Jones diameter and crossing number of knots

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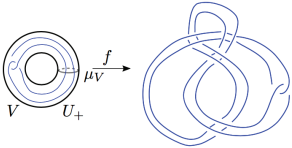
Crossing numbers:

- Given a knot K, the crossing number $c(K)$ is the smallest number of crossings over all knot diagrams representing K.
- Hard to calculate for arbitrary knots.
- Behavior under basic topological operations (e.g. connected sum, satellite operations) still poorly understood.
- **Well known Conjectures:**
- Crossing number is additive under connected sum (open).
- If K is satellite with companion J, then $c(K) > c(J)$ (open).
- **Figure:** *Untwisted Whitehead double* of figure-8 knot: $W(K)$, $K = 4_1$ Is this a minimum crossing number diagram? Is $c(W(K)) = 18$?
- Untwisted:=the framing of the satellite construction is zero.



Known results

- **General bounds:**
- (Lakenby, 2005) For any knots $K_1, \dots K_n$ we have

$$c(K_1) + \cdots + c(K_n) \geq c(K_1 \# \dots \# K_n) \geq \frac{c(K_1) + \cdots + c(K_n)}{152},$$

  so crossing number increases when number connected summands does.

- (Lakenby, 2005) If K is satellite with companion J then,

$$c(K) \geq 10^{-13} c(J).$$

- General bounds not good enough to be used for determination of crossing numbers of any knots.
- **Exact results for classes of knots/links:**
- (Murasugi) Torus knots: For $p, q > 0$, $T_{(p,q)} = (p, q)$-torus knots, then

$$c(T_{(p,q)}) = \min((p-1)q, (q-1)p).$$

Exact results for classes, cont.

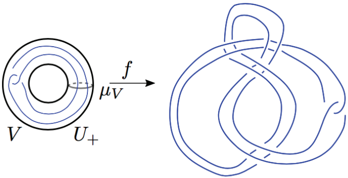
- (Kauffman, Murasugi, Thistlethwaite, 80's) Alternating Knots (more generally or adequate knots).
- (Tait Conjecture) A reduced alternating diagram (or adequate) of K realizes $c(K)$.
- Additivity Conjecture holds for adequate knots (Kauffman, Murasugi, Thistlethwaite)
- The writhe number of an adequate diagram $D = D(K)$ is invariant of K.
- (Lickorish-Thistlethwaite, 80's) Crossing numbers for Montesinos knots.
- In above cases a "special" diagram of K gives $c(K)$.
- (K.-Lee, '21) Crossing nos of first infinite families of prime satellites:

Theorem

Let $W(K)$=untwisted Whitehead double of a knot K. If K is adequate with writhe number zero, then $c(W(K)) = 4.c(K) + 2$.

Exact results for classes, cont.

- "Doubling" an adequate diagram $D = D(K)$, with writhe zero, produces a minimum crossing number of $W(K)$.
- Crossing number of untwisted Whitehead doubles of figure-8 is 18.



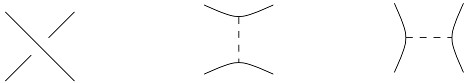
- Plenty of adequate knots with zero writhe number:

Corollary

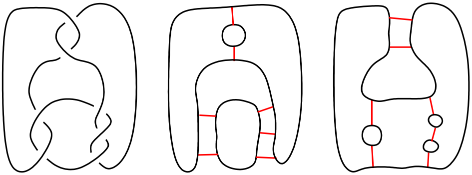
If K is adequate, with mirror image K^*, then $c(W(K \# K^*)) = 8.c(K) + 2$.

Alternating/Adequate knots

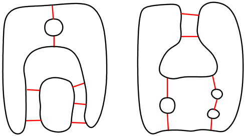
Two choices for each crossing, of knot diagram D: A or B resolution.



- A Kauffman *state* $\sigma(D)$ is a choice of A or B resolutions for all crossings.
- $\sigma(D)$: *state circles*.
- Form a *fat graph* H_σ by adding edges at resolved crossings.



- K is called *A–adequate* if has a diagram $D = D(K)$ where the all-A state graph $H_A = H_A(D)$ has no 1-edge loops.
- Similarly we have B-adequate
- Left: graph from adequate state. Right: Graph from inadequate state.



- K is *adequate* if it admits a diagram that is both *A and B–adequate*.
- Introduced by (Lickorish–Thistlethwaite, 80's). Alternating knots are adequate but there is more.
- Properties of the Jones polynomial were used to determine crossing numbers of adequate knots and to prove the Tait Conjectures: The degree span of the Jones polynomial of an alternating knot gives the crossing number.
- For adequate knots, the crossing number is determined by looking at Jones polynomials of the "parallels" of a knot.

The colored Jones polynomial

- For non-adequate knots (with Lee) we use the *colored Jones polynomials*.
- Colored Jones function: sequence $\{J_K(n)\}_n$ of Laurent pol. t.
- The Jones polynomial corresponds to $n = 2$.
- (Garoufalidis - Le, 2005) $\{J_K(n)\}$ satisfies a l linear recurrence relation

$$a_d(t^{2n}, t)J_K(n+d) + \cdots + a_0(t^{2n}, t)J_K(n) = 0$$

  for all n, where $a_j(u, v) \in [u, v]$. *q-holonomicity*.
- Example: for the trefoil the colored Jones polynomial is

$$J_K(n) = t^{-6(n^2-1)} \sum_{j=-\frac{n-1}{2}}^{\frac{n-1}{2}} t^{24j^2+12j} \frac{t^{8j+2} - t^{-(8j+2)}}{t^2 - t^{-2}}.$$

- Recurrence relation

$$(t^{8n+12} - 1)J_K(n+2) + (t^{-4n-6} - t^{-12n-10} - t^{8n+10} + t^{-2})J_K(n+1)$$
$$-(t^{-4n+4} - t^{-12n-8})J_K(n) = 0.$$

Impact of q-holonomicity on the degree of CJP

- Let $d_+[J_K(n)]$ and $d_-[J_K(n)]$ denote the maximal and minimal degree of $J_K(n)$ in t, and set

$$d[J_K(n)] := 4d_+[J_K(n)] - 4d_-[J_K(n)] := s_2(n)n^2 + s_1(n)n + s_0(n),$$

$$s_i : \mathbb{N} \longrightarrow \mathbb{Q}, \quad i = 0, 1, 2.$$

- " q-holonomicity" implies that the set of cluster points $\{s_2(n)\}'_{n\in\mathbb{N}}$ is finite.
- Point with the largest absolute value, denoted by dj_K, is called the *Jones diameter* of K.

Theorem

(Lickorish-Thistlethwaite, 80's) For any knot we have

$$dj_K \leq 2c(K),$$

where $c(K)$ is the crossing number of K.
If K is adequate then we have equality.

- With Lee we prove the converse: $dj_K = 2c(K)$, implies K is adequate.

Knots of maximal Jones diameter

- K.-Lee, 2021:

Theorem

Let K be a knot with Jones diameter dj_K and crossing number $c(K)$. Then,

$$dj_K \leq 2c(K),$$

with equality $dj_K = 2c(K)$ if and only if K is adequate.

- In fact, we show:
- Suppose a knot K admits a diagram $D = D(K)$, with $c := c(D)$, crossings and such that $dj_K = 2c(D)$. Then D must be an adequate adequate diagram.
- So if D realizes $c(K)$ and $dj_K = 2c(D) = 2c(K) =$, for some knot K, then D is adequate.

Crossing number application

- Theorem has immediate corollary: A diagram with number of crossings "too close" to the Jones diameter gives the crossing number of the knot!!

Corollary

Suppose K is a non-adequate knot admitting a diagram $D = D(K)$ such that

$$dj_K = 2(c(D) - 1).$$

Then we have $c(D) = c(K)$.

Proof. Since K is non-adequate, Theorem gives that $2c(K) > dj_K$. Hence we get $c(D) \geq c(K) > \frac{dj_K}{2} = c(D) - 1$, giving $c(D) = c(K)$. $\square$

- **Example.** For $K = W(figure - 8)$, by Baker-Motegi-Takata, $dj_K = 34 = 2.17 = 2(18 - 1)$.
- Doubling the standard diagram of figure-8 produces a diagram of 18 crossings.

Proof ideas/tools:

- Masbaum-Vogel fusion theory of the $SU(2)$-quantum invariants for knots and trivalent graphs.
- If $D = D(K)$ is adequate then $dj_K = 2c(D)$: If $D = D(K)$ is non-adequate, then state graphs have loop edges.
- Understand contribution to the degree of CJP of crossings of D producing edge loops. Show that $d[J_K(n)] \leq (2c(D) - q(D))n^2 + O(n)$, for some $q := q(D) > 0$.
- Crossing number applications come from Corollary.
- Start with $D = D(K)$ adequate diagram of zero writhe. The "Double" of D is a diagram for the Whitehead double $W(K)$ with $x := 4c(D) + 2$ crossings.
- Use a result of Baker-Motegi-Takata (2019) to calculate the Jones diameter of $W(K)$. It is equal to $2(x - 1)$.
- Show that the Whitehead double $W(K)$ is not an adequate knot (the tricky part).
- Apply Corollary to conclude that $c(W(K)) = 4c(D) + 2$.

Doubles of amphicheiral knots

- If K is amphicheiral adequate knot then $\mathrm{wr}(K) = 0$.

Corollary

Suppose that K is an amphicheiral adequate knot with crossing number $c(K)$. Then $c(W(K)) = 4c(K) + 2$.

- For any even $n > 0$ there are alternating, amphicheiral knots $c(K) = n$.
- K= figure-8 knot is the 1st example: We have

$$c(W(\#_m K)) = 16m + 2.$$

- Prime amphicheiral adequate knots with $C(K) \leq 12$. (Knotinfo Cha-Livingston-Moore).

4_1	8_{18}	10_{43}	$12a_{435}$	$12a_{506}$	$12a_{1105}$	$12a_{1275}$
6_3	10_{17}	10_{45}	$12a_{471}$	$12a_{510}$	$12a_{1127}$	$12a_{1281}$
8_3	10_{33}	10_{99}	$12a_{477}$	$12a_{1019}$	$12a_{1202}$	$12a_{1287}$
8_9	10_{37}	10_{123}	$12a_{499}$	$12a_{1039}$	$12a_{1273}$	$12a_{1288}$

- Out of the 2977 prime knots with up to 12 crossings, 1851 are listed as adequate on Knotinfo and thus Corollary applies to $K\#K^*$.