Seifert manifolds and the Chen-Yang volume conjecture

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Quantum Topology and Hyperbolic Geometry, Phu Quoc Island, Vietnam, June 2-6, 2025

Settings and talk theme

- *3-manifolds:* M=compact, orientable, with $\partial M = \emptyset$ or ∂M=tori (*Empty or toroidal boundary*).
- *Geometric invariants:* Simplicial volume of M.
- *Quantum invariants:* Turaev-Viro invariants of M.
- *Chen-Yang VC:* The asymptotics of the Turaev-Viro invariants of M determine its simplicial volume.

Talk theme:

- Discuss the (generalized) volume conjecture and survey related progress.
- Discuss older and recent work on the behavior of Turaev-Viro invariants (and their asymptotics) under gluing 3-manifolds along a boundary torus.
- Derive the volume conjecture for Seifert fibered 3-manifolds with non-empty boundary and for large classes of manifolds obtained by gluing Seifert fibered manifolds. (*Plumbed 3-manifolds*).

Canonical decomposition:

Theorem (Knesser, Milnor, Jaco-Shalen, Johanson, Thurston + Perelman)

M=oriented, compact, with empty or toroidal boundary.

1. *There is a unique collection of 2-spheres that decompose M*

$$M = M_1 \# M_2 \# \ldots \# M_p \# (\# S^2 \times S^1)^k,$$

   where M_i are irreducible 3-manifolds.

2. *For M=irreducible, there is a unique collection of disjointly embedded essential tori $\mathcal{T}$ such the components of M cut along $\mathcal{T}$ are either Seifert fibered manifolds or hyperbolic.*

- *Hyperbolic:* Interior admits complete, hyperbolic metric of finite volume.
- *Simplicial Volume of M:* $\mathrm{Vol}(M)$:= sum of volumes of hyperbolic pieces.
- By Mostow rigidity and uniqueness geometric decomposition, $\mathrm{Vol}(M)$ is a topological invariant of M.
- *Seifert fibered manifolds:* Admit S^1-actions. Have zero simplicial volume.

Chen-Yang Volume Conjecture

- *Turaev-Viro invariants, 1990:* Real valued invariants

$$TV_r(M) := TV_r(M, q),$$

  depending on integers $r > 0$ and a $2r$-th root of unity.

- Here we consider $r > 1$ odd integer and $q = e^{\frac{2\pi i}{r}}$.
- $TV_r(M, q)$ are combinatorially defined invariants and can be computed from triangulations of M by a *state sum* formula. Sums involve *quantum $6j$-sympols*.
- Chen-Yang adapted the construction for ideal triangulations.
- **Volume Conjecture**(Q. Chen- T. Yang, 2015) For M compact, orientable

$$LTV(M) := \limsup_{r \to \infty} \frac{2\pi}{r} \log(TV_r(M, e^{\frac{2\pi i}{r}})) = \mathrm{Vol}(M).$$

- The choice of root q is important! The growth of the invariants $TV_r(M, e^{\frac{\pi i}{r}})$ is polynomial in r. So in this case $LTV(M, e^{\frac{\pi i}{r}}) = 0$, for all 3-manifolds! (more later in the talk)

What do we know:

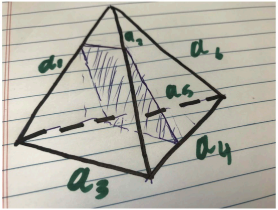
- *Detcherrry-K.-Yang, 2016*: (First examples) of hyperbolic links in S^3: The complement of figure-eight knot and of the Borromean rings. Also all knots of zero volume in S^3.
- *Detcherry-K, 2017* : All zero volume links in S^3 and in connected sums of copies of $S^1 \times S^2$.
- *(Ohtsuki, 2017)* : Infinite families of closed hyperbolic 3-manifolds. All hyperbolic manifolds obtained by (integral) Dehn fillings on figure-eight knot.
  Method: Complex analysis, Fourier analysis, Saddle Point Method.
- *Belletti-Detcherry-K- Yang, 2018* : *Fundamental Shadow links* Infinite family of cusped hyperbolic 3-manifolds that are link complements connected sums of copies of $S^1 \times S^2$. By *Costantino-D. Thurston* these links produce all 3-manifolds by Dehn filling.
- *K.-H. Wong, 2019*: Whitehead chains complements: Octahedral link complements in S^3. (Ohtsuki's method.)
- *Belleti, 2019*: More families of octahedral links in connected sums of copies of $S^1 \times S^2$.

What do we know: Cont'

- *K.-H. Wong-Yang, 2020*: All hyperbolic manifolds obtained by (rational) Dehn fillings on figure-eight . (Ohtsuki's method).
- *(Kumar, 2019, K.-H. Wong-Yang)*: Infinite families of octahedral links in S^3, including all the fully augmented octahedral links. A different proof for the later class is given by *Ibarra- McQuire-Purcell, 2025*).
- *(Kumar-Melby, 2021)*: Infinite families of closed manifolds with arbitrarily large number of hyperbolic pieces in their geometric decomposition. (BDKY $6j$-asymptotics).
- *Chen-Zhou, 2023*: Complements of most twist knots in S^3 and most integral Dehn surgeries along them (Ohtsuki's method).
- *Ge- Meng-Wang- Y. Yang*: Most rational Dehn fillings along twist knots (Ohtsuki's method).
- *Kumar- Melby, 2022*: The VC is closed under "cabling operations" of knots (that produce knots).
- *Detcherry-K.- Marasinghe, 2025*: The VC is "closed" under gluing a Seifert fibered 3-manifold (more later). Prove, VC for Seifert fibered 3-manifolds with boundary and "plumbed" 3-manifolds (TQFT features).

Building blocks of TV invariants relate to volumes!

- Color the edges of a triangulation with certain “quantum ” data

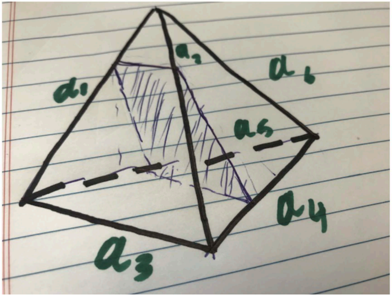


- Colored tetrahedra get “6j-symbol” $\mathbf{Q} := Q(a_1, a_2, a_3, a_4, a_5, a_6)$= function of the a_i and r. $TV_r(M)$ is a weighted sum over all tetrahedra of triangulation (*State sum*).
- **Good news?** Asymptotics of quantum 6j-symbols relate to volumes of geometric tetraherda...
- Ok,.. can we deduce the Chen-Yang volume conjecture from these relations?

Colors assigned to each tetrahedron satisfy certain admissibility conditions (admissible 6-tuples)

Faces:

$$F_1 = (a_1, a_2, a_3), \quad F_2 = (a_2, a_4, a_6), \quad F_3 = (a_1, a_5, a_6) \text{ and } F_4 = (a_3, a_4, a_5).$$



Faces : $T_1 = \dfrac{a_1 + a_2 + a_3}{2}, \quad T_2 = \dfrac{a_1 + a_5 + a_6}{2}, \quad T_3 = \ldots \text{ and } T_4 = \ldots.$

Quadrilaterals:

$$Q_1 = \frac{a_1 + a_2 + a_4 + a_5}{2}, \quad Q_2 = \frac{a_1 + a_3 + a_4 + a_6}{2} \text{ and } Q_3 = \frac{a_2 + a_3 + a_5 + a_6}{2}.$$

Quantum 6j-symbol? Given admissible 6-tuple $\alpha := (a_1, a_2, a_3, a_4, a_5, a_6)$,

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{vmatrix} = \Delta(\alpha) \times \sum_{z=\max\{T_1,T_2,T_3,T_4\}}^{\min\{Q_1,Q_2,Q_3\}} \frac{(-1)^z \{z+1\}!}{\prod_{j=1}^4 \{z-T_j\}! \prod_{k=1}^3 \{Q_k - z\}!} \qquad (1)$$

Quantum integer: $r \geqslant 3$ odd integer and $q = e^{\frac{2i\pi}{r}}$.

$$\{n\} = q^n - q^{-n} = 2\sin(\frac{2n\pi}{r}) = 2\sin(\frac{2\pi}{r})[n]$$

where

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}} = \frac{2\sin(\frac{2n\pi}{r})}{2\sin(\frac{2\pi}{r})}.$$

Quantum factorial: $\{n\}! = \prod_{i=1}^{n} \{i\}$.

where

$$\Delta(\alpha) := \text{"even more quantum factorials."}$$

Volumes of hyperbolic trancated tetrahedra?

- With “correct” parametrizations we can translate the “admissible” colorings on the edges of the tetrahedron into “appropriate” dihedral angle values of “geometric” tetrahedra.
- For example, in appropriate setting, the a_i’s in $\mathbf{Q} := Q(a_1, a_2, a_3, a_4, a_5, a_6)$, will translate to dihedral angles of hyperbolic truncated tetrahedra (these occur in ideal triangulations of cusped hyperbolic 3-manifolds). Then, the large r-asymptotics of Q, give the volume of the tetrahedra.
- When the angles are all zero we have regular ideal hyperbolic octahedra.
- *(BDKY, 2018)* Generalizing work of *J. Murakami-Yano and Costantino, Chen-Murakami*:

$$\frac{2\pi}{r} \log(\mathbf{Q}) \leqslant v_{\mathrm{oct}} + O\left(\frac{\log r}{r}\right),$$

  where $v_{\mathrm{oct}} \cong 3.66..$ is the volume of the regular ideal hyperbolic octahedron.
- The state sums giving the Turaev- Viro invariants are highly “alternating” and understanding the asymptotics of each term is not enough to give the volume conjecture (e.g. for all cusped 3-manifolds built by octahedra).

How do we view the TV invariants?

- Hard to use above relations to prove the VC from the definition of Turaev-Viro invariants as state sums on triangulations!
- In all cases the VC is verified the invariants are viewed through their relation to the Reshetikhin-Turaev TQFT! (more later)
- In this setting, for example, the Turaev-Viro invariants of the complement of a FSL, $TV_r(M_L)$ are given as a sum of *all positive terms* each of which is a product of absolute values of quantum $6j$-sympols.
- Above estimate gives

$$\lim_{r\to\infty} \frac{2\pi}{r} \log(TV_r(M_L)) \leq 2n(L) \cdot v_{\mathrm{oct}} = \mathrm{Vol}(M_L),$$

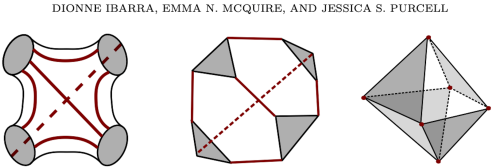
where $n(L)$=integer encoding intrinsic topology of L.

- We show that above bound is realized by a term in our sum.
- We get

$$\lim_{r\to\infty} \frac{2\pi}{r} \log(TV_r(M_L)) = 2n(L) \cdot v_{\mathrm{oct}} = \mathrm{Vol}(M_L).$$

FSL: Construction/Geometry

Topological building block: 3-ball with 4 discs on its boundary, and 6 arcs connecting them.



- Glue n blocks along discs to get a genus $n+1$ handlebody with the arcs forming a link L on the boundary.
- Take the (oriented) double of the handlebody to get a closed 3-manifold M that is the connect sum of $n+1$ copies of $S^2 \times S^1$ with a link L in it.
- Geometrically, the complement of L, M_L, is decomposed into n regular ideal hyperbolic octahedra and we have

$$\mathrm{Vol}(M_L) = 2n \cdot v_{\mathrm{oct}}.$$

Detour: TQFT Highlights

- Reshetikhin-Turaev SO_3-TQFT following skein-theoretic framework of *Blanchet,-Habegger,-Masbaum-Vogel*.
- For odd $r \geqslant 3$ and q a $2r$-root of unity TQFT functor RT_r, assigns
  - a finite dimensional Hermitian $\mathbb{C}$-vector space $RT_r(\Sigma)$, to closed, oriented, surface Σ, where for disjoint unions $\Sigma \coprod \Sigma'$ one has
  $$RT_r(\Sigma \coprod \Sigma') = RT_r(\Sigma) \otimes RT_r(\Sigma').$$
  - an invariant $RT_r(M) \in \mathbb{C}$ for closed M and a vector $RT_r(M)$ in $RT_r(\partial M)$ otherwise.
  - a linear map
  $$RT_r(M): \ RT_r(\Sigma) \to RT_r(\Sigma'),$$
  to a cobordism (M, Σ, Σ'), s.t. that compositions of cobordisms are sent to compositions of linear maps (up to powers of q)
- if $\Sigma := T^2$ a torus, the Hermitian pairing is positive definite on $RT_r(T^2)$, for all choices of the root q.

Relation to Turaev-Viro invariants

- *Roberts, Benedetti- Petronio*: If ∂M is tori, then

$$TV_r(M, q^2) = \langle RT_r(M, q), \overline{RT_r(M, q)} \rangle = ||RT_r(M, q)||^2, \tag{2}$$

where $|| \cdot ||$ is the natural Hermitian norm on $RT_r(\partial M)$.

- **Note 1.** Back to the choice of root $q = e^{\frac{2\pi i}{r}}$ for the Chen-Yang volume conjecture:
- We want to choose a root where the corresponding Reshetikhin-Turaev TQFT is non-unitary.
- Why? Because at roots where the TQFT is unitary (e.g. $q = e^{\frac{\pi i}{r}}$) the r-growth of Turaev-Viro invariants is polynomial.

- **Note 2.** In all cases where the volume conjecture is verified $TV_r(M, q^2)$ is views through Equation (2).

For Example: Links complements in S^3.

- For link complements $TV_r(S^3 \smallsetminus K, e^{\frac{2\pi i}{r}})$ are obtained from colored Jones link polynomial.
- (Detcherry-K.-Yang, 2017) For $K \subset S^3$ a knot and $r = 2m + 1$ there is a constant η_r independent of K so that

$$TV_r(S^3 \smallsetminus K, e^{\frac{2\pi i}{r}}) = \eta_r^2 \sum_{n=1}^{m} |J_K^n(e^{\frac{4\pi i}{r}})|^2.$$

- For links, above expression uses the multicolored Jones polynomial.
- For links in S^3 the asymptotics of the Turaev-Viro invariants are studied through the asymptotics of the colored Jones polynomial: e.g.
  1. Figure eight, Whitehead link, Borromean rings, Twist knots.
  2. Twist knots
  3. Volume zero knots...

Deviation: Dominating summand.

- Working with examples (of both knots and links) we noted that the large r asymptotics of $TV_r(S^3 \setminus K, e^{\frac{2\pi i}{r}})$ were dominated and realized by the term $J_{L,m}(e^{\frac{2\pi i}{m+\frac{1}{2}}})$. We asked:
- **Question** Is it true that for any hyperbolic link L in S^3, we have

$$\lim_{m \to +\infty} \frac{2\pi}{m} \log |J_{L,m}(e^{\frac{2\pi i}{m+\frac{1}{2}}})| = \mathrm{Vol}(S^3 \setminus L)?$$

- For all links/knots where the Chen-Yang Conjecture has been verified above Questions has affirmative answer.
- **Note.** Recall that the Kashaev-Murakami-Murakami VC is about $J_{L,m}(e^{\frac{2\pi i}{m}})$.

Glueing along tori:

- $S = (B; \frac{q_1}{p_1} \dots \frac{q_n}{p_n})$, Seifert fibered 3-manifold, with orbifold B, and $p_1, \cdots, p_n$ are the multiplicities of exceptional fibers.
- Recall M has toroidal boundary.

  Detcherry-K.- Marasinghe, 2025:

Theorem

(Theorem A) Let S be as above and with at least two boundary components. Then, for any 3-manifold M' obtained by gluing S along a component of $T' \subset \partial S$ to a component of ∂M,

$$\frac{r^{-K}}{A} TV_r(M) \leqslant TV_r(M') \leqslant A r^K TV_r(M),$$

for some constants A and $K > 0$ and r coprime to $p_1, \cdots, p_n$.

- Theorem implies that if M' satisfies the volume conjecture them M' does as well.

More specifically...

- Some care is needed with the $\limsup$. But, for example,
- If

$$LTV(M) := \lim_{r\to\infty} \frac{2\pi}{r} \log(TV_r(M, e^{\frac{2\pi i}{r}})) = \mathrm{Vol}(M).$$

- then

$$LTV(M') := \limsup_{r\to\infty} \frac{2\pi}{r} \log(TV_r(M', e^{\frac{2\pi i}{r}})) = \mathrm{Vol}(M').$$

- For hyperbolic M the conjecture is stated with “limit” and this is how it is verified for the classes of hyperbolic 3-manifolds for which it is know.
- Apply to satellite operations for links in S^3:
- *Example.*If L is a link obtained as a satellite of the figure-eight with pattern an iterated torus link, then

$$LTV(S^3 \setminus L) = \mathrm{Vol}(S^3 \setminus L) \approx 2.0298832.$$

Other implications:Volume zero case

- If $\mathrm{Vol}(M) = 0$, then

$$LTV(M) = 0 \quad \text{iff} \quad LTV(\mathrm{M}') = 0.$$

- *Corollary 1*: Suppose that S is an oriented Seifert fibered 3-manifold that either has a non-empty boundary, or it is closed and admits an orientation reversing involution. Then we have

$$LTV(S) = \limsup_{r\to\infty,\ r \text{ odd}} \frac{2\pi}{r} \log |TV_r(S)| = \mathrm{Vol}(S) = 0.$$

- *Plumbed 3-manifold*: Decomposes into Seifert fibered 3-manifolds along tori with decomposition graph a tree:
- *Corollary 2*: Let G be plumbed manifold with non-empty boundary and with an associated tree $T(G)$ where all but at most one leaf is a 3-manifold with at least one boundary component coming from ∂G. Then,

$$LTV(G) = \mathrm{Vol}(G) = 0.$$

LTV upper bounds

- *Detcherry, K. 2017*:
  1. For any compact orientable 3-manifold M with empty or toroidal boundary we have
  $$LTV(M) \leqslant C\mathrm{Vol}(M),$$
  where $C > 0$ is a universal constant.
  2. If M' is obtained by glueing any M and M_1 along a boundary torus then:
  $$TV_r(M') \leqslant TV_r(M) \cdot TV_r(M_1) \quad \text{and so} \quad LTV(M') \leqslant LTV(M) + LTV(M_1).$$

- **Conclusion.** If $M_1 = S$= Seifert fibered 3-manifold, then $TV_r(S)$ has at most exponential exponential growth and we get
$$TV_r(M') \leqslant Ar^K TV_r(M),$$
for some constants A, $N > 0$ and for all for all odd $r > 1$.

- **Aside.** Simplicial volume is also subadditive: $\mathrm{Vol}(M') \leqslant \mathrm{Vol}(M') + \mathrm{Vol}(M_1)$ and if the torus we glue is incompressible then Vol is additive.

Invertible TQFT operators:

- (S, T^2, T^2) a cobordism from torus to torus. (For simplicity, think that M has one boundary component.)
- By TQFT properties

$$RT_r(M') = RT_r(S)(RT_r(M)),$$

where $RT_r(S) : RT_r(T^2) \to RT_r(T^2)$, is the TQFT linear operator.

- **Observation.** If $RT_r(S)$ is invertible write $RT_r(S)^{-1}(RT_r(M')) = RT_r(M)$, and hence

$$||RT_r(M)|| \leqslant |||RT_r(S)^{-1}||| \cdot ||RT_r(M')||,$$

where $|||\cdot|||$ = the operator norm of linear map:

$$|||RT_r(S)^{-1}||| := \max_{||x||=1} ||RT_r(S)^{-1}(x)||, \quad x \in RT_r(T^2).$$

- We get

$$|||RT_r(S)^{-1}|||^{-1} \cdot ||RT_r(M)|| \leqslant ||RT_r(M')||.$$

- Knowing the r-growth of $|||RT_r(S)^{-1}|||$, will give info about the growth of $||RT_r(M')||$!

Back to Seifert fibered spaces:

- Key Step:

Theorem

For $S = S(B; \frac{q_1}{p_1} \dots \frac{q_n}{p_n})$ a Seifert fibered 3-manifold, with two boundary components, the linear map $RT_r(S): \ RT_r(T) \to RT_r(T')$, is invertible for all odd r coprime to $p_1, \cdots, p_n$. Furthermore, there are constants C and $N > 0$ such that

$$|||RT_r(S)^{-1}||| \leqslant C\, R^N.$$

- Hence, get Theorem A

$$\frac{r^{-N}}{C} TV_r(M) \leqslant TV_r(M').$$

- **Idea:** Decompose the cobordism $S : T^2 \longrightarrow T^2$ into f "nice" cobordisms

$$\{S_i : T^2 \longrightarrow T^2\}_i.$$

- Reduce the proof to proving the result for the TQFT maps $RT_r(S_i)$.

What are "nice" cobordisms?

- $S = S(B; \frac{q_1}{p_1} \dots \frac{q_n}{p_n})$, Seifert manifold viewed as cobordim $T^2 \longrightarrow T^2$.
- Find a collection $\mathcal{T}$ of term *essential* embedded tori in S, such that
- each is *vertical* with respect to the Seifert fibration,
- Each component S_i of S cut along $\mathcal{T}$ is one of the following types:

  1. The trivial S^1 bundle over a torus with two holes.
  2. The twisted S^1-bundle over the Klein bottle with two holes.
  3. The twisted S^1-bundle over the Mobius band with one hole.
  4. S_i admits a Seifert fibration over an annulus with one exceptional fiber.

- We use particular SO_3-TQFT properties to compute the eigenvalues of $RT_r(S_i)$, and or cases (1)-(3) show invertibility (for all r) and that

$$|||RT_r(S_i)^{-1}|||$$

  has at most polynomial growth as $r \to \infty$.
- For (4) these properties hold for r not divisible by the multiplicity of the exceptional fiber.

- We choose $\mathcal{T}$ so that cobordism $S : T^2 \longrightarrow T^2$ is a composition of

$$S = S_n S_{n_1} \dots S_0,$$

where $S_i : T^2 \longrightarrow T^2$ are "nice" cobordisms as above.

- Now (up to a power of q)

$$RT_r(S) = RT_r(S_m) \circ \cdots \circ RT_r(S_1) \circ RT_r(S_0),$$

- and, hence, the map $RT_r(S) : \ RT_r(T^2) \longrightarrow RT_r(T^2)$ is invertible

$$|||RT_r(S)^{-1}|||$$

has at most polynomial growth. □

- **Note.** The case when ∂M has more than one component and ∂S has more than two components needs more delicate but similar arguments.

Hyperbolic Cobordisms?

- Let $N : T^2 \longrightarrow T^2$, a hyperbolic cobordism.
- Suppose the TQFT map $RT_r(S) : RT_r(T^2) \to RT_r(T^2)$, is invertible and let M a 3-manifold with toroidal incompressible boundary.
- Glue N to M along a boundary torus to obtain a 3-manifold M'.
- Recall that $\mathrm{Vol}(M') = \mathrm{Vol}(N) + \mathrm{Vol}(M)$.
- Recall that

$$|||RT_r(N)^{-1}|||^{-1} \cdot ||RT_r(M)|| \leqslant ||RT_r(M')|| \leqslant ||RT_r(N)|| \cdot ||RT_r(M)||.$$

- We'd like to have cobordisms with exponential r-growth and ideally

$$\log(|||RT_r(N)^{-1}|||^{-1}) \sim rLTV(N), \quad \text{for} \quad r \to \infty,$$

hoping to get a lower bound

$$LTV(M') \geq LTV(M) + LTV(N).$$

Hyperbolic Cobordisms: Cont'n

- **Some bad news:** We can't hope to have this for all hyperbolic cobordisms. Why?
- Suppose that there is a slope **s** on one of the components of ∂N so that the 3-manifold $N(\mathbf{s})$ obtained by Dehn filling N along **s** has zero simplicial volume. (e.g. its a Seifert fibered 3-manifold).
- Then, TQFT arguments imply that $|||RT_r(M)^{-1}|||^{-1}$ grows at most polynomially in r!
- Constructions of hyperbolic 3-manifolds N where all Dehn fillings along any single component of ∂M produce hyperbolic manifolds are abundant! Need make sure that all the slopes on each cusp have length ¿6.
- Take N to be the complement of any "highly twisted", prime link in S^3!

Problem. Construct hyperbolic cobordisms $N : T^2 \longrightarrow T^2$ such that $RT_r(N): \ RT_r(T^2) \to RT_r(T^2)$ is invertible for all prime r such that and $|||RT_r(S)^{-1}|||^{-1}$ grows exponentially with r.

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