SEIFERT COBORDISMS AND THE CHEN-YANG VOLUME CONJECTURE

R. DETCHERRY, E. KALFAGIANNI AND S. MARASINGHE

Abstract. We study the large r asymptotic behavior of the Turaev-Viro invariants $TV_r(M; e^{\frac{2\pi i}{r}})$ of 3-manifolds with toroidal boundary, under the operation of gluing a Seifert-fibered 3-manifold along a component of ∂M. We show that the Turaev-Viro invariants volume conjecture is closed under this operation. As an application we prove the volume conjecture for all Seifert fibered 3-manifolds with boundary and for large classes of graph 3-manifolds.

*K*eywords: simplicial volume, plumbed manifold, Seifert fibered manifold, Turaev-Viro invariants, volume conjecture.

1. Introduction

Given a compact 3-manifold M the Turaev-Viro invariants $TV_r(M; q^2)$ are a family of real-valued invariants depending on an odd integer $r \geq 3$ and a primitive $2r$-th root of unity q. In this paper, we are concerned with the case of $q = e^{\frac{\pi i}{r}}$. The invariants were originally constructed via state sums on triangulations of 3-manifolds [24] and were later related to skein-theoretic quantum constructions of Reshetikhin-Turaev invariants [2, 3, 12, 21]. In this paper we will follow this viewpoint. We will view $TV_r(M; q)$ through its relation to the skein theoretic SO_3-TQFT as constructed by Blanchet, Habegger, Masbaum and Vogel [4, 5].

All the 3-manifolds considered in this paper will be orientable and either closed or with boundary consisting of tori (i.e. toroidal boundary) We prove the following:

Theorem 1.1. *Let S be a Seifert fibered 3-manifold with at least two boundary components and let M be any 3-manifold with toroidal boundary. Then, for any 3-manifold M' obtained by gluing S along a component of ∂S to a component of ∂M, there exist constants A and $K > 0$, and a finite set of integers I, such that*

$$\frac{r^{-K}}{A} TV_r(M) \leqslant TV_r(M') \leqslant A r^K TV_r(M),$$

for all odd r not divisible by any $p \in I$.

Some of the most prominent open problems in quantum topology are the volume conjectures, asserting that geometric invariants of 3-manifolds (e.g. hyperbolic volume) are determined by quantum invariants. Theorem 1.1 has applications to the Turaev-Viro invariants volume conjecture. The conjecture, that is a natural 3-manifold

MSC 57K31, 57K16, 57M25.

generalization of the well known Kashaev, Murakami and Murakami [7] conjecture, asserts that the large r asymptotics of the Turaev-Viro invariants determine the simplicial volume of 3-manifolds. Specifically, Chen and Yang [8] conjectured that for hyperbolic manifolds of finite volume, the growth rate of the Turaev-Viro invariants is exponential and it determines the hyperbolic volume of the manifold.

By the geometrization theorem, any 3-manifold M with empty or toroidal boundary admits a canonical decomposition, along essential spheres and tori, into pieces that are either hyperbolic or Seifert fibered spaces. We will refer to this as the geometric decomposition of M. The simplicial volume $\mathrm{Vol}(M)$ of M is defined as the sum of the volumes of the hyperbolic pieces in this decomposition and it is equal to its Gromov norm times $v_3 \approx 1.01494$ [23], which is the hyperbolic volume of a regular ideal hyperbolic tetrahedron. The simplicial volume is additive under disjoint unions and connected sums of 3-manifolds as well us under gluing along essential tori. The following generalization of the Chen-Yang Conjecture was stated in [11].

Conjecture 1.2. *For every compact orientable 3-manifold M, with empty or toroidal boundary, we have*

$$LTV(M) := \limsup_{r\to\infty,\ r\ odd} \frac{2\pi}{r} \log |TV_r(M)| = \mathrm{Vol}(M),$$

where r runs over all odd integers.

The upper inequality of Theorem 1.1 follows from [11] and in fact it holds for all odd $r \geq 3$. The theorem implies that if $\mathrm{Vol}(M) = 0$, then we have $LTV(M) = LTV(M')$. As a corollary we have the following:

Corollary 1.3. *Suppose that S is an oriented Seifert fibered 3-manifold that either has a non-empty boundary, or it is closed and admits an orientation reversing involution. Then we have*

$$LTV(S) = \limsup_{r\to\infty,\ r\ odd} \frac{2\pi}{r} \log |TV_r(S)| = \mathrm{Vol}(S) = 0.$$

Theorem 1.1 generalizes to large families of 3-manifolds obtained by gluing together Seifert fibered 3-manifolds (Corollary 4.5). As a result in Corollary 5.3 we also verify Conjecture 1.2 for these manifolds.

We note that if M satisfies Conjecture 1.2 with "limsup" in the definition of $LTV(M)$ is actually a limit, then Theorem 1.1 implies that M' also satisfies the conjecture. For hyperbolic M, the Chen-Yang conjecture is stated for $LTV(M)$ being the limit and in this form it has been verified for large families 3-manifolds with cusps. We note however that the restriction to "$\limsup$" for general 3-manifolds is necessary. Indeed for Seifert fibered spaces the invariants $TV_r(M; q)$ can vanish for infinitely many integers r. See, for example, [19].

The hyperbolic links in S^3 for which the volume conjecture has been verified include the figure-eight knot, the Borromean rings [12], the twist knots [9], the Whitehead chains [25], and large families of octahedral links in S^3 including the octahedral augmented links [16, 26]. The conjecture has also been verified for fundamental shadow links [2] which form a class of hyperbolic links in connected sums of $S^1 \times S^2$ that gives all orientable 3-manifolds that are either closed or with toroidal boundary by Dehn filling, and for additional families of hyperbolic links in connected sums of $S^1 \times S^2$ [1]. For infinite families of non-hyperbolic links where Conjecture 1.2 holds when $LTV(M)$ the limit, see [17]. Theorem 1.1 can be applied to any of these families of links to produce new families of satellite links satisfying Conjecture 1.2.

As an example we state the following:

Corollary 1.4. *If L is a link obtained as an iterated satellite of the figure-eight with patterns torus links, then*

$$LTV(S^3 \setminus L) = \mathrm{Vol}(S^3 \setminus L) \approx 2.0298832.$$

Forming a satellite of a knot K with pattern a torus link amounts to gluing a Seifert fibered 3-manifold with at least two boundary components to the boundary torus of the knot complement [6]. Thus, Corollary 1.4 follows from Theorem 1.1 and above discussion.

The proofs of many results in the area rely at least partially on analytic estimates and direct analysis of the asymptotics of the Reshetikhin-Turaev and Turaev -Viro invariants. See, for example, [2, 1, 9, 17, 26, 19] and references therein. In contrast our proofs in this paper rely heavily on TQFT properties and 3-manifold topology. In the process, we discuss an approach that could potentially lead to new progress towards understanding the behavior of the asymptotics of the Turaev-Viro invariants under hyperbolic Dehn filling. For details, the reader is referred to Section six.

Acknowledgement. The research of E.K. and S.M. is partially supported by NSF grant DMS-2304033. The research of R.D. is partially supported by the ANR project “NAQI-34T” (ANR-23-ERCS-0008) and by the project “CLICQ” of the Région Bourgogne-Franche Comté.

2. TQFT and Turaev-Viro invariants

In this section we recall how to obtain the Turaev-Viro invariants from the Reshetikhin-Turaev SO_3-TQFT of [20]. We begin by summarizing some basic features of the SO_3-TQFT following skein-theoretic framework of [4, 5].

2.1. **Preliminaries.** For an odd integer $r \geqslant 3$ and a primitive $2r$-th root of unity q, the SO_3-TQFT functor, denoted by RT_r, associates a finite dimensional Hermitian $\mathbb{C}$-vector space $RT_r(\Sigma)$, to any closed oriented surface Σ, such that:

(a) For a disjoint union $\Sigma \coprod \Sigma'$ one has $RT_r(\Sigma \coprod \Sigma') = RT_r(\Sigma) \otimes RT_r(\Sigma')$.
(b) For a closed oriented 3-manifold M, the value $RT_r(M) \in \mathbb{C}$ is the SO_3-Reshetikhin-Turaev invariant and if $\partial M \neq \emptyset$, $RT_r(M)$ is a vector in $RT_r(\partial M)$.
(c) If (M, Σ, Σ') is a a cobordism from a surface Σ to a surface Σ', then

$$RT_r(M): \ RT_r(\Sigma) \to RT_r(\Sigma'),$$

is a linear map such that compositions of cobordisms are sent to compositions of linear maps (up to powers of q).

(d) The 3-manifold invariants RT_r are multiplicative under disjoint union and for connected sums we have

$$RT_r(M\#M') = \eta_r^{-1} RT_r(M) RT_r(M'),$$

where $\eta_r = \dfrac{2\sin(\frac{2\pi}{r})}{\sqrt{r}}$. Furthermore we have $RT_r(S^2 \times S^1) = 1$.

Remark 2.1. In this paper we will be concerned with the question of whether the maps $RT_r(M)$ are invertible, and in the case we have inverses, we will study the r-growth rate of the operator norm of the inverses. Since these properties are not affected by multiplication by a power of q in the sequel we will assume that compositions of cobordisms are sent to compositions of linear maps RT_r.

The spaces $RT_r(\Sigma)$ are certain quotients of Kauffman bracket skein modules (at q) of a handlebody bounded by Σ. In this paper we are interested in the case where Σ is a the 2-torus T^2. In this case, [5], views T^2 as the boundary of a solid torus $D^2 \times S^1$ and obtains elements $e_i \in RT_r(T^2)$ by taking the core $\{0\} \times S^1$ of the solid torus decorated with the $i-1$ Jones-Wenzl idempotent. For $r = 2m+1$ and q a $2r$-th root of unity, this process gives a family $e_1, \ldots, e_{2m-1}$ of elements in $RT_r(T^2)$ [5, Lemma 3.2]. We have the following:

Theorem 2.2. [5, Theorem 4.10] *For $r = 2m+1 \geqslant 3$, the Hermitian pairing of $RT_r(T^2)$ is positive definite and the family $e_1, e_2, \ldots e_m$ is an orthonormal basis. Moreover, for $0 \leqslant i \leqslant m-1$, we have $e_{m-i} = e_{m+1+i}$.*

The Turaev-Viro invariants of compact oriented 3-manifolds originally were defined as state sums over triangulations of manifolds (see [24]). In this paper however, we will only use the relation between the Turaev-Viro invariants and Reshetikhin-Turaev invariants. This relation was first proved by Roberts [21] in the case of closed 3-manifolds, and extended to manifolds with boundary by Benedetti and Petronio [3]. We state it only in the case of manifolds with toroidal boundary, which is what we need.

Theorem 2.3. *Let M be a compact oriented manifold with toroidal boundary, let $r \geqslant 3$ be an odd integer and let q be a primitive $2r$-th root of unity. Then,*

$$TV_r(M, q^2) = ||RT_r(M, q)||^2$$

where $||\cdot||$ is the natural Hermitian norm on $RT_r(\partial M)$.

Since for T^2 a torus, the natural Hermitian form on $RT_r(T^2)$ is definite positive for any q, the invariants $TV_r(M)$ are non-negative.

Remark 2.4. Given a finite dimensional Hermitian $\mathbb{C}$-vector space V, with a positive Hermitian pairing $\langle ., . \rangle : V \times V \longrightarrow \mathbb{C}$, as above, we use $||.||$ to denote the norm induced by the Hermitian pairing (i.e. $||x||^2 := \langle x, x \rangle$). Given a linear map $A : V \longrightarrow V$, we will use $|||A|||$ to denote the norm of the operator, that is

$$|||A||| := \max_{||x||=1} ||A(x)||.$$

where $x \in RT_r(T')$. We also define

$$n(A) := \min_{||x||=1} ||A(x)||.$$

If we assume that A is invertible, then for any $x \in V$, one has

$$||x|| = ||A^{-1}(Ax)|| \leq |||A^{-1}||| \cdot ||Ax||,$$

with equality for some choice of x. Inverting the inequality, one gets:

$$n(A) = |||A^{-1}|||^{-1}.$$

Finally, if A is Hermitian, that is, $\langle Ax, y\rangle = \langle x, Ay\rangle$ for any $x, y \in V$, then A has an orthonormal basis of diagonalization, and

$$|||A||| = \max_{\lambda \in \mathrm{Spec}(A)} (|\lambda|) \quad \text{and} \quad n(A) = \min_{\lambda \in \mathrm{Spec}(A)} (|\lambda|).$$

2.2. **Genus one mutation.** Consider $T^2 \simeq \mathbb{R}^2/\mathbb{Z}^2$ as the quotient of $\mathbb{R}^2$ where $\pi_1(T^2) \simeq \mathbb{Z}^2$ acts by covering translations. The elliptic involution ι on T^2 is defined by

$$\begin{array}{cccccc} \iota & : & T^2 \simeq \mathbb{R}^2/\mathbb{Z}^2 & \longrightarrow & T^2 \simeq \mathbb{R}^2/\mathbb{Z}^2 \\ & & (x,y) & \longrightarrow & (-x,-y). \end{array},$$

and its isotopy class defines an element in the mapping class group of the torus $\Gamma(T^2)$.

Given an element φ in $\Gamma(T^2)$ consider

$$M_\varphi := [0,1] \times T^2 \bigcup_{(x,1)\sim\varphi(x)} T^2,$$

the mapping cylinder of φ. Now $RT_r(M_\varphi)$ is a vector in $RT_r(T)^2 \otimes \overline{RT_r(T^2)}$. The latter space can be identified with $\mathrm{End}(RT_r(T^2))$ as $RT_r(T^2) \simeq RT_r(T^2)^*$ by the natural Hermitian form. The assignment $\rho_r(\varphi) = RT_r(M_\varphi)$, defines a projective representation

$$\rho_r : \Gamma(T^2) \longrightarrow \mathrm{End}(RT_r(T^2)). \tag{1}$$

Definition 2.5. A compact oriented 3-manifold M' is said to be obtained from another compact oriented 3-manifold M by genus one mutation, if M' is obtained from M by cutting along an embedded torus in M and regluing using the elliptic involution of T^2.

We will make use of the following fact, which was proved by [22] for the 3-manifold invariants, and which we state for cobordism invariants:

Lemma 2.6. *If C and C' are two cobordisms, and C' is obtained from C by genus one mutation, then $RT_r(C) = RT_r(C')$ for any odd integer $r \geq 3$.*

Proof. Let $T \simeq T^2$ be a torus embedded in C such that C' is obtained by cutting C along T and regluing using the elliptic involution. Equivalently, considering a regular neighborhood N of T in C, one can say that C' is obtained from C by replacing a trivial cylinder $N \simeq T \times [0,1]$ by the mapping cylinder M_ι of the elliptic involution ι on T. Since the elliptic involution is in the kernel of the representation ρ_r (see for example [13]), the mapping cylinder of the elliptic involution and the trivial cylinder have the same image by RT_r. Moreover, $C \setminus N$ and $C' \setminus M_\iota$ are equivalent cobordisms.

Since RT_r is a TQFT, the maps $RT_r(C)$ and $RT_r(C')$, are obtained from $RT_r(C \setminus N)$ and $RT_r(T^2 \times [0,1])$ (resp. $RT_r(C \setminus N)$ and $RT_r(M_\iota)$) by tensor contraction, and therefore are the same map. □

3. TQFT maps of Seifert fibered spaces

Let $S = S(B;\ \frac{q_1}{p_1} \dots \frac{q_n}{p_n})$ denote the orientable Seifert fibered 3-manifold with fiber space 2-orbifold B and fiber invariants $(q_1, p_1) \dots (q_n, p_n)$ in the notation of [15]. Recall that , for $i = 1, \cdots n$, q_i is co-prime to p_i and that the integers $p_1, \cdots, p_n$ are called the multiplicities of the exceptional fibers. In particular, if $p_i = 1$, for all $1, \cdots, n$, then S is an S^1-bundle over the surface B. If the surface B has boundary then S also has boundary which is union of tori. If ∂S has two components, say T and T', then as discussed in Section 2, the Reshetikhin-Turaev SO_3-TQFT gives a linear map

$$RT_r(S):\ RT_r(T) \to RT_r(T'),$$

for any odd integer $r \geqslant 3$ and a primitive $2r$-th root of unity q. As earlier we use $|| \cdot ||$ to denote is the norm induced by the Hermitian form on $RT_r(\partial S)$ and we use $||| \cdot |||$ to denote the operator norm of linear maps between TQFT spaces. That is, in the case that $RT_r(S)$ is invertible, we will have

$$|||RT_r(S)^{-1}||| := \max_{||x||=1} ||RT_r(S)^{-1}(x)||,$$

where $x \in RT_r(T')$.

Our main result in this section is the following:

Theorem 3.1. *For $S = S(B;\ \frac{q_1}{p_1} \dots \frac{q_n}{p_n})$ a Seifert fibered 3-manifold, with $\partial S = T \cup T'$, the linear map $RT_r(S):\ RT_r(T) \to RT_r(T')$, is invertible for all odd r coprime to $p_1, \cdots, p_n$. Furthermore, there are constants C and $N > 0$ such that*

$$|||RT_r(S)^{-1}||| \leqslant C R^N.$$

The last part of Theorem 3.1 says that the operator norm of the inverse of $RT_r(M)$ grows at most polynomially. Next we prove couple of lemmas that we need for the proof of the theorem.

Lemma 3.2. *If $S_p := S_p(A;\ \frac{q}{p})$ fibers over an annulus A with an exceptional fiber of multiplicity $p > 1$, the linear map $RT_r(S_p)$ is invertible for all r coprime to p and the operator norm $|||RT_r(S_p)^{-1}|||$ grows at most polynomially in r.*

Proof. It is known that S_p is a cable space and for such spaces the operators $RT_r(S_p)$ and the growth of their norm were explicitly computed by Kumar and Melby [18, Theorem 1.7]. □

We will use $\Sigma_{g,n}$ (resp. $P_{g,n}$) to denote an orientable (resp. non-orientable), compact surface of genus g and n boundary components. The second lemma we need is the following:

Lemma 3.3. *Suppose that S is one of the following Seifert fibered 3-manifolds:*

(a) The trivial S^1 bundle over a torus with two holes $\Sigma_{1,2}$.
(b) The twisted S^1-bundle over the Klein bottle with two holes $P_{1,2}$.
(c) The twisted S^1-bundle over the Mobius band with one hole $P_{0,2}$.
Then, the linear map $RT_r(S)$ is invertible for all odd $r \geq 3$ and the operator norm $|||RT_r(S)^{-1}|||$ grows at most polynomially in r.

Proof. Let $S := S^1 \times \Sigma_{1,2}$ and $\partial S = T \cup T'$. For $r = 2m+1$, the operator $RT_r(S)$ is exactly the operator K computed in [5, Section 5.10]. It is self-adjoint since it is symmetric with respect to the orthonormal basis $e_1, \cdots, e_m$ of $RT_r(T^2)$. The eigenvalues of K have been computed in there and show to be

$$\lambda_j := \frac{(-r)}{(q^{2j} - q^{-2j})^2} = \frac{r}{4\sin^2(\frac{2\pi j}{r})}, \quad \text{where} \quad j = 1, \cdots, m,$$

and $q = e^{\frac{2\pi i}{r}}$.

Since $\lambda_j \neq 0$, the operator is invertible for all odd $r \geq 3$ and the eigenvalues of the inverse are $\lambda_j^{-1} := \frac{(q^{2j}-q^{-2j})^2}{(-r)}$. Since the operator is self-adjoint, to bound $|||RT_r(S)^{-1}|||$, it is enough to bound the eigenvalues. Since

$$|\lambda_j^{-1}| \leqslant \frac{4}{r},$$

the result follows. This finishes the proof of (a).

Let us now prove (b). Let $S' := S^1 \tilde{\times} P_{1,2}$, be the twisted S^1-bundle over the Klein bottle. We claim that S' is obtained from $S^1 \times \Sigma_{1,2}$ by genus one mutation. To see this, consider an orientation reversing simple closed curve γ on $P_{1,2}$. Cutting $P_{1,2}$ along γ and regluing by a diffeomorphism of S^1 that reverses the orientation, we get back $\Sigma_{1,2}$. In the same way, cutting S' along $S^1 \times \gamma$ and regluing by the elliptic involution, we get S, which finishes the proof of the claim. Now by Lemma 2.6 we get $RT_r(S') = RT_r(S)$, and the desired conclusion follows from part (a).

Next we prove (c). Let $S'' := S^1 \tilde{\times} P_{0,2}$, be the twisted S^1-bundle over the Mobius band with one hole. Note that gluing two copies of $P_{0,2}$ together along a boundary component, one gets $P_{1,2}$. We also have that the square of S'' as a cobordism $T^2 \longrightarrow T^2$ satisfies $S'' \circ S'' = S'$. Therefore,

$$RT_r(S'' \circ S'') = RT_r(S') = RT_r(S).$$

Since $RT_r(S)$ is self-adjoint, it is diagonalizable in a Hermitian basis of $RT_r(T^2)$. However, since the eigenvalues λ_j of $RT_r(S)$ are all distinct, $RT_r(S'')$ must be diagonalizable in the same basis, and its eigenvalues μ_j are square roots of the λ_j's. Note that the λ_j's are positive, hence the μ_j's are all real and $RT_r(S'')$ is self-adjoint. Therefore, we get the inequality

$$|\mu_j|^{-1} \leqslant \frac{2}{\sqrt{r}},$$

which implies $|||RT_r(S'')^{-1}||| \leq \frac{2}{\sqrt{r}}$.

$\square$

We are now ready to prove Theorem 3.1.

Proof. Let $S = S(B;\ \frac{q_1}{p_1} \dots \frac{q_n}{p_n})$ be Seifert fibered 3-manifold, with $\partial S = T \cup T'$. The surface B can be cut along a disjoint union of simple closed curves Γ into annuli each of which contains exactly one orbifold point of the fibration, or two-holed tori or one-holed Mobius bands that contain no orbifold point. Moreover, the curves of Γ may be chosen so that each is separating in B. The inverse image of Γ under the Seifert fibration map $S \longrightarrow B$ is a collection $\mathcal{T}$ of tori in S each of which is vertical with respect to the fibration. By construction, for each component S_i of $S \setminus \mathcal{T}$ there are the following possibilities:

(a) S_i fibers over an annulus with one exceptional fiber.
(b) S_i fibers over $\Sigma_{1,2}$ with no exceptional fibers. Hence, it is the trivial S^1 bundle over $\Sigma_{1,2}$.
(c) S_i fibers over the Mobius band with one hole and no exceptional fibers. Hence, it is the twisted S^1-bundle over $P_{0,2}$.

Moreover, since each curve of Γ is separating, S as a cobordism is a composition of the cobordisms S_i. Since RT_r is a TQFT operator, and with the understanding of Remark 2.1, we have

$$RT_r(S) = RT_r(S_m) \circ \cdots \circ RT_r(S_1) \circ RT_r(S_0).$$

By Lemma 3.3 if S_i is as in (b)-(c) above, then $RT_r(S_i)$ is invertible for all r and if S_i is as in case (a) then by Lemma 3.2 $RT_r(S_i)$ is invertible for all r coprime to the multiplicity of the exceptional fiber. It follows that, for all odd r coprime to $p_1, \cdots, p_n$, $RT_r(S)$ is invertible with inverse

$$RT_r(S)^{-1} = RT_r(S_0)^{-1} \circ RT_r(S_1)^{-1} \circ \cdots \circ RT_r(S_m)^{-1}.$$

Also by Lemmas 3.3 and 3.2, for $i = 0, \cdots n$, the operator norm of the inverses $|||RT_r(S_i)^{-1}|||$ grows at most polynomially in r. Since $|||\cdot|||$ is sub-multiplicative under composition of linear operators it follows that there are constants C and $N > 0$ such that

$$|||RT_r(S)^{-1}||| \leqslant CR^N.$$

□

4. Gluing Theorems

4.1. **Seifert cobordisms.** In this section we prove the following theorem, which in particular implies Theorem 1.1 of the Introduction.

Theorem 4.1. *Let $S = (B;\ \frac{q_1}{p_1} \dots \frac{q_n}{p_n})$ be a Seifert fibered 3-manifold with at least two boundary components and let M be any 3-manifold with toroidal boundary. Then, for any 3-manifold M' obtained by gluing S along a component of $T' \subset \partial S$ to a component of ∂M, there exist constants A and $K > 0$ such that*

$$\frac{r^{-K}}{A} TV_r(M) \leqslant TV_r(M') \leqslant Ar^K TV_r(M).$$

Here the upper inequality holds for all odd r, while the lower inequality holds for all r coprime to $p_1, \cdots, p_n$.

In particular, if the limsup in the definition of $LTV(M)$ is actually a limit, then $LTV(M') = LTV(M)$.

By the classification theorem of manifolds that admit Seifert fibrations manifolds with more than two boundary components admit unique such fibrations. Hence the integers $p_1, \cdots, p_n$ are uniquely determined by the the 3-manifold and vice versa. Setting $I := \{p_1, \cdots, p_n\}$, we obtain Theorem 1.1.

4.2. **Invertible cable spaces.** For the proof of Theorem 4.1 we need to recall the notion of an invertible cable space from [11].

Definition 4.2. Let S be a 3-manifold with toroidal boundary with a distinguished torus boundary component T and such that ∂S has at least three boundary components. S is called an invertible cabling space if it has zero simplicial volume (i. e. $\mathrm{Vol}(S) = 0$) and there is a Dehn filling along some components of ∂S distinct from T that produces a 3-manifold homeomorphic to $T \times [0, 1]$.

Corollary 4.3. [11, Corollary 8.3] *Let M be a 3-manifold with toroidal boundary and S be an invertible cabling space. Let M' be obtained by gluing a component of $\partial S \setminus T$ to a component of ∂M. Then, there exist constants A and $K > 0$ such that*

$$TV_r(M) \leqslant TV_r(M') \leqslant Ar^K TV_r(M)$$

for all odd $r \geq 3$. In particular, we have $LTV(M) = LTV(M')$.

We will need the following:

Lemma 4.4. *For any $n \geq 3$, $S := S^1 \times \Sigma_{0,n}$ is an invertible cable space.*

Proof. Since S is an S^1-bundle, $\mathrm{Vol}(S) = 0$. Designate one component $T \in \partial S$ as the distinguished component. Now, the trivial Dehn filling along all but one of the tori in $\partial S \setminus T$, produces the trivial S^1- bundle over an annulus, that is $S^1 \times S^1 \times [0, 1]$, where $T = S^1 \times S^1$.

□

4.3. **Proof of Theorem 4.1.** By construction, M' is obtained by gluing a boundary component of S to a component of ∂M. Since S is a Seifert fibered manifold, by [11, Theorem 5.2] (and its proof) we have

$$TV_r(M') \leqslant TV_r(S) \cdot TV_r(M),$$

for all odd integers $r \geq 3$. On the other hand, since S is a Seifert fibered 3-manifold, by [11, Theorem 5.2], there are constants A and $N > 0$ such that $TV_r(S') \leqslant Ar^N$, for all for all odd integers $r \geq 3$. Thus, we have $LTV(S) \leqslant 0$. Hence, the upper inequality in the statement of the theorem follows, and in particular, we have

$$LTV(M') \leqslant LTV(M).$$

For the proof of the lower inequality we will distinguish two cases:

Case 1. Suppose that S has exactly two boundary components, say T and T'. By Theorem 3.1 the linear map $RT_r(S): \ RT_r(T) \to RT_r(T')$, is invertible for all odd r coprime to $p_1, \cdots, p_n$. Furthermore, there are constants C and $K > 0$ such that

$$|||RT_r(S)^{-1}||| \leqslant Cr^N. \tag{2}$$

If M has only one boundary component, then $RT_r(M)$ is a vector in $RT_r(T^2)$, and by the TQFT properties, we have that

$$RT_r(M') = RT_r(S)(RT_r(M)).$$

Now we can write $RT_r(S)^{-1}(RT_r(M')) = RT_r(M)$, and hence

$$||RT_r(M)|| \leqslant |||RT_r(S)^{-1}||| \cdot ||RT_r(M')||,$$

which in turn gives

$$|||RT_r(S)^{-1}|||^{-1} \cdot ||RT_r(M)|| \leqslant ||RT_r(M')||.$$

The last inequality combined with (2) gives

$$\frac{r^{-N}}{C} TV_r(M) \leqslant TV_r(M'). \tag{3}$$

Finally, by adjusting the constants K, N, A, C we get the desired result.

If M has more than one boundary components, let T_1 denote the one that is used to glue S. Then, M' may equivalently be seen as obtained from M by gluing the cobordism

$$S' := S \coprod (\partial M \setminus T_1) \times [0,1]$$

onto ∂M. The latter is a cobordism $\partial M \to \partial M$, and by the TQFT properties, $RT_r(S')$ is invertible. We claim that $|||RT_r(S')^{-1}||| = |||RT_r(S)^{-1}|||$. Indeed, RT_r is a monoidal functor, so

$$RT_r(S') = RT_r(S) \otimes \mathrm{id}_{RT_r(\partial M \setminus T_1)}.$$

Moreover, the operator norm $|||\cdot|||$ is multiplicative under tensor product of Hermitian vector spaces and maps. The remaining of the claim follows exactly as before.

Case 2. Suppose that S has $n \geq 3$ boundary components and n exceptional fibers of orders $p_1, \ldots, p_n$. Pick a curve γ in the orbifold B that separates it into a surface $\Sigma_{0,n}$ (containing no orbifold point) and an orbifold B' with exactly two boundary components. We can furthermore assume that the boundary component of S glued onto M corresponding to a curve in B'. Taking the pre-images under the Seifert fibration, we see that M' is obtained from M by first gluing a Seifert manifold with two boundary components on a torus boundary component of M, obtaining a 3-manifold M_0, and then gluing $S^1 \times \Sigma_{0,n}$ on a boundary component of M_0.

By Case 1, there exists constants $A, K > 0$ such that

$$\frac{1}{Ar^K} TV_r(M) \leqslant TV_r(M_0) \leqslant Ar^K TV_r(M)$$

for any r coprime to all of the integers p_i. By Corollary 4.3, there exists constants $B, L > 0$ such that

$$\frac{1}{Br^L} TV_r(M_0) \leqslant TV_r(M') \leqslant Br^L TV_r(M_0)$$

for any odd $r \geq 3$. Therefore, we get the desired inequalities in this case as well. □

4.4. **Plumbed Cobordisms.** A *graph manifold* G is a 3-manifold that can be decomposed into Seifert fibered spaces by cutting along a collection $\mathcal{T}$ of incompressible tori. To any graph manifold G we associate a graph $T(G)$ with vertices corresponding to components of $G \setminus \mathcal{T}$ and each edge corresponds to a torus in $\mathcal{T}$ along which the manifolds corresponding are glued. The leaves are vertices of valence one. In the case that above graph is a tree we will say that M is a *plumbed manifold*. The following generalizes Theorem 4.1 to plumbed 3-manifolds.

Corollary 4.5. *Let G be a plumbed 3-manifold such that each leaf on $T(G)$ has at least one boundary component coming from a component of ∂G, and let M be any 3-manifold with non-empty toroidal boundary. Then, for any 3-manifold M' obtained by gluing G along a component $T' \subset \partial G$ to a component of ∂M, there exist constants A and $K > 0$ and a finite set I of non-zero integers such that*

$$\frac{r^{-K}}{A} TV_r(M) \leqslant TV_r(M') \leqslant Ar^K TV_r(M)$$

Here the upper inequality holds for all odd integers $r > 2$ and the lower inequality holds for all r not divisible by any of the numbers in I.

In particular, if the limsup in the definition of $LTV(M)$ is actually a limit, then $LTV(M') = LTV(M)$.

Proof. The proof is by induction on the number of edges of $T(G)$. If there are no edges, the conclusion follows from Theorem 4.1 where the set I is the set of multiplicities of the exceptional fibers of G_1.

Otherwise, remove from $T(G)$ an edge e that ends to a leaf S to obtain a tree $T(G_1)$, associated to a graph manifold G_1. By hypothesis S has a boundary component which comes from ∂G. Suppose that the edge e corresponds to a torus $T' \in \mathcal{T}$. Now cutting G along T' we obtain two 3-manifolds: One is the graph manifold G_1 above and the second is the Seifert fibered manifold S, which by hypothesis has at least two boundary components. Let M_1 denote the 3-manifold obtained by gluing G_1 to M along the component of ∂G_1 that corresponds to the component of ∂G glued along ∂M in the construction of M'. Now M' is obtained by gluing S to M_1 along a boundary component. The result follows by applying the induction hypothesis to M and G_1 and Theorem 4.1 to M_1 and S.

□

5. Volume Conjecture applications

In this section, we discuss applications of Theorems 4.1 and 4.5 to Conjecture 1.2. We will use properties about the behavior of the Gromov norm (and hence simplicial volume) under the operation of gluing 3-manifolds along spheres and tori. For details we refer the reader to [23].

Theorem 5.1. *Let M be a 3-manifold with non-empty boundary such that*

$$\lim_{r\to\infty,\ r\ \text{odd}} \frac{2\pi}{r} \log |TV_r(M)| = \mathrm{Vol}(M). \tag{4}$$

Suppose that M' is obtained from M by gluing to a component $T' \subset \partial M$, either
(a) a Seifert fibered space S as in Theorem 4.1; or
(b) a plumbed 3-manifold G as in Corollary 4.5.
Then $\mathrm{LTV}(M') = \mathrm{Vol}(M')$.

Proof. By Theorem 4.1, we have $LTV(M') = LTV(M)$ in the case of (a) and by Corollary 4.5, we have $LTV(M') = LTV(M)$ in the case of (b). So in both cases we only need to prove that $\mathrm{Vol}(M') = \mathrm{Vol}(M)$. We will discuss the details for (a). The proof of (b) is completely analogous.

The manifold M' is the gluing of M and the Seifert manifold S along a torus T. Since S has at least two boundary components, in particular it is not a solid torus and the torus T is incompressible in S. There are thus two cases:

Case 1: The torus T is also incompressible in M. Then the torus T is also incompressible in M' and we have $\mathrm{Vol}(M') = \mathrm{Vol}(M) + \mathrm{Vol}(S) = \mathrm{Vol}(M)$ since the simplicial volume is additive under gluing along an incompressible torus, and S is a Seifert manifold.

Case 2: The torus T is compressible in M. Then M is the connected sum of a solid torus V and another 3-manifold M_0. Since the simplicial volume is additive under disjoint union, connected sums we have $\mathrm{Vol}(M_0) = \mathrm{Vol}(M)$. Now we can obtain M' as a connected sum $M' = M_0 \# S'$ where S' is obtained by gluing S to the solid torus V. Since S' is a Seifert fibered manifold, $\mathrm{Vol}(S'') = 0$. Again by by additivity of the simplicial volume under connected sum (and disjoint unions), we have

$$\mathrm{Vol}(M') = \mathrm{Vol}(M_0) + \mathrm{Vol}(S') = \mathrm{Vol}(M_0) = \mathrm{Vol}(M),$$

giving the desired result. □

Remark 5.2. Note that if M in Theorem 5.1 has zero simplicial volume the conclusion of the theorem follows if the limit in Equation (4) is replaced by suplim. Indeed, Theorem 4.1 and Corollary 4.5 imply that if $\mathrm{LTV}(M) = 0$ then $\mathrm{LTV}(M') = 0$. On the other hand, since the Gromov norm is subadditive under gluing 3-manifolds along tori gluing manifolds of simplicial volume zero produces volume zero manifolds.

Next we have two results that prove the volume conjecture for Seifert fibered 3-manifolds with non-empty boundary and for large classes of graph manifolds. Our first result is the following:

Corollary 1.3. *Suppose that S is an oriented Seifert fibered 3-manifold that either has a non-empty boundary, or it is closed and admits an orientation reversing involution. Then we have*

$$LTV(S) = \limsup_{r\to\infty,\ r\ odd} \frac{2\pi}{r} \log |TV_r(S)| = \mathrm{Vol}(S) = 0.$$

Proof. First suppose that $\partial S \neq \emptyset$. Removing from S the neighborhood of a regular fiber of S, which is a solid torus $M := D^2 \times S^1$, we obtain a Seifert manifold S' that has at least two boundary components and one of them will be glued to ∂M. On the other hand, by Theorem 2.3, we have

$$TV_r(M) = TV_r(D^2 \times S^1) = RT_r(S^2 \times S^1) = 1,$$

and hence we obtain $LTV(M) = \mathrm{Vol}(M) = 0$. Now the result follows by part (a) of Theorem 5.1.

Next suppose that S is closed and there is an orientation reversing involution $i : S \longrightarrow S$. Then S is the double of a Seifert fibered manifold S_1, where S_1 has non empty boundary. This is if we let $\bar{S_1}$ denote S_1 with the opposite orientation, then S is obtained by identifying $\bar{S_1}$ and S_1 along their boundary. On one hand we have $\mathrm{Vol}(S) = 0 = \mathrm{Vol}(S_1)$. On the other hand, by Theorem 2.3,

$$TV_r(S, q^2) = ||RT_r(S_1), q)||^2 = TV_r(S_1, q^2),$$

and hence $LTV(S) = LTV(S_1) = 0$.

□

Now we turn to the second result that considers plumbed 3-manifolds.

Corollary 5.3. *Let G be plumbed manifold with non-empty boundary and with an associated tree $T(G)$ where all but at most one leaf is a 3-manifold with at least one boundary component coming from ∂G. Then,*

$$LTV(G) = \mathrm{Vol}(G) = 0.$$

Proof. Remove from G the neighborhood of a regular fiber of a leaf S. Then proceed as in the proof of Corollary 1.3 using part (b) of Theorem 5.1. □

Remark 5.4. Some cases of Corollary 1.3 were also verified by the third author of this paper using using different methods [19]. Note that in this paper for the sequence of integers $r \to \infty$ used to establish that $LTV(S) = 0$, r is co-prime to the multiplicities of the exceptional fibers of Seifert fibrations. In contrast to that, in [19] the sequence of integers $r \to \infty$ is when r is divisible by the multiplicities of all fibers.

6. Hyperbolic cobordisms

In the view of our results here it is reasonable to ask what is the behavior of the TQFT operator maps for cobordisms with non-zero simplicial volume. For example, let M be a 3-manifold with two torus boundary components, $\partial M = T \cup T'$ whose interior admits a hyperbolic structure. As before we get operators $RT_r(M)$:

$RT_r(M) \to RT_r(T')$. If $RT_r(M)$ is invertible, one would hope that the operator norm $|||RT_r(S)^{-1}|||$ grows exponentially as $r \to \infty$. However, as we will see below this is not always the case

With M as above, on the torus $T' \subset \partial M$ take a simple closed curve representing slope $\mathbf{s}$, and let $M(\mathbf{s})$ denote the 3-manifold obtained by Dehn filling M along $\mathbf{s}$. If the length of the geodesic representing $\mathbf{s}$ on T' is large enough, then $M(\mathbf{s})$ is also hyperbolic [23]. However, for slopes represented by shorter the resulting manifold can be exceptional (i.e. non-hyperbolic) and in particular $M(\mathbf{s})$ can be a Seifert fibered 3-manifold. For example, M is the complement of the Whitehead link in S^3 then a Dehn filling along one of the components of ∂M produces a solid torus which has volume 0. The next proposition shows that in these cases the operator norm $|||RT_r(M)^{-1}|||^{-1}$ grows at most polynomially.

Proposition 6.1. *Let M be a cobordim from T to T' as above, and suppose the the map $RT_r(M)$ is invertible. Suppose that M admits a Dehn filling with slope s along a component of ∂M so that $M(s)$ is a 3-manifold of zero simplicial volume. Then, the operator norm $|||RT_r(M)^{-1}|||^{-1}$ grows at most polynomially.*

Proof. By assumption $M(\mathbf{s})$ has boundary a single torus T and $RT_r(M(\mathbf{s}))$ is a vector in $RT_r(T)$. Since $M(\mathbf{s})$ has zero simplicial volume, by [11, Theorem 11] its norm with respect to the Hermitian pairing on $RT_r(T)$, the norm $||RT_r(M(\mathbf{s}))||$ grows at most polynomially in r.

By Remark 2.4 we have

$$|||RT_r(M)^{-1}|||^{-1} = \min_{||x||=1} ||RT_r(M)(x)||, \tag{5}$$

where $x \in RT_r(T')$. On the other hand, by the TQFT properties,

$$RT_r(M(\mathbf{s})) = RT_r(M)(e_r(\mathbf{s})), \tag{6}$$

where $e_r(\mathbf{s}) \in RT_r(T')$ is the vector the TQFT-functor RT_r assigns to the solid torus where the meridian is the curve representing the slope $\mathbf{s}$.

We claim that $e_r(\mathbf{s})$ is a vector of Hermitian norm 1. Indeed, $e_r(\mathbf{s})$ is the RT_r-vector of a solid torus $D^2 \times S^1$ but with the meridian of $D^2 \times S^1$ identified with the curve of slope $\mathbf{s}$ on T^2. Hence, it is the image of the basis vector e_1 introduced in Theorem 2.2 by $\rho_r(\phi_{\mathbf{s}})$, where ρ_r is the quantum representation of the mapping class group class group of T^2 (see Equation (1) in Section 2) and ϕ_s is any mapping class that sends the meridian of T^2 to the curve of slope $\mathbf{s}$. Since the image of the quantum representation ρ_r consists only of unitary maps, $e_r(\mathbf{s})$ has norm 1.

Now Equations (5) and (6) and the discussion in the beginning of the proof imply

$$|||RT_r(M)^{-1}|||^{-1} \leqslant ||RT_r(M(\mathbf{s}))|| \leqslant A \cdot r^N, \tag{7}$$

for some constants A, $N > 0$. $\square$

Note that the first part of inequality (7) implies that if $|||RT_r(M)^{-1}|||^{-1}$ grows exponentially with r, then the invariants $TV_r(M(\mathbf{s}), q^2) = ||RT_r(M(\mathbf{s}), q)||^2$ grow exponentially. Tools that allow to establish exponential growth of the Turaev-Viro invariants of Dehn fillings are highly desirable as they will lead to progress on the volume conjecture as well as on another important conjecture in quantum topology; the AMU conjecture [10]. We ask the following question:

Problem 6.2. *Construct examples of hyperbolic cobordisms $M : T \longrightarrow T'$ such that $RT_r(M): \ RT_r(M) \to RT_r(T')$ is invertible and $|||RT_r(S)^{-1}|||^{-1}$ grows exponentially with r.*

In the view of Proposition 6.1 one has to look at hyperbolic cobordisms $M : T \longrightarrow T'$, such that all the 3-manifolds by filling one of the components of ∂M have non-zero simplicial volume. One way to obtain such cobordisms is to consider complements of two component highly twisted links in S^3 (see [14] and references therein). In these cases all the Dehn fillings of either of the two boundary components produce hyperbolic 3-manifolds.

References

[1] G. Belletti. An upper bound conjecture for the Yokota invariant. *Algebraic & Geometric Topology, preprint at arXiv:2307.12963*, 2019.

[2] G. Belletti, R. Detcherry, E. Kalfagianni, and T. Yang. Growth of quantum $6j$-symbols and applications to the volume conjecture. *J. Differential Geom.*, 120(2):199–229, 2022.

[3] R. Benedetti and C. Petronio. On Roberts' proof of the Turaev-Walker theorem. *J. Knot Theory Ramifications*, 5(4):427–439, 1996.

[4] C. Blanchet, N. Habegger, G. Masbaum, and P. Vogel. Three-manifold invariants derived from the Kauffman bracket. *Topology*, 31(4):685–699, 1992.

[5] C. Blanchet, N. Habegger, G. Masbaum, and P. Vogel. Topological quantum field theories derived from the Kauffman bracket. *Topology*, 34(4):883–927, 1995.

[6] G. Burde and K. Murasugi. Links and Seifert fiber spaces. *Duke Math. J.*, 37:89–93, 1970.

[7] A. Champanerkar, O. Dasbach, E. Kalfagianni, I. Kofman, W. Neumann, and N. Stoltzfus, editors. *Interactions between hyperbolic geometry, quantum topology and number theory*, volume 541 of *Contemporary Mathematics*. American Mathematical Society, Providence, RI, 2011.

[8] Q. Chen and T. Yang. Volume conjectures for the Reshetikhin-Turaev and the Turaev-Viro invariants. *Quantum Topol.*, 9(3):419–460, 2018.

[9] Q. Chen and S. Zhu. On the asymptotic expansions of various quantum invariants I: the colored Jones polynomial of twist knots at the root of unity $e^{\frac{2\pi\sqrt{-1}}{n+\frac{1}{2}}}$. *arXiv preprint at arXiv:2307.12963*, 2023.

[10] R. Detcherry and E. Kalfagianni. Quantum representations and monodromies of fibered links. *Adv. Math.*, 351:676–701, 2019.

[11] R. Detcherry and E. Kalfagianni. Gromov norm and Turaev-Viro invariants of 3-manifolds. *Ann. Sci. Éc. Norm. Supér. (4)*, 53(6):1363–1391, 2020.

[12] R. Detcherry, E. Kalfagianni, and T. Yang. Turaev-Viro invariants, colored Jones polynomials, and volume. *Quantum Topol.*, 9(4):775–813, 2018.

[13] M. H. Freedman, K. Walker, and Z. Wang. Quantum SU(2) faithfully detects mapping class groups modulo center. *Geom. Topol.*, 6:523–539, 2002.

[14] D. Futer, E. Kalfagianni, and J. S. Purcell. Dehn filling, volume, and the Jones polynomial. *J. Differential Geom.*, 78(3):429–464, 2008.

[15] M. Jankins and W. D. Neumann. *Lectures on Seifert manifolds*, volume 2 of *Brandeis Lecture Notes*. Brandeis University, Waltham, MA, 1983.
[16] S. Kumar. Fundamental shadow links realized as links in S^3. *Algebr. Geom. Topol.*, 21(6):3153–3198, 2021.
[17] S. Kumar and J. M. Melby. Asymptotic additivity of the Turaev-Viro invariants for a family of 3-manifolds. *J. Lond. Math. Soc. (2)*, 106(4):3043–3068, 2022.
[18] S. Kumar and J. M. Melby. Turaev-Viro invariants and cabling operations. *Internat. J. Math.*, 34(11):Paper No. 2350065, 22, 2023.
[19] S. Marasinghe. Seifert fibered 3-manifolds and turaev-viro invariants volume conjecture. *arXiv preprint at 2504.10682*, 2025.
[20] N. Reshetikhin and V. G. Turaev. Invariants of 3-manifolds via link polynomials and quantum groups. *Invent. Math.*, 103(3):547–597, 1991.
[21] J. Roberts. Skein theory and Turaev-Viro invariants. *Topology*, 34(4):771–787, 1995.
[22] Y. W. Rong. Mutation and Witten invariants. *Topology*, 33(3):499–507, 1994.
[23] W. P. Thurston. *The geometry and topology of three-manifolds.* Princeton University Math Department Notes, 1979.
[24] V. G. Turaev and O. Y. Viro. State sum invariants of 3-manifolds and quantum $6j$-symbols. *Topology*, 31(4):865–902, 1992.
[25] K. H. Wong. Asymptotics of some quantum invariants of the Whitehead chains. *arXiv preprint at arXiv:1912.10638*, 2019.
[26] K. H. Wong and T. Yang. Relative Reshetikhin-Turaev invariants, hyperbolic cone metrics and discrete Fourier transforms I. *Comm. Math. Phys.*, 400(2):1019–1070, 2023.

Université Bourgogne Europe, CNRS, IMB UMR 5584, F-21000 Dijon, France
Email address: renaud.detcherry@u-bourgogne.fr

Department of Mathematics, Michigan State University, East Lansing, MI, 48824, USA
Email address: kalfagia@msu.edu

Department of Mathematics, Michigan State University, East Lansing, MI, 48824, USA
Email address: marasin1@msu.edu