Skein modules, character varieties and essential surfaces of 3-manifolds

joint w/ R. Detcherry and A. Sikora.

E. Kalfagianni, Michigan State University

Quantum Topology, Quantum Information and connections to Mathematical Physics (in honor of Professor Arthur Jaffe's 85th birthday), May 27 - 31, 2024 Texas A&M .

Kauffman Bracket Skein Module

- M= oriented, compact 3-manifold.
- R= commutative ring R with a distinguished invertible element $A \in R$
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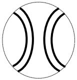
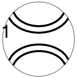
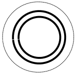
K1:  $= A$  $+ A^{-1}$ 

K2: $L \sqcup$  $= (-A^2 - A^{-2})L$

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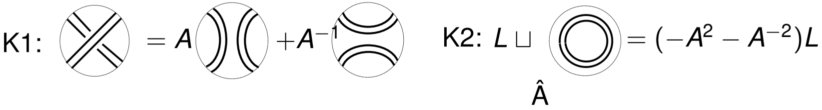

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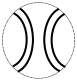
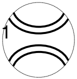
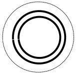
K1:  $= A$  $+A^{-1}$    K2: $L \sqcup$  $= (-A^2 - A^{-2})L$

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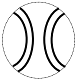
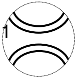
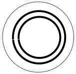
K1:  $= A$  $+A^{-1}$  K2: $L \sqcup$  $= (-A^2 - A^{-2})L$

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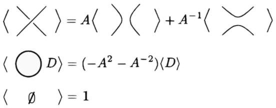
- *Surface skein algebra*: $\mathcal{S}(M)$, for $M = \text{surface} \times I$.
  - Used in constructions of Witten-Reshetikhin -Turaev TQFT theories.
  - Relations with character varieties, cluster algebras, quantum field theories, hyperbolic geometry.....
- In this talk: M= closed 3-manifold, and $R = \mathbb{Z}[A^{\pm 1}]$ or $R = \mathbb{Q}(A)$.

Examples

- $\mathcal{S}(S^3, R) \cong R$, generated by the *empty link*:
- *Kauffman bracket:* $\langle\ \rangle$: link diagrams $\longrightarrow R$ such that



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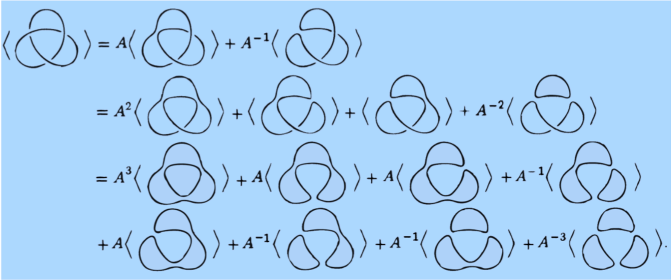

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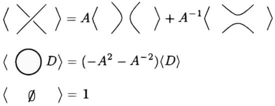

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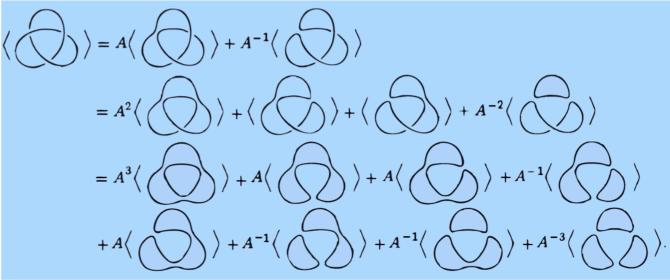
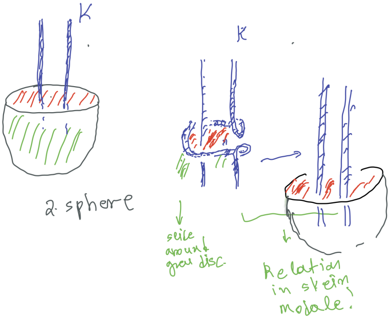


- We obtain: $K = J(A, A^{-1}) \cdot \emptyset$, $J(A, A^{-1})$= Jones polyn. of framed K.

What about other 3-manifolds?

- How is $\pi_1(M)$ reflected in $\mathbb{Z}[A^{\pm 1}]$? more later
- Embedded surfaces: e.g. non separating spheres?



- Such spheres create torsion in $\mathcal{S}(S^2 \times S^1, \mathbb{Z}[A^{\pm 1}])$ (Hoste-Przytycki, 90's). Similar phenomena with other π_1-injective surfaces.

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- (Gilmer-Masbaum, Detcherry-Wolf, 2020) Σ_g=genus g surface

$$\dim_{\mathbb{Q}(A)} S(\Sigma_g \times S^1, \mathbb{Q}(A)) = 2^{2g+1} + 2g - 1.$$

Character variety connection

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- **Fact 2:** $\rho, \rho' : \pi_1(M) \to SL(2, \mathbb{C})$ are identified in $X(M)$ iff $tr\rho = tr\rho'$.

Character variety connection, con't

- The skein module "at $A = -1$ gives is the coordinate ring of the character variety".
- Specifically: Let

$$S_{-1}(M) := S(M, \mathbb{Z}[A^{\pm 1}]) \otimes_{\mathbb{Z}[A^{\pm 1}]} \mathbb{C},$$

where the $\mathbb{Z}[A^{\pm 1}]$-module structure of $\mathbb{C}$ is given by sending A to -1.

Theorem (Przytycki-Sikora, 2000)

$S_{-1}(M)$ has the structure of $\mathbb{C}$-algebra that is isomorphic to the coordinate ring $\mathbb{C}[\mathcal{X}(M)]$.

- The isomorphism:

$$\psi : S_{-1}(M) \longrightarrow \mathbb{C}[\mathcal{X}(M)] \;\; \text{sends} \;\; K \;\; \longrightarrow \;\; -t_K,$$

for any knot.

- *Trace function*: $t_K : \mathbb{C}[\mathcal{X}(M)] \longrightarrow \mathbb{C}, \quad t_K([\rho]) = Tr\rho([K])$, for all $\rho : \pi_1(M) \longrightarrow \mathrm{SL}_2(\mathbb{C})$.

Two questions:

- Rest of the talk:
- **Question 1:** When is $\mathcal{S}(M)$ finitely generating over $\mathbb{Z}[A^{\pm 1}]$?
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- However:
- there are M containing essential surfaces but $X(M)$ is finite!

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- Examples of M with essential surfaces and finite $X(M)$

Theorem (Mroczkowski, 2011, Belletti-Detcherry 2024)

$\mathcal{S}(M, \mathbb{Z}[A^{\pm 1}])$ is not finitely generated $M := \mathbb{R}P^3 \# L(2p, 1)$, for any $p > 1$.

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- Conjecture A is true for M with infinite $X(M)$:

Theorem (Detcherry-K.-Sikora, 2023)

If $X(M)$ is infinite, then $\mathcal{S}(M, \mathbb{Z}[A^{\pm 1}])$ is not finitely generated.

- Examples of M with essential surfaces and finite $X(M)$

Theorem (Mroczkowski, 2011, Belletti-Detcherry 2024)

$\mathcal{S}(M, \mathbb{Z}[A^{\pm 1}])$ is not finitely generated $M := \mathbb{R}P^3 \# L(2p, 1)$, for any $p > 1$.

- (Detcherry-K.-Sikora, 2024): Conjecture A for all *Seifert fibered manifolds*

Theorem (*Theorem B*)

A Seifert 3-manifold M contains no essential surfaces if and only if $\mathcal{S}(M, \mathbb{Z}[A^{\pm 1}])$ is finitely generated.

Applications: What is $\dim_{\mathbb{Q}(A)} \mathcal{S}(M)$?

- Recall: At $A = -1$, the skein module $S_{-1}(M)$ is the coordinate ring the $\mathrm{SL}_2(\mathbb{C})$-character variety of M.
- **Question.**How does the skein module $\mathcal{S}(M) := \mathcal{S}(M, \mathbb{Q}(A))$ relate to $\mathcal{X}(M)$ and $X(M)$ for generic values of A?

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Theorem (Detcherry-K.-Sikora, 2023)

If M is a closed 3-manifold with fiinitely generated $S(M, \mathbb{Z}[A^{\pm 1}])$, then

$$|X(M)| \leq \dim_{\mathbb{Q}(A)} \mathcal{S}(M) \leq \dim_{\mathbb{C}} \mathbb{C}[\mathcal{X}(M)].$$

In particular, if $\mathcal{X}(M)$ is reduced, then $\dim_{\mathbb{Q}(A)} \mathcal{S}(M) = |X(M)|$.

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- If $\mathcal{X}(M)$ is reduced we also.
  1. obtain information about torsion in $S(M, \mathbb{Z}[A^{\pm 1}])$;
  2. we have a method to obtain a basis of $\mathcal{S}(M)$ from one of $\mathbb{C}[X(M)]$.

Applications cont'

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- Computed $\mathcal{X}(M)$ for all *Seifert 3-manifolds* with out essential surfaces, determined when it is reduced and computed $\dim_{\mathbb{Q}(A)} \mathcal{S}(M)$.

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- For instance: $M := \Sigma(p_1, p_2, p_3)$ is a Brieskorn spheres

$$\dim_{\mathbb{Q}(A)} \mathcal{S}(M) = 1 + \frac{(p_1 - 1)(p_2 - 1)(p_3 - 1)}{4}.$$

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- It is conjectured that this is always true.

Outline of proof of Theorem B:

- M=Seifert fibered 3-manifold

$\mathcal{S}(M, \mathbb{Z}[A^{\pm 1}])$ is finitely generated $\Leftrightarrow$ M contains no essential surfaces.

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$$\mathcal{S}(M, \mathbb{Z}[A^{\pm 1}]) \text{ finitely generated} \Rightarrow X(M) \text{ is finite}$$

This is true for all 3-manifolds! The proof relies on

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- *implication* $\Leftarrow$:

1. Use topological and character variety properties/results of Seifert fibered 3-manifolds to reduce the problem to a special class of Seifert fibered 3-manifolds: They fiber over S^2 with three exceptional spheres and have non-zero Euler number.
2. Use Skein-theoretic techniques to prove that $\mathcal{S}(M, \mathbb{Z}[A^{\pm 1}])$ is finitely generated, for the special class of 3-manifolds.

The manifolds $M = M(S^2; \frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3})$

- Start with $S_{0,3} \times S^1$ ($S_{0,3}$="pair of pants")
- Obtain M by attaching solid V_1, V_2, V_3 tori to ∂N with meridians attached to curves of slopes $\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3}$.
- Euler number $e(M) := \frac{q_1}{p_1} + \frac{q_2}{p_2} + \frac{q_3}{p_3}$.
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- Using the Frohman-Gelca basis for skein algebras of tori the skein module $\mathcal{S}(S_{0,3} \times S^1, \mathbb{Z}[A^{\pm 1}])$ corresponds to a subspace of $\mathbb{Z}^6$.
- Adding the solid tori V_i leads to between generators of $\mathcal{S}(S_{0,3} \times S^1, \mathbb{Z}[A^{\pm 1}])$ and a presentation of $\mathcal{S}(M, \mathbb{Z}[A^{\pm 1}])$.
- This perspective allows to obtain an a complexity on $\mathcal{S}(M, \mathbb{Z}[A^{\pm 1}])$ that under the hypothesis that $e(M) \neq 0$, can be used to reduce the set of generators to finitely many.

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