY AND COMPANION TORI

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ABSTRACT. We define a knot K to be *surgery n-trivial* if there exists a collection of m-disjoint *crossing circles* for K such that twisting along any $0 < m \leq n$ of them turns K into the unknot. We use techniques from the theory of sutured 3-manifolds to study the question of whether, in the case that K is a non-trivial satellite, these crossing circles can be chosen to be disjoint from a companion torus of K. We show that if the twists are taken to have order bigger than one the answer to this question is *yes*. As a consequence of this we show that, under the hypothesis that the twists used have order bigger than one, the only knot that is surgery n-trivial for all $n \in \mathbf{N}$, is the unknot.

Mathematics Subject Classification: 57M25, 57N10.

1. INTRODUCTION

The notion of *n-triviality* is a multiplex unknotting operation that was introduced independently by Gussarov and Ohyama ([G], [O]). Roughly speaking, a knot is n-trivial if it can be unknotted in $2^n - 1$ different ways by *multiple crossing changes.* In [G], it is shown that a knot is n-trivial precisely when all of its finite type knot invariants of orders $< n$ vanish. It is known that certain satellite knots contsructed out of *companion* or *model* knots with vanishing finite type invariants, also have vanishing finite type invariants. The research in this paper is motivated by the question of the extent to which this process can be reversed. More specifically, one can ask the following:

Question 1.1. *Suppose that K is a satellite knot that is n-trivial. Do there exist numbers $m, n \in \mathbf{N}$ with $n = m + k$ such that K can be exhibited as a satellite with companion an m-trivial knot $\hat{K}$ and model a knot Q that is k-trivial inside the companion solid torus?*

An affirmative answer to this question would clarify the interplay n-triviality and satellite structures of knots and would place the finite type invariants into the framework of toroidal decompositions of knot complements. Furthermore, as the results of this paper suggest, it should allow the reduction of the conjecture that the unknot is the only knot all of whose finite type knot invariants vanish, to the corresponding statement for *simple* knots.

This research is partially supported by the NSF through grant DMS-0104000.

In the current paper, we consider the notion of surgery n-triviality. This is a strengthened version of n-triviality where each multiple crossing change, required in the Gussarov-Ohyama definition, can be achieved by twisting along a crossing circle. We employ results of Gabai ([Ga]) and Lackenby ([La1]) to study the positions of the companion tori of surgery n-trivial knots relatively the crossing changes used to unknot them and to obtain a partial affirmative answer to Question 1.1. We show that, provided that the twists involved have order bigger than one, there exists a companion torus of a surgery n-trivial knot K that is disjoint from all the crossing circles that exhibit the surgery n-triviality of K. Combining this with the work of B. Freedman and M. Freedman ([FF] and a result of [La2] we show that provided that the unknotting twists are taken to have order bigger than one, the only knot that is surgery n-trivial for all $n \in \mathbf{N}$, is the unknot. It should be noted that although, as it follows from the definitions, surgery n-triviality is stronger than n-triviality, the precise relation between the two notions is not clear at this time.

The paper is organized as follows: In Section 2 we provide the basic definitions and state our main results. In Section 3 we prove some preliminary results needed in subsequesnt sections. Sections 4 and 5 contain the proofs of the main results of the paper. Finally, Section 6 provides contsructions of surgery n-trivial knots.

2. Notions of n-triviality and n-collections

2.1. **Definitions.** Throughout the paper we work in the PL or the smooth category. Before we can state our results we need some terminology. We begin by recalling the following definition:

Definition 2.1. ([G] [O]) *A knot K is called n-trivial iff it admits a projection that contains a collection of n disjoint sets of crossings $\{C_1, \ldots, C_n\}$, such that, for every $0 < m \leq n$, changing all the crossings in any m of these sets simultaneously, yields a projection of the unknot.*

A *crossing disc* for a knot $K \subset S^3$, is an embedded disc $D \subset S^3$ such that K intersects D only in its interior with zero algebraic number. Let $q \in \mathbf{Z}$. Performing $\frac{1}{q}$- surgery on $K_1 := \partial D$, changes K to another knot $K^{'} \subset S^3$. We will say that $K^{'}$ is obtained from K by *a surgery modification* of order q about K_1. In particular, if $q = 0$ then $K^{'} = K$. As it is illustrated in Figure 1, $K^{'}$ is obtained from K by inserting q-full twists in the strings of K piercing D.

An n-collection for a knot K is a pair $(\mathcal{D}, \bar{q})$, such that:

i) $\mathcal{D} := \{D_1, \ldots, D_n\}$ is a set of n mutually disjoint crossing discs for K;

ii) $\bar{q} := \{\frac{1}{q_1}, \ldots, \frac{1}{q_n}\}$, with $q_i \in \mathbf{Z} - \{0\}$;

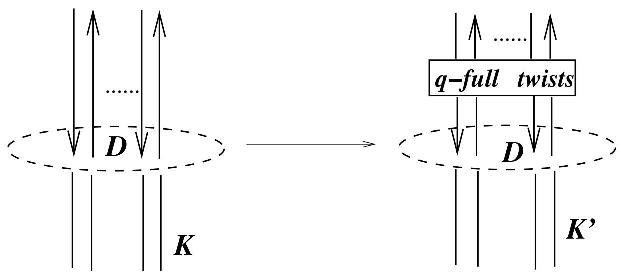

FIGURE 1. Performing $\frac{1}{q}$- surgery on ∂D inserts q-full twists on the strings of K piercing $\mathrm{int}(D)$.

iii) the knots $K_1, \dots, K_n$ are labeled by $\frac{1}{q_1}, \dots, \frac{1}{q_n}$, respectively. Here, for $i = 1, \dots, n$, $K_i := \partial D_i$.

We will say that $(\mathcal{D}, \bar{q})$ is *non-unitary* (resp. *unitary*) iff $|q_i| > 1$ (resp. $|q_i| = 1$), for every $i = 1, \dots, n$. Given a knot K and an n-collection $(\mathcal{D}, \bar{q})$, for $j = 1, \dots, n$, let $i_j \in \{0,\ 1\}$ and

$$\bar{i} := (i_1, \dots, i_n).$$

For every $\bar{i}$, we will denote by $K(\bar{i})$ the knot obtained from K by a surgery modification of order q_i (resp. 0), along each ∂D_j for which $i_j = 1$ (resp. $i_j = 0$).

Definition 2.2. *A knot K is called surgery n-trivial if there exists an n-collection $(\mathcal{D}, \bar{q})$ for K, such that the knot $K(\bar{i})$ is the unknot for every $\bar{i} \neq (0, \dots, 0)$. The link $L := K_1 \cup \ldots \cup K_n$ is called an n-trivializer of K. An n-trivializer is called non-unitary (resp. unitary) iff the corresponding n-collection is non-unitary (resp. unitary). A non-unitary n-trivializer is called positive iff $q_i > 1$.*

Lemma 2.3. *If K is surgery n-trivial then K is n-trivial.*

Proof: Suppose that K is surgery n-trivial, for some $n \in \mathbf{N}$, and let $(\mathcal{D}, \bar{q})$ be an n-collection of K that satisfies Definition 2.2. For every $i = 1, \dots, n$, modify the strings of K piercing $\mathrm{int}(D_i)$ as follows: Insert q full twists followed by $-q$ full twists so that the resulting embedding is isotopic to K. Note that inserting q full twists amounts to inserting an appropriate pure braid. For $i = 1, \dots, n$, choose C_i to be a set of crossings that trivializes the pure braid realizing the $-q$ full twists corresponding to D_i. It follows from Definition 2.2 that the collection $\{C_1, \dots, C_n\}$ satisfies Definition 2.1. □

Remark 2.4. Let K be a knot embedded in the interior of a solid torus $V \subset S^3$. Throughout the paper, we will use the term "K is surgery n-trivial inside V" to mean the following: There exists an embedding of K in $\mathrm{int}(V)$ and an n-collection $(\mathcal{D}, \bar{q}) \subset \mathrm{int}(V)$ such that, for every $\bar{i} \neq (0, \dots, 0)$, $K(\bar{i})$ is an unknotted curve that can be isotoped inside a 3-ball in int(V).

2.2. **The main results.** Next we turn our attention to the study of companion tori of surgery n-trivial knots. Let $V \subset S^3$ be a standard solid torus and let $Q \subset V$ be a knot that is geometrically essential in V. Suppose that Q is embedded in such a way so that it is k-trivial inside V. Also let $\hat{K}$ be a knot that is m-trivial. It is know that certain satellite knots formed with *companion* $\hat{K}$ and *model* Q are $(m+k)$-trivial. Such contractions can be found in [P]. Furthermore, if Q is surgery m-trivial inside V and $\hat{K}$ is surgery k-trivial then these satellites will be $(m+k)$-surgery trivial. Examples of such constructions are given in Section 6. The following proposition, that gives a partial afirmative answer to Question 1.1, shows that in many cases these are the most general constructions of satellite surgery n-trivial knots.

Proposition 2.5. *Let $n \in \mathbf{N}$. Suppose that a non-trivial satellite knot K is shown to be surgery n-trivial by a positive n-trivializer $(L, \bar{q})$. Then there exist numbers $m, k \in \mathbf{N}$, sublinks $L_1, L_2 \subset L$ and knots $\hat{K}$, Q such that:*
(i) We have $k + m = n$ and $L_1 \cup L_2 = L$.
(ii) $\hat{K}$ is a companion knot and Q is a model knot for K.
(iii) $\hat{K}$ is surgery m-trivial. Furthermore, L_2 lies in the exterior of the companion solid torus V_1 and forms a positive m-trivializer for $\hat{K}$.
(iv) L_1 lies in $\mathrm{int}(V_1)$ and forms a positive k-trivializer that shows K to be surgery k-trivial inside V_1. Furthermore, L_1 gives rise to a positive k-trivializer that shows Q to be surgery k-trivial inside a standard solid torus in S^3.

Our next result shows that, under the hypotheses of Proposition 2.5, the question of whether there exits a non-trivial knot that is surgery n-trivial for every $n \in \mathbf{N}$ is reduced to the corresponding question for simple knots:

Theorem 2.6. *Suppose that there exists a non-trivial knot K that is surgery n-trivial for every $n \in \mathbf{N}$. Suppose, moreover, that, for every $n \in \mathbf{N}$, K is shown to be surgery n-trivial by a non-unitary n-trivializer. Then there exists a simple knot with the same properties.*

As already mentioned in the Introduction, Gussarov has shown that a knot is n-trivial iff all of its finite type invariants of orders $< n$ vanish. It is a conjecture of Vassiliev ([V]), as well as an implication of the Volume Conjecture ([MM]), that the only knot all of whose finite type invariants vanish is the unknot. In the light of Gussarov's result, this conjecture is equivalent to the statement that the only knot that is n-trivial for all $n \in \mathbf{N}$ is the unknot. To that respect, combining Theorem 2.6 with a result of [La2], we have the following:

Corollary 2.7. *Let K be a knot that is surgery n-trivial, for all $n \in \mathbf{N}$. Suppose, moreover, that for every $n \in \mathbf{N}$, K admits a non-unitary n-trivializer. Then, K is the unknot.*

A knot is called n-adjacent to the unknot (or strongly n-trivial) iff it is surgery n-trivial and it admits an n-collection $(\mathcal{D}, \bar{q})$ such that the interior of each disc in $\mathcal{D}$ intersects K exactly twice. For such knots a stronger version of Proposition 2.5 was shown in [KL1]. Furthermore, Howards and Luecke ([HL]) have shown that the only knot that is n-adjacent to the unknot for all $n \in \mathbf{N}$, is the unknot.

3. n-COLLECTIONS FOR THE UNKNOT

For a link $\bar{L} \subset S^3$ we will use $\eta(\bar{L})$ to denote a regular neighborhood of $\bar{L}$. Let $(\mathcal{D}, \bar{q})$ be an n-collection for a knot K and let the notation be as in Section 2. Since the linking number of K and every component of L is zero, K bounds a Seifert surface in $M_L := S^3 \setminus \eta(K \cup L)$.

Proposition 3.1. *Suppose that K is the unknot and let $(\mathcal{D}, \bar{q})$ be an n-collection that exhibits K as surgery n-trivial. Suppose, moreover, that S is a Seifert surface that is of minimum genus among all surfaces bounded by K in M_L. Then, S is a disc and $K \cup L$ is the unlink of $n+1$ components.*

Proof: The proof is by induction on n.

Suppose $n = 1$; thus $L = K_1$ and $\mathcal{D} = \{D_1\}$. If M_L is reducible the conclusion of the lemma is clearly true since $K \cup L$ is a split link of two unknotted components. We will show that this is always the case. Suppose, on the contrary, that M_L is irreducible and let S be a Seifert surface that is of minimum genus among all surfaces bounded by K in M_L. By assumption, K remains unknotted in the 3-manifolds obtained by $\frac{1}{q_1}$-Dehn filling or ∞-Dehn filling on $\partial\eta(K_1)$. By Corollary 2.4 of [Ga], genus(S) = genus(K) and thus S is a disc.

Among all discs bounded by K_1 let D be one that minimizes the intersection $|D \cap K|$. We will show that $|D \cap K| = 0$. If $|D \cap S| = 0$ then clearly $|D \cap K| = 0$ and there is nothing to prove. First, suppose that $D \cap S$ contains closed components. Let $C \subset D \cap S$ be such a component that is innermost on S; that is C bounds a disk $\bar{D} \subset S$ with $\mathrm{int}(\bar{D}) \cap D = \emptyset$. Now, C also bounds a disk $D^* \subset D$. Note that $D^* \cap K = \emptyset$. For, otherwise, the disc D', obtained from D by replacing the interior of D^* with that of $\bar{D}$, would have the property that $|K \cap D'| < |K \cap D|$. By an isotopy of S and D we can eliminate all closed components of $D \cap S$. Thus $D \cap S$ consists of a number of arcs. Let $\alpha \in (D \cap S)$ be an outermost component; that is α separates S into two discs the interior of one of which, say D_2, is disjoint from D. Let R be a regular neighborhood of D_2. We have $\partial R = D_1^* \cup D_2^*$, where

$$D_1^* \subset \mathrm{int}(D)\ ,\ D_1^* \cap D_2^* = \partial D_1^* = \partial D_2^* \text{ and } \mathrm{int}(D_2^*) \cap D = \emptyset.$$

Let D'' be the disc obtained from D by replacing $\mathrm{int}(D_1^*)$ with $\mathrm{int}(D_2^*)$. Clearly, we have $|D'' \cap K| < |D \cap K|$. But this contradicts our minimality assumption. Thus $|D \cap K| = 0$, which, in turn, contradicts our assumption that M_L is irreducible.

Inductively suppose that the conclusion of the proposition is true for all $n \leq m$ and let $(\mathcal{D}, \bar{q})$ be an $(m+1)$-collection with the properties of the statement of the proposition. We claim that M_L is reducible. For, suppose on the contrary that M_L is irreducible. Let $L_1 := L \setminus K_1$ and $M_L(q_1)$ (resp. $M_L(\infty)$) denote the 3-manifold obtained from M_L by $\frac{1}{q_1}$-Dehn filling (resp.∞-Dehn filling) on $\partial\eta(K_1)$. By Corollary 2.4 of [Ga], S remains of minimum genus in at least one of $M_L(\infty)$, $M_L(q_1)$ say in $M_L(\infty)$. By induction, S is a disc and $K \cup K_2 \cup \ldots \cup K_{m+1}$ is the unlink. Thus $K_2, \ldots, K_{m+1}$ bound disjoint discs that are disjoint from S. We can easily choose these discs to be disjoint from K_1 which contradicts our assumption that M_L is irreducible. Hence, we conclude that M_L is reducible. Then at least one component of L, say K_{m+1}, bounds a disc in the complement of $\bar{L} := K \cup K_1 \cup \ldots \cup K_m$. By the induction hypothesis, K bounds a disc $\hat{D}$ in the complement of $K_1 \cup \ldots \cup K_m$ and $\bar{L}$ is the unlink of m+1 components. We can choose $\hat{D}$ as well as the discs bounded by $K_1, \ldots, K_m$ to be disjoint from the disc bounded by K_{m+1}. Thus S is a disc and $K \cup L$ is the unlink. $\square$

4. Surgery n-triviality of satellite knots

The purpose of this section is to prove Proposition 2.5. We begin by defining a notion of complexity on the set of knots. Let M be an orientable, compact, irreducible 3-manifold with ∂M a union of incompressible tori. A collection $\mathcal{T}$ of disjointedly embedded, pairwise non-parallel, essential tori in M will be called an *admissible* collection. Given an admissible collection $\mathcal{T}$, let $|\mathcal{T}|$ denote the number of components of $\mathcal{T}$. By the Haken finiteness theorem ([H], Lemma 13.2) the number

$$\tau(M) = \max\{|\mathcal{T}| \mid \mathcal{T} \text{ admissible collection of tori in } M\},$$

is well defined. For a knot K we define $\tau(K) := \tau(M)$, where $M := S^3 \setminus \eta(K)$. The knot K is *simple* iff $\tau(K) = 0$. Let $V_1 \subset S^3$ a solid torus, with core $\hat{K}$, such that $K \subset \mathrm{int}(V_1)$. K is a non-trivial satellite, with companion $\hat{K}$, iff: i)K is geometrically essential in V_1; ii) K is not the core of V_1; and iii) V_1 is knotted. Thre is a standardly embedded solid torus $V \subset S^3$, a geometrically essential knot $Q \subset \mathrm{int}(V)$ that is not the core of V, and a homeomorphism $f : V \longrightarrow V_1$ such that $f(Q) = K$. The knot Q is called a model of K. The following lemma follows directly from the definitions:

Lemma 4.1. *Let K be a non-trivial satellite with companion $\hat{K}$ and model Q. We have $\tau(\hat{K}) < \tau(K)$ and $\tau(Q) < \tau(K)$.*

Our next lemma will allow us to choose an n-trivializer of a non-trivial satellite knot to be disjoint from a companion torus.

Lemma 4.2. *Let K be a knot with $\tau(K) > 0$. Suppose that K is surgery n-trivial and that it admits a positive n-trivializer $(L, \bar{q})$. Then, there exists an essential embedded torus $T_L \subset S^3 \setminus \eta(K)$ such that after an ambient isotopy of L in $S^3 \setminus \eta(K)$ we have $T_L \cap L = \emptyset$.*

Proof: The proof of the lemma is by induction on n.

The induction base: Suppose that $n = 1$; thus $L = K_1$. Let $M := S^3 \setminus \eta(K)$ and let S be a Seifert surface for K that is taut in the Thurston norm in $N := M \setminus \eta(K_1)$. Since K is non-trivial and twisting along L unknots K, it follows that N is irreducible. Now Assumptions 1.1 and 1.3 of pages 616-617 of [La1] hold for M, L and F a companion torus of K. Let $N(q_1)$ (resp. $K(q_1)$) denote the 3-manifold (resp. knot) obtained from N (resp. K) by filling in $\partial\eta(K_1)$ with slope $\dfrac{1}{q_1}$. Since K is non-trivial and $K(q_1)$ is the unknot, S doesn't remain taut in $N(q_1)$.

Case 1: Suppose that M is S_{K_1}-atoroidal (see Definition 1.6 of [Ga]). Since $\tau(K) > 0$, there exists an essential torus $F \subset M$. By Theorem 1.4(b) of [La1], after an ambient isotopy of K_1 in M we have

$$|K_1 \cap F| = \frac{-\chi(F)}{q_1 - 1},$$

where $\chi(F)$ is the Euler characteristic of F and $|K_1 \cap F|$ denotes the geometric intersection of K_1 and F. Since $\chi(F) = 0$, it follows that we have $K_1 \cap F = \emptyset$. Thus, setting $T_L := F$ we have the desired conclusion in this case.

Case 2: Suppose that M is not S_{K_1}-atoroidal. Then, by Proposition 2.3 of [La1], there exists an essential torus $T \subset M$ that separates $S \cup \partial M$ from K_1. In fact, T bounds a solid torus $V \subset S^3$ such that $S \subset \mathrm{int}(V)$ and $K_1 \subset S^3 \setminus V$. Now setting $T_L := T$ we have the desired conclusion.

The inductive step: Inductively, suppose that for any $m < n$ and any positive m-trivializer $(L_1, \bar{q}_1)$, there exists an essential torus $T_{L_1} \subset S^3 \setminus \eta(K)$ such that, after an ambient isotopy of L_1 in $S^3 \setminus \eta(K)$, we have $T_{L_1} \cap L_1 = \emptyset$. Let $(L, \bar{q})$ be a positive n-trivializer. Set $L_1 := \cup_{i=2}^{n} K_i$, $\bar{q}_1 := (q_2, \dots, q_n)$ and $M := S^3 \setminus \eta(K \cup L_1)$. Since L_1 is an $(n-1)$-trivializer of K it follows that M is irreducible. For, an essential 2-sphere $\Sigma \subset M$ must separate at least one component of L_1, say K_n, from $K \cup (L_1 \setminus K_n)$. But then no surgery modification along K_n can unknot K, contrary to our assumption. Let S be a Seifert surface for K that is taut in the Thurston norm in $N := M \setminus \eta(K_1)$. Since a surgery modification of order q_1 along K_1 unknots K, by Proposition 3.1, K bounds a disc in $N(q_1)$; the 3-manifold obtained from $M \setminus \eta(K_1)$ by $\dfrac{1}{q_1}$-Dehn filling of $\partial\eta(K_1)$. Thus S doesn't remain taut in $N(q_1)$.

Case 1: Suppose that M is S_{K_1}-atoroidal. By the induction hypothesis, there exists an essential torus $T_{L_1} \subset S^3 \setminus \eta(K)$ such that, after an ambient isotopy of L_1 in $S^3 \setminus \eta(K)$, we have $T_{L_1} \cap L_1 = \emptyset$. Thus $T_{L_1} \subset M$. Again Theorem 1.4(b) of [La1] applies for M, S, $F := T_{L_1}$, $M' := N(q_1)$ and the surgery curve $K_1 \subset M$. We conclude that K_1 can be isotoped in M so that we have

$$|K_1 \cap F| = \frac{-\chi(F)}{q_1 - 1},$$

where $\chi(F)$ is the Euler characteristic of F and $|K_1 \cap F|$ denotes the geometric intersection of K_1 and F. Since $\chi(F) = 0$, it follows that we have $K_1 \cap F = \emptyset$ and the conclusion, in this case, follows.

Case 2: Suppose that M is not S_{K_1}-atoroidal. Then, by Proposition 2.3 of [La1], there exists a torus T that is essential in M and separates $S \cup \partial M$ from K_1. Clearly we have $T \cap L = \emptyset$. We claim that T remains essential in $S^3 \setminus \eta(K)$. For, suppose not and let $V \subset S^3$ be a solid torus with $\partial V = T$ and $S \cup K \cup L_1 \subset \operatorname{int}(V)$. Since T compresses in $S^3 \setminus \eta(K)$, K must lie inside a 3-ball $B \subset \operatorname{int}(V)$. Since $K_1 \subset (S^3 \setminus V)$, K_1 can be isotoped so that it bounds a disc disjoint from B. Thus a surgery modification along K_1 cannot unknot K, contrary to our assumption. Therefore, T is essential in $S^3 \setminus \eta(K)$ and by setting $T_L := T$ we are done. □

Proof of Proposition 2.5: Let $(L\ \bar{q})$ be a positive n-trivializer for K. By Lemma 4.2 there is an essential torus $T_1 \subset S^3 \setminus \eta(K)$ such that $T_1 \cap L = \emptyset$. Now T_1 exhibits K as a non-trivial satellite. Let $\hat{K}$ denote the companion knot of K and let V_1 denote the companion solid torus. Let Q be a model knot for this satellite. There is a standard torus V such that $Q \subset V \subset S^3$ and an embedding $f : V \longrightarrow S^3$ with $f(Q) = K$ and $f(V) = V_1$.

Case 1: Suppose that the winding number of K in V_1 is non-zero. Suppose that for some $i = 1, \ldots, n$, K_i lies in $\operatorname{int}(V_1)$. Then the knot obtained from K by $\dfrac{1}{q_i}$-surgery on K_i is a satellite with non-zero winding number and companion $\hat{K}$. Since $\hat{K}$ is non-trivial, K is a non-trivial satellite and, thus, it cannot be the unknot. Thus all the K_i's must lie in $S^3 \setminus V_1$. Using the notation of Definition 2.2,

for $j = 1, \ldots, n$, let $i_j \in \{0,\ 1\}$ and let $\bar{i} := (i_1, \ldots, i_n)$. By assumption, $K(\bar{i})$ is the unknot for every $\bar{i}_n \neq (0, \ldots, 0)$. On the other hand, $K(\bar{i})$ is a satellite with companion $\hat{K}(\bar{i})$ and model Q. In order for $K(\bar{i})$ to be the unknot, for every $\bar{i} \neq (0, \ldots, 0)$, $\hat{K}(\bar{i})$ must be the unknot. Hence $(L\ \bar{q})$ is a positive n-trivializer for $\hat{K}$ and the conclusion of the theorem is true for $m := n$, $k := 0$, $L_2 := L$ and $L_1 := \emptyset$.

Case 2: Suppose that the winding number of K in V_1 is zero. In this case, it is possible to unknot K inside V_1. Let $L_1 \subset L$ (resp. $L_2 \subset L$) denote the sublink of L whose components lie in $\operatorname{int}(V_1)$ (resp. $S^3 \setminus V_1$). Let k (resp. m) denote the number of components of L_1 (resp. L_2). Clearly, L_1 (resp. L_2) gives rise to a positive k-trivializer (resp. m-trivializer) of K. By the argument used in the proof of Case 1, L_2 gives rise to a positive m-trivializer for $\hat{K}$. To continue note that, since V_1 is knotted, in order to unknot K by a surgery modification inside V_1, this surgery modification must turn K into an unknotted curve that can be isotoped in a 3-ball inside V_1. From this observation it follows that L_1 gives rise to a positive k-trivializer that shows

K to be surgery k-trivial inside V_1. Since T_1 is incompressible in $\overline{S^3 \setminus V_1}$, the discs in $\mathcal{D}$ bounded by the components of L_1 can be isotoped in $\mathrm{int}(V_1)$. Now it is easy to see that $f^{-1}(L_1)$ gives rise to a positive k-trivializer that shows Q to be surgery k-trivial inside V. □

Corollary 4.3. *Let K be a knot that is a non-trivial satellite. Suppose that for every $n \in \mathbf{N}$, K is shown to be surgery n-trivial by a positive n-trivializer. There exists a companion knot $\hat{K}$ and a model knot Q of K such that at least one of the following is true:*
(i) $\hat{K}$ is surgery n-trivial for all $n \in \mathbf{N}$. Furthermore, for every $n \in \mathbf{N}$, $\hat{K}$ admits a positive n-trivializer.
(ii) K has zero winding number inside the companion solid torus. Furthermore, for every $n \in \mathbf{N}$, Q admits a positive n-trivializer that shows it to be surgery n-trivial inside a standard solid torus $V \subset S^3$.

Proof: By Proposition 2.5, for every $n \in \mathbf{N}$, there is a number $m_n \in \mathbf{N}$ and a companion knot $\hat{K}_n$ such that $\hat{K}_n$ is shown to be surgery m_n-trivial by a positive n-trivializer. Let V^n denote the companion torus for $\hat{K}_n$. Let $k_n := n - m_n$ and let $f_n : V \longrightarrow V_n$ denote the homeomophism that realizes K as a satellite with companion $\hat{K}_n$. Since a given knot admits only finitely many companions ([S1]), we may, without loss of generality, assume that all the knots $\hat{K}_n$ are isotopic to a knot $\hat{K}$. By the proof of Proposition 2.5, for every $n \in \mathbf{N}$, K admits a positive k_n-trivializer that shows it surgery k_n-trivial inside V_n. For $n > 1$, the image of this k_n-trivializer under the faithful homeomorphism $g_n : V_n \longrightarrow V_1$ shows $g_n(K)$ to be k_n-trivial inside V_1. But since the cores of V_n, V_1 are isotopic, $J_n := g_n(K)$ also represents K. We conclude that, for every $n \in \mathbf{N}$, K is shown to be k_n-trivial inside V_1 by a positive k_n-trivializer. Since $\hat{K}$ is knotted, it is impossible to have satellites of $\hat{K}$ with different framings that are isotopic (see Theorem 2.1 of [KMS]). Thus the homeomorphisms $g_n \circ f_n : V \longrightarrow V_1$ define the same framing for all $n > 1$; let $f : V \longrightarrow V_1$ denote this framing. Let $Q_n \subset V$ be a model knot of the satellite such that $f(Q_n) = J_n$. As in the proof of Proposition 2.5 it follows that if $k_n > 0$, then Q_n has zero winding number and it is shown to be surgery k_n-trivial inside V by a positive k_n-trivializer. This k_n-trivializer is obtained by using f to pull back a positive k_n-trivializer of J_n in V_1. By the arguments of [S2] (see also [Mo]) it follows that infinitely many of the knots $\{Q_n\}_{n \in \mathbf{N}}$ are isotopic in V. For our purposes we may, without loss of generality, assume that all of $\{Q_n\}_{n \in \mathbf{N}}$ are isotopic in V to a knot Q. Now notice that if $\{m_n\}_{n \in \mathbf{N}}$ has an increasing subsequence then clearly $\hat{K}$ is surgery m-trivial for all $m \in \mathbf{N}$, and part (i) of the statement of the corollary is true. Otherwise, $\{k_n\}_{n \in \mathbf{N}}$ must have an increasing subsequence and from our discussion above, Q satisfies part (ii) of the statement of the corollary. □

5. n-TRIVIALITY IN THE SOLID TORUS; THE REDUCTION TO SIMPLE KNOTS.

In this section we prove Theorem 2.6. The proof of the theorem uses Corollary 4.3 and the following lemma that is concerned with knots that are unknotted in S^3 and surgery n-trivial in a standard solid torus.

Lemma 5.1. *Let Q be a knot embedded in a standard solid torus $V \subset S^3$ so that it is geometrically essential and has zero winding number. Suppose that Q is unknotted in S^3 and that it is surgery n-trivial in V, for some $n > 0$. Suppose, moreover, that Q admits an n-trivializer $(L, \bar{q})$ such that no component of L bounds an embedded disc inside V that is disjoint from Q. Then, there exists a constant $c = c(Q)$ such that $n \leq c$.*

Proof: We will assume that n is arbitrarily large and arrive at a contradiction. Let $(\mathcal{D}, \bar{q})$ be an n-collection corresponding to $(L, \bar{q})$ such that, for $i = 1, \dots, n$, $D_i \subset \operatorname{int}(V)$. Choose $(\mathcal{D}, \bar{q})$ so that among all such n-collections it minimizes the geometric intersection number $|\Delta \cap Q|$ in V. Here $\Delta := \cup_{i=1}^{n} D_i$. Since no component of L bounds a disc in V that is disjoint from Q, we have $|\Delta \cap Q| > 0$. Set

$$\overline{\mu} := \max\{|D_i \cap Q| \mid , i = 1, \dots, n\}.$$

Every D_i decomposes Q into a collection of arcs each of which has its endpoints in $\operatorname{int}(D_i)$ and its interior is disjoint from D_i. By Proposition 3.1, the components of L bound disjoint discs in the complement of Q; let D'_i denote the component bounded by K_i. From these observations, and our minimality assumption, it follows that each of these arcs must intersect every meridian disc of V (i.e. it must have non-zero wrapping number). It follows that there is a constant $a = a(Q)$, depending on the wrapping number of Q in V, such that

$$0 < \overline{\mu} \leq a. \tag{5.1}$$

By (5.1), if $n >> a$, the collection $D_1, \dots, D_n$ decomposes into subcollections such that all the discs in each sub collection intersect Q at the same number of points. Let m_n denote the largest of the cardinalities of these sub collections. Since, clearly, m_n increases as n, for our purposes we may without loss of generality assume that

$$|D_i \cap Q| = |D_j \cap Q|, \text{ for every } 1 \leq i, j \leq n. \tag{5.2}$$

Assume, moreover, that Δ, Δ' are isotoped so that $\Delta \cap \Delta'$ is minimum in V. It follows that $\Delta \cap \Delta' = \emptyset$. For, otherwise it is easy to see that either $\Delta \cap \Delta'$ is not minimum in V or (5.2) is not true. Now the 2-spheres $\{\Sigma_i := D_i \cup D'_i\}_{i=1}^{n}$ are mutually disjoint in S^3. Set $\Sigma := \cup_{i=1}^{n} \Sigma_i$. Set $N := \overline{V \setminus \eta(Q)}$ and $\Gamma := \Sigma \cap N$. Now Γ is a collection of disjoint planar surfaces, properly embedded in N. For $1 \leq i \leq n$, let $C_i \in \Gamma$ denote the surface containing K_i and set $\Gamma' := \{C_1, \dots, C_n\}$.

Claim 1: There can be at most one surface in Γ' that is boundary parallel in N.

Proof of Claim 1: Suppose that two of $C_1, \dots, C_n$, say C_1, C_2, are boundary parallel in N. First we claim that both of C_1, C_2 can be isotoped on ∂V. For, suppose for example that C_1, and thus K_1, can be isotoped on $\partial\eta(Q)$. Since the linking number of K_1 and Q is zero, K_1 must be a longitude of $\partial\eta(Q)$. Since K_1 bounds an embedded disc in V we conclude that Q must bound an embedded disc in V. However this contradicts our assumption that Q is geometrically essential in V. Thus both of C_1, C_2 can be isotoped on ∂V. Since K_1, K_2 bound discs in V we conclude that each is isotopic to a meridian of V. Let

$$N' := N \setminus \eta(K_1) \text{ and } N'' := N \setminus \eta(K_1 \cup K_2).$$

For $r, s \in \mathbf{Z}$ let $N'(r)$ (resp. $N''(r,s)$) denote the 3-manifold obtained from N' (resp. N'') by filling in $\partial\eta(K_i)$ (resp. $\partial\eta(K_i \cup K_j)$) with slope $\frac{1}{r}$ (resp. slopes $\frac{1}{r}, \frac{1}{s}$). Since K_1, K_2 are isotopic in V we see that $N''(q_1, q_2) = N'(q_1 + q_2)$. Since Q has zero winding number in V and zero linking number with K_1, K_2, Q bounds a Seifert surface in N'. Let S denote such a surface that is taut in the Thurston norm. By assumption, S doesn't remain taut in any of $N'(q_1)$, $N'(q_1 + q_2)$. Since clearly $q_1 + q_2 \neq q_1$, we get two distinct Dehn fillings of N' along $\partial\eta(K_1)$ under which S doesn't remain taut. However this contradicts Corollary 2.4 of [Ga]. This finishes the proof of Claim 1.

Claim 2: The surfaces $C_1, \dots, C_n$ are incompressible. Furthermore, there exists a constant b, depending only on N, such that $b(C_i) \leq b$. Here $b(C_i)$ denotes the first Betti number of C_i.

Proof of Claim 2: By construction, ∂C_i consists of two components that are longitudes of ∂V and one component for each point of $Q \cap \Delta$. By (5.1), it follows that there is a constant b, depending only on Q and thus on N, such that the number of components in ∂C_i is bounded from above by b. Since $C_1, \dots, C_n$ are planar surfaces their first Betti number is also bounded as desired. The claim that $C_1, \dots, C_n$ are incompressible follows from our assumption that $|\Delta \cap Q|$ was chosen to be minimum in the isotopy classes of Δ and Q in V.

Let us now finish the proof of the proposition: Since no component of L bounds a disc in N, none of the surfaces $C_1, \dots, C_n$ can be a disc. We claim that none of $C_1, \dots, C_n$ can be an annulus. For, suppose that C_i is an annulus. Since K_i bounds a disc in V, C_i has to be homotopically trivial in C_i and thus, contrary to our assumption, it bounds a disc in N. By these observations and Claims 1,2, Theorem 1 of [FF] applies. We conclude that there exists a constant γ depending on N, and thus on Q, such that either $n \leq \gamma$ or some of $C_1, \dots, C_n$ are parallel in N. Note that if $n >> \gamma$ then $i(\frac{n}{\gamma})$ of $C_1, \dots, C_n$ are parallel; where $i(x)$ denotes the integral part of x. Since $i(\frac{n}{\gamma})$ increases with n, for our purposes we may without loss of generality assume that all of $C_2, \dots, C_n$ are parallel to C_1 in N. For $k > 1$, let $\bar{K}_k$ denote the image of K_k on C_1 ender the cobordism from C_k to C_1. By our assumptions it follows that each of $K_1, \bar{K}_2, \ldots, \bar{K}_n$ represents a word in $\pi_1(C_1)$ in which each of the free generators (and their inverses) represented by components of ∂C_1 appears at most once. Since there are finitely many such words in $\pi_1(C_1)$, if $n >> 0$, two of $K_1, \bar{K}_2, \ldots, \bar{K}_n$ must be isotopic in C_1. It follows that two of $K_1, \ldots, K_n$ are isotopic in N. By an argument similar to this in the proof of Claim 1 this last assumption is seen to lead to a contradiction. □

Proof of Theorem 2.6: It is enough to show that there exists a non-trivial knot K' with $\tau(K') = 0$ and such that, for every $n \in \mathbf{N}$, K' is shown to be surgery n-trivial by a non-unitary n-trivializer. Let K be as in the statement of the theorem and let K^* denote the mirror image of K. For every $n \in \mathbf{N}$, let $(L_n, \ \bar{q}_n)$ be a non-unitary n-trivializer for K. Recall that $\bar{q}_n = (q_{1n}, \ldots, q_{nn})$, where $|q_{in}| > 1$. Let p_n (resp. m_n) denote the number of q_{in}'s with $q_{in} > 1$ (resp. $q_{in} < -1$). Clearly, at least one of the sequences $\{p_n\}_{n \in \mathbf{N}}$, $\{m_n\}_{n \in \mathbf{N}}$ must have an increasing subsequence. Suppose that $\{p_n\}_{n \in \mathbf{N}}$ doesn't have such a subsequence. Then $\{m_n\}_{n \in \mathbf{N}}$ must have an increasing subsequence. Notice that K^* is surgery n-trivial for all $n \in \mathbf{N}$ and that $(L_n, \ -\bar{q}_n)$ is a non-unitary n-trivializer for K, where $-\bar{q}_n := (-q_{1n}, \ldots, -q_{nn})$. Thus K^* admits a positive m-trivializer for every $m \in \mathbf{N}$. Hence, without loss of generality, we can restrict ourselves to positive n-trivializers.

Let $\bar{K}$ be a knot such that $\tau(\bar{K}) > 0$ and for every $n \in \mathbf{N}$, $\bar{K}$ is shown to be surgery n-trivial by a positive n-trivializer. Suppose, moreover, that among all knots that have these properties, $\bar{K}$ has the smallest value of τ. By Corollary 4.3, $\bar{K}$ is a non-trivial satellite with companion $\hat{K}$ and model Q such that one of (i)-(ii) of the statement of the corollary is satisfied.

Case 1: Suppose that (i) of Corollary 4.3 is satisfied. Then $\hat{K}$ is surgery m-trivial for all $m \in \mathbf{N}$. Moreover, for every $m \in \mathbf{N}$, $\hat{K}$ admits a positive m-trivializer. By Lemma 4.1, and our assumptions on $\bar{K}$, $\tau(\hat{K}) = 0$ and thus $\hat{K}$ is simple. Since $\hat{K}$ is a companion of $\bar{K}$, $\hat{K}$ cannot be the unknot. Thus the desired conclusion in this case is true for $K' := \hat{K}$.

Case 2: Suppose that (ii) of Corollary 4.3 is satisfied. Then, for every $m \in \mathbf{N}$, Q admits a positive m-trivializer $(L^m, \bar{q}^m)$ that shows Q to be surgery m-trivial in a standard solid torus V in S^3. Let us consider the n-trivializer's constructed in the proof of Corollary 4.3. By construction, it follows that if some component of L^m bounds a disc inside V that is disjoint from Q then K is the unknot. Thus, since we assumed that $\tau(K) > 0$, no component of L^m bounds a disc inside V that is disjoint from Q. On the other hand, and since Q has zero winding number, if Q is the unknot then Lemma 5.1 applies to give that there is a constant c, depending on Q, such that $m \leq c$. But this contradicts our assumption that m can be arbitrarily large. Thus, Q is not the unknot. By assumption, and since $\tau(Q) < \tau(K)$, we have $\tau(Q) = 0$. Thus, in this case, the conclusion follows for $K' := Q$. □

Now Corollary 2.7 follows from Theorem 2.6 and the following lemma:

Lemma 5.2. *Let K be a knot that is surgery n-trivial, for all $n \in \mathbf{N}$. Suppose, moreover, that K is simple and that for every $n \in \mathbf{N}$ there exists a non-unitary n-collection that shows K to be surgery n-trivial. Then, K is the unknot.*

Proof: Let K be a knot shown to be surgery n-trivial by a non-unitary n-collection $(\mathcal{D}, \bar{q})$. By Theorem 1.1 of [La2], there exists constants $b = b(K)$ and $l = l(K)$ such that:

(1) If $n > b$ then there are $i, j \in \{1, \ldots, n\}$, such that the surgery curves $K_i := \partial D_i$ and $K_j := \partial D_j$ are isotopic in the complement of K.

(2) If $n > l$ then there are $i, j \in \{1, \ldots, n\}$ such that $q_i = q_j$.

Suppose now that $n >> bm$, for some $m \in \mathbf{N}$. Then, by (1), there are surgery curves $K_{i_1} := \partial D_{i_1}, \ldots, K_{i_m} := \partial D_{i_m}$ that are all isotopic in the complement of K. Thus, for every $1 < j \leq m$, performing a surgery modification of order q_{i_1} along K_{i_1} and one of order $q_{i_2}, \ldots, q_{i_j}$ along $K_{i_2}, \ldots, K_{i_j}$, respectively, is equivalent to performing a surgery modification of order $q^j := q_{i_1} + q_{i_2} + \ldots + q_{i_j}$ along K_{i_1}. If K is not the unknot, then $0 \neq q^j \neq q_{i_1}$. By (2), $m \leq l$ and therefore n cannot be arbitrarily large. $\square$

6. Constructions of surgery n-trivial knots

For an n-collection $(\mathcal{D}, \bar{q})$ of a knot K set $\overline{\mu}(\mathcal{D}, \bar{q}) := \max\{|D_i \cap K| \mid i = 1, \ldots, n\}$, and $\underline{\mu}(\mathcal{D}, \bar{q}) := \min\{|D_i \cap K| \mid i = 1, \ldots, n\}$. Given an n-trivializer $(L, \bar{q})$ of a knot K, we define the multiplicity of $(L, \bar{q})$ by

$$\overline{\mu} := \min\{\overline{\mu}(\mathcal{D}, \bar{q}) \mid (\mathcal{D}, \bar{q}) \text{ an } n\text{-collection for } (L, \bar{q})\}.$$

For a knot K let $S_Q(K)$ denote the satellite of K with model Q and zero framing and let w denote the wrapping number of $S_Q(K)$ in the companion solid torus.

Proposition 6.1. *Suppose that K is n-adjacent to the unknot and that Q is the unknot. Then, $S_Q(K)$ is surgery n-trivial. Moreover, $S_Q(K)$ admits an n-trivializer of multiplicity $2w$.*

Proof: Let K be a knot that is n-adjacent to the unknot. Let $D(K)$ be a projection of K containing sets of crossings $C_1, \ldots, C_n$ satisfying Definition 2.1. By Theorem 4.4 and Remark 5.5 of [AK], we can assume that, for $i = 1, \ldots, n$, the q_i crossings in C_i are part of q_i clasps as indicated in the left hand side picture of Figure 2. (Note: The figure illustrates the case $q_i = -1$.)

An n-trivializer $(L, \bar{q})$ for K, is constructed by letting, for every $i = 1, \ldots, n$, D_i be a crossing disc for C_i and choosing q_i as follows: $|q_i|$ will be the cardinality of C_i and $q_i > 0$ or $q_i < 0$ according to whether the crossings in C_i are negative or positive. For example, in the case shown in Figure 2 we would take $q_i = -1$. Let $K_i = \partial D_i$ and $L := K_1 \cup \ldots \cup K_n$. We can obtain a projection of $S_Q(K)$, by first adjusting $D(K)$ to realize the zero

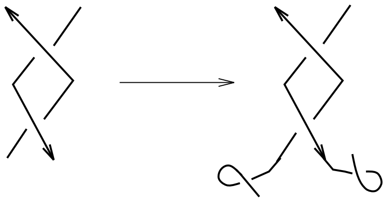

FIGURE 2. A negative full twists has been added for each of the two positive crossings of the clasp to adjust the framing.

framing and then applying the model Q. While adjusting the framing on $D(K)$ one adds one full twist (a kink) for every crossing of $D(K)$; the sign of the kink should be opposite of that of the crossing (see Figure 2). Each clasp c on $D(K)$ gives rise to w parallel clasps on $S_Q(K)$ that intersect the interior of a disc $D_i \in \mathcal{D}$ in $2w$ points. The case $w = 2$ is shown in Figure 3.

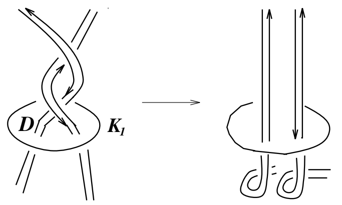

FIGURE 3. Performing -1 surgery on K_1 untwists the clasp and inserts two positive full twists.

By construction, surgery on K_i untwists the multiple clasp and adds a q_i-full twists on each of its branches. These twists will cancel with the ones inserted on $D(K)$ to account for the $2q_i$ crossings of c. We conclude that, in the notation of Definition 2.2, for every $\bar{i} \neq (0, \dots, 0)$ the knot $K(\bar{i})$ is a zero framed satellite with companion and model knot the unknot. Thus, $K(\bar{i})$ is the unknot and it follows that $(L, \bar{q})$ is an n-trivializer that shows $\bar{K}$ to be surgery n-trivial. To finish the proof note that, by construction, $\overline{\mu}(L, \bar{q}) = 2w$. □

Remark 6.2. Combining the results of [AK] with Proposition 6.1 one obtains that, for every $n, m \in \mathbf{N}$, there exist many non-trivial knots that are shown to be surgery n-trivial by an n-trivializer of multiplicity $2m$. Clearly, by the proof of Proposition 6.1, if the knot K to begin with admits a non-unitary (resp. unitary) n-trivializer then $S_Q(K)$ admits a non-unitary (resp. unitary) n-trivializer.

For a knot K let $\mathrm{Wh}(K)$ denote any untwisted Whitehead double of K and let $\mathrm{Wh}^{(n)}(K)$ denote any nth iterated untwisted Whitehead double of K. The following is shown in [K].

Proposition 6.3. *For any knot K and any $n \in \mathbf{N}$, $\mathrm{Wh}^{(n)}(K)$ is surgery n-trivial. Moreover, $\mathrm{Wh}^{(n)}(K)$ admits an n-trivializer of multiplicity 2^n.*

Since the genus of $\mathrm{Wh}^{(n)}(K)$ is one the main result of [HL] implies that $\mathrm{Wh}^{(n)}(K)$ can be at most 2-adjacent to the unknot. Thus we have:

Corollary 6.4. *For every $n \in \mathbf{N}$, there exist infinitely many knots that are surgery n-trivial but are not m-adjacent to the unknot, for any $m > 2$.*

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