

PANTS COMPLEX, TQFT AND HYPERBOLIC GEOMETRY

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ABSTRACT. We introduce a coarse perspective on relations of the $SU(2)$ -Witten-Reshetikhin-Turaev TQFT, the Weil-Petersson geometry of the Teichmüller space, and volumes of hyperbolic 3-manifolds. Using data from the asymptotic expansions of the curve operators in the skein theoretic version of the $SU(2)$ -TQFT, we define the quantum intersection number between pants decompositions of a closed surface. We show that the quantum intersection number admits two sided bounds in terms of the geometric intersection number and we use it to obtain a metric on the pants graph of surfaces. Using work of Brock we show that the pants graph equipped with this metric is quasi-isometric to the Teichmüller space with the Weil-Petersson metric and that the translation length of our metric provides two sided linear bounds on the volume of hyperbolic fibered manifolds. We briefly discuss how these relations are interpreted from the view point of $SU(2)$ -character varieties of 3-manifolds.

We also obtain a characterization of pseudo-Anosov mapping classes in terms of asymptotics of the quantum intersection number under iteration in the mapping class group and relate these asymptotics with stretch factors. We also discuss how these results fit with a conjecture of Andersen, Masbaum and Ueno about quantum representations of mapping class groups.

1. INTRODUCTION

The purpose of the paper is to present a coarse perspective on relations of asymptotic aspects of the unitary $SU(2)$ -Witten-Reshetikhin-Turaev TQFT to hyperbolic geometry. These connections are motivated and facilitated by the fact that pant decompositions of surfaces play prominent roles in the skein theoretic construction of the $SU(2)$ -TQFT, given by Blanchet, Habegger, Masbaum and Vogel [4, 5], and in the construction of a combinatorial model for the Weil-Petersson geometry on Teichmüller spaces given by Brock [8].

A *pants decomposition* P of a closed oriented surface Σ of genus $g > 1$ is a collection of $3g - 3$ distinct isotopy classes of simple closed curves on Σ . The *pants graph* $C_P^1(\Sigma)$, first considered by Hatcher and Thurston[22], is an abstract graph with set of vertices $C_P^0(\Sigma)$ isotopy classes of pants decompositions. Two vertices P and P' are connected by an edge in $C_P^1(\Sigma)$ if P' is obtained from P by an *elementary move*. See Definition 2.12. The pants graph plays important roles in the study of mapping class groups, Teichmüller theory, and hyperbolic geometry of surfaces.

For a positive integer r and a primitive $2r$ -root of unity, the $SU(2)$ -Witten-Reshetikhin-Turaev TQFT [33, 5] assigns to Σ a finite dimensional Hermitian vector space $V_r(\Sigma)$. Given

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a pants decomposition P , for any $r \geq 3$, [5] constructs a basis $\mathcal{B}_P := \mathcal{B}_P^r$ of $V_r(\Sigma)$ that is orthonormal with respect to the Hermitian pairing. For any $Q \in C_P^0(\Sigma)$, they also construct a family of Hermitian operators $T_r^Q : V_r(\Sigma) \rightarrow V_r(\Sigma)$, known as *curve operators*, so that \mathcal{B}_Q consists of eigenvectors of the operator T_r^Q .

In this paper we work with the *TQFT* theory associated to the roots of unity $\zeta_r = -e^{\frac{\pi i}{2r}}$. In this case, the large- r asymptotic behavior of the curve operators was studied by Detcherry in [15]. His results imply that given Q, P there is an integer $n := n(Q, P) \geq 0$, so that, for all r larger enough, the matrix representing T_r^Q in the basis \mathcal{B}_P has exactly n diagonals (excluding the main diagonal) that contain non-zero entries. We call $n(Q, P)$ the *quantum intersection number* of Q with respect to P .

Since pants decompositions of surfaces play important roles in both hyperbolic geometry and in TQFT, it is natural to ask what geometric properties of the pants graph are captured by quantities or invariants arising from TQFT. In this paper, we show that the quantum intersection number captures the coarse geometry relations of the pants graph to the Teichmüller space established in [8].

The spectral radii of the operators T_r^Q are uniformly bounded in terms of the genus of Σ , and their norms only depend on the orbit of Q under the action of the mapping class group and not on the Hermitian bases of the spaces $V_r(\Sigma)$. Intuitively speaking, this suggests that relations of large- r asymptotics of T_r^Q with the geometry of $C_P^1(\Sigma)$, should be reflected in the pairing defined by the quantum intersection number. As we will discuss in Section 8, relations of the quantum intersection number to geometry, also translate to information about the geometric content of the large- r asymptotics of the curve operator norms.

We consider the quantum intersection number as a function $n : C_P^0(\Sigma) \times C_P^0(\Sigma) \rightarrow \mathbb{N}$ and we use a general process we call *metrification* to promote it to a metric d_{qt} on $C_P^0(\Sigma)$ such that the mapping class $\text{Mod}(\Sigma)$ acts by isometries on $(C_P^0(\Sigma), d_{qt})$. Roughly speaking, the distance $d_{qt}(P, Q)$ is the smallest total quantum intersection number over all finite sequences in $C_P^0(\Sigma)$ from P to Q .

We prove the following theorem showing that the 0-skeleton of pants graph equipped with the metric d_{qt} derived from TQFT, coarsely records the geometry of Teichmüller space $\mathcal{T}(\Sigma)$ equipped with its Weil-Petersson metric d_{WP} .

Theorem 1.1. *The metric space $C_P^0(\Sigma)$ with the metric d_{qt} is quasi-isometric to the Teichmüller space with its Weil-Petersson metric d_{WP} .*

The *Bers constant* $L = L(g)$ is a number so that any hyperbolic surface X topologically equivalent to Σ , admits at least one pants decomposition P_X with the length of all curves at most L . By construction, Theorem 1.1 gives that for any $P, Q \in C_P^0(\Sigma)$, the distance $d_{qt}(P_X, P_Y)$ is within bounded ratio from the distance $d_{WP}(X, Y)$, and the bounds depend only on Σ . For more details, and for the definitions of terms and objects in the statement of Theorem 1.1, the reader is referred to Section 7.

Theorem 1.1 leads to new relations of asymptotic elements of the Witten-Reshetikhin-Turaev $SU(2)$ -TQFT to volumes of hyperbolic 3-manifolds. Recall that the translation length of an isometry ϕ on a metric space (X, d) is defined by

$$L^d(\phi) = \inf\{d(\phi(x), x) \mid x \in X\}.$$

Since mapping classes $\phi \in \text{Mod}(\Sigma)$ act as isometries on $(C_P^0(\Sigma), d_{qt})$ and on $(\mathcal{T}(\Sigma), d_{WP})$ we can consider the translation lengths $L^{d_{qt}}(\phi)$ and $L^{d_{WP}}(\phi)$, respectively. Theorem 1.1 implies that for any $\phi \in \text{Mod}(\Sigma)$ the translation lengths $L^{d_{qt}}(\phi)$ and $L^{d_{WP}}(\phi)$ are within bounded ratios from each other, with bounds depending only on the topology of Σ .

If ϕ is a pseudo-Anosov mapping class, the mapping torus $M_\phi := F \times [0, 1] / (x, 1) \sim (\phi(x), 0)$ is a hyperbolic 3-manifold that fibers over the circle. In this case, Brock [9] proved that the hyperbolic volume $\text{vol}(M_\phi)$ and the translation length $L^{d_{WP}}(\phi)$ are within bounded ratios from each other, with bounds depending only on Σ . Combining this result with Theorem 1.1 we obtain the following:

Theorem 1.2. *There exist a positive constant N , depending only on Σ , so that for any pseudo-Anosov mapping class $\phi \in \text{Mod}(\Sigma)$ we have*

$$\frac{1}{N} L^{d_{qt}}(\phi) \leq \text{vol}(M_\phi) \leq N L^{d_{qt}}(\phi).$$

Theorems 1.1 and 1.2 provide general new relations between features of the $SU(2)$ -TQFT and hyperbolic geometry, approached from the view point of coarse geometry of the Teichmüller space. In particular, Theorem 1.2 relates 2-dimensional TQFT aspects to 3-dimensional hyperbolic geometry which is compatible with the TQFT philosophy that the quantum invariants of mapping tori are determined by invariants of the corresponding monodromies.

Open conjectures in quantum topology, predict finer and exact relations between features of the $SU(2)$ -TQFTs and 3-manifold hyperbolic geometry. For instance, the volume conjecture of Kashaev asserts that large- r asymptotics of values of the colored Jones polynomial of hyperbolic knots determine the volume of the knot complement. More recently, Chen and Yang [13] conjectured that similar asymptotics of the Witten-Reshetikhin-Turaev and the related Turaev-Viro invariants, determine the volumes of hyperbolic 3-manifolds. Similarly, a conjecture of Bonahon-Wong-Yang predicts that the volume of hyperbolic mapping tori is determined by asymptotics of certain $SL_2(\mathbb{C})$ -quantum invariants of pseudo-Anosov mapping classes [7, 6]. In the last couple of decades, work by several authors has verified and refined these conjectures [19, 11, 3, 17, 35, 24] for special families of 3-manifolds but general approaches seem to be currently missing. Theorems 1.1 and 1.2, and corollaries stated in Section 7, are compatible with and support the general picture of relations predicted by these conjectures.

1.1. Geometric and quantum intersection numbers. Given an integer $r \geq 3$, let $\mathcal{C}_r := \{1, 2, \dots, r-1\}$ and let $\zeta_r = -e^{\frac{\pi i}{2r}}$. Given a pants decomposition and a *dual banded graph*

Γ , with set of edges E , [5] considers the set U_r of *admissible colorings*, which are certain functions $\mathbf{c} : E \rightarrow \mathcal{C}_r$. To an admissible coloring \mathbf{c} , they associate a vector $\phi_{\mathbf{c}}$ in $V_r(\Sigma)$ so that the set $\mathcal{B}_P = \{\phi_{\mathbf{c}}\}_{\mathbf{c}}$ is an orthonormal basis of $V_r(\Sigma)$. For any $\mathbf{c} \in U_r$, the function $\frac{\mathbf{c}}{r}$ is an element in the space \mathbb{R}^E of functions from E to $(0, 1)$.

For any admissible coloring \mathbf{c} , the pair $(\frac{\mathbf{c}}{r}, \frac{1}{r})$ is an element of $\mathbb{R}^E \times [0, 1]$. Detcherry [15] showed that, for any pants decompositions Q, P , there is an open set $V_r \subset \mathbb{R}^E \times [0, 1]$, and analytic functions

$$G_{\mathbf{k}}^Q = G_{\mathbf{k}}^Q\left(\frac{\mathbf{c}}{r}, \frac{1}{r}\right) : V_r \rightarrow \mathbb{C},$$

parametrized by functions $\mathbf{k} : E \rightarrow \mathbb{Z}$, so that

$$(1) \quad T_r^Q(\phi_{\mathbf{c}}) = \sum_{\mathbf{k}} G_{\mathbf{k}}^Q\left(\frac{\mathbf{c}}{r}, \frac{1}{r}\right) \phi_{\mathbf{c}+\mathbf{k}}.$$

Furthermore, for given P, Q and Γ , the family $\{G_{\mathbf{k}}^Q\}_{\mathbf{k} \neq 0}$ contains finitely many non-zero functions. The number of non-zero functions, denoted by $n(Q, P)$, is the quantum intersection number of Q with respect to P .

A key ingredient in the proofs of Theorems 1.1 and 1.2 is the following result.

Theorem 1.3. *Given pants decompositions P and Q , let $I(Q, P)$ denote their total geometric intersection number. Then,*

$$\frac{I(Q, P)}{3g-3} \leq n(Q, P) \leq (I(Q, P) + 1)^{3g-3} - 1.$$

We stress that the definition of $n(Q, P)$ is not symmetric in P, Q . Nevertheless, using the process of metrification, described in Appendix A, we define a metric d_{qt} on $C_P^0(\Sigma)$ out of the quantum intersection number.

The *path metric* d_{π} on the pants graph $C_P^1(\Sigma)$ is defined by assigning the value 1 to each edge of $C_P^1(\Sigma)$. Work of Mazur and Minsky on the geometry of curve complexes, and Theorem 1.3 can be used to show that the distances $d_{\pi}(P, Q)$ and $d_{qt}(P, Q)$ are within bounded ratio from each other. See Theorem 6.4. This, in turn, combined with Brock's result stating that $(C_P^0(\Sigma), d_{\pi})$ is quasi-isometric with the Teichmüller space $\mathcal{T}(\Sigma)$ equipped with its Weil-Petersson metric d_{WP} , gives Theorem 1.1.

The proof of Theorem 1.3 relies on quantum topology techniques. A key part is to use the Masbaum-Vogel fusion theory underlying the $SU(2)$ skein theory to find *state-sum* formulae for the functions $G_{\mathbf{k}}^Q(\theta, 0)$. The proof is given in Sections 4 and 5.

As a corollary of the proof of Theorem 1.2 we get the following result.

Corollary 1.4. *There is a constant $N > 0$, only depending on the topology of Σ , so that for any pseudo-Anosov mapping class $\phi \in \text{Mod}(\Sigma)$ and any $P \in C_P^0(\Sigma)$ we have*

$$n(P, \phi(P)) \geq N \text{vol}(M_{\phi}).$$

The quantum intersection number $n(\gamma, P)$ is defined for any multicurve γ on Σ and $P \in C_P^0(\Sigma)$, and Theorem 1.3 holds. In this setting, it follows that quantum intersection numbers

can be used to detect the Nielsen-Thurston types of mapping classes and to estimate stretch factors of pseudo-Anosov mapping classes. The reader is referred to Section 7.

1.2. Relation to the AMU conjecture. Recall that for any $r \geq 3$ and $\zeta_r = -e^{\frac{\pi i}{2r}}$ the $SU(2)$ -TQFT also supplies a linear representation ρ_r of $\text{Mod}(\Sigma)$ on the space $V_r(\Sigma)$. An intriguing open conjecture of Andersen, Masbaum and Ueno [1] asserts that for any pseudo-Anosov $\phi \in \text{Mod}(\Sigma)$, the image $\rho_r(\phi)$ has infinite order for all large enough levels r .

For a multicurve γ and $P \in C_P^0(\Sigma)$, let $r_0(\gamma, P) > 0$ denote the smallest integer such that for any lever $r \geq r_0(\gamma, P)$ the matrix representing T_r^γ in the basis \mathcal{B}_P , realizes $n(\gamma, P)$.

Corollary 1.5. *Let ϕ be a pseudo-Anosov mapping class. If there is a multicurve γ and $P \in C_P^0(\Sigma)$ so that $\limsup_{m \rightarrow \infty} [r_0(\phi^m(\gamma), P)] < \infty$, then, ϕ satisfies the AMU conjecture.*

To the best of our knowledge, at this writing, there isn't even a single example of a pseudo-Anosov mapping classes acting on a closed surface for which the AMU conjecture is verified. The authors of this paper, used their work on the asymptotics of Turaev-Viro 3-manifold invariants to give constructions of vast families of pseudo-Anosov mapping classes, acting on any surface with boundary, that satisfy the AMU conjecture [3, 16, 18, 23]. However, these techniques have not so far been successful for the case of closed surfaces. Corollary 1.5, which follows from the general version of Theorem 1.3, shows that the problem can be approached through the study of quantum intersection numbers.

1.3. Fourier coefficients of trace functions and volume. The works of Sikora-Przytycki [10, 28] and Turaev [32] provide a way to view the TQFT spaces $\{V_r(\Sigma)\}_r$ as quantizations of the character variety $\mathcal{M}(\Sigma) = \text{Hom}(\pi_1(\Sigma), \text{SU}(2))/\text{SU}(2)$ along the Goldman Poisson structure. In this setting, the curve operators $\{T_r^Q\}_r$ form quantizations, in the sense of [15], of the trace functions f_Q defined by

$$f_Q(\rho) = \prod_{\gamma \in Q} (-\text{Tr}(\rho(\gamma))),$$

for any $\rho \in \mathcal{M}(\Sigma)$. More specifically, the composition and the commutator of operators correspond to the product and Goldman's Poisson bracket of the trace functions. A pants decomposition P , viewed as a family of $3g - 3$ curves, induces a natural action of a torus on $\mathcal{M}(\Sigma)$, which is the hamiltonian flow of the moment map

$$h_P : \rho \longmapsto \left(\frac{1}{\pi} \arccos(\text{Tr}(\rho(\alpha))) \right)_{\alpha \in P},$$

defined in [21]. Given P, Q , the trace function f_Q has a Fourier decomposition with respect to the torus action defined by P and by [15], the coefficients $G_k^Q(\theta, 0)$ correspond to the Fourier coefficients f_Q . In this setting, Theorem 1.3 shows that the number of non-zero Fourier coefficients of the trace functions f_Q grows as the intersection number of P, Q and it improves the bound of 2^{3g-3} that was previously given in [12]. From this view point,

Theorem 1.2 reveals a previously unnoticed relation of Goldman’s picture to volumes of hyperbolic 3-manifolds, while the proof relies on purely quantum topological techniques.

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2. TQFT CURVE OPERATORS AND THEIR COEFFICIENT FUNCTIONS

In this section, after recalling the setting from [5] and the results we need from [15], we will give the precise definition of the quantum intersection number, state a general version of Theorem 1.3 and outline the strategy of its proof.

2.1. Preliminaries and definitions. Let Σ be a closed surface of genus $g \geq 1$. A *multi-curve* on Σ is a collection of disjoint simple closed curves on Σ and a *pants decomposition* is a multicurve with $3g - 3$ curves that decomposes Σ into pants (3-holed 2-spheres). Given a pants decomposition P and a trivalent graph Γ embedded on Σ we will say that P and Γ are *dual* (or Γ is dual to P) if each vertex of Γ lies in exactly one pants of P , and each edge of Γ intersects a single curve of P exactly once. When we want to stress the fact that Γ is dual to P , we will use the notation Γ_P .

Given a pants decomposition P with dual graph Γ let E denote the set of edges of Γ . We will use the notation $P = \{\alpha_e \mid e \in E\}$, where α_e denotes the curve in P that intersects the edge e of Γ . Following [5], we assume that Γ is *banded* (i.e. thickened with a cyclic ordering at each vertex) and that Σ is obtained as a double of Γ .

Recall that for any integer $r \geq 3$, we defined $\mathcal{C}_r = \{1, 2, \dots, r - 1\}$. For $\zeta_r = -e^{\frac{\pi i}{2r}}$, the $SU(2)$ -Witten-Reshetikhin-Turaev TQFT [29, 33, 34] assigns to Σ a finite dimensional \mathbb{C} -vector space $V_r(\Sigma)$ with a non-degenerate Hermitian pairing

$$\langle \cdot \rangle : V_r(\Sigma) \times V_r(\Sigma) \longrightarrow \mathbb{C}.$$

Definition 2.1. An *r-admissible coloring* of Γ is a function $\mathbf{c} : E \longrightarrow \mathcal{C}_r$ such that for any triple of edges e, f, g adjacent to the same vertex of Γ , we have

- (i) $c_e + c_f + c_g < 2r$ and $c_e + c_f + c_g$ an odd number; and
- (ii) $c_e < c_f + c_g$, $c_f < c_e + c_g$ and $c_g < c_e + c_f$.

Let $U_r := U_r(\Gamma)$ denote the set of all r -admissible colorings of Γ .

Remark 2.2. We warn the reader that, compared to [5], we shifted all colors by 1, following the conventions of [25] for example. Hence, for instance, the color 2 corresponds to the fundamental $U_q(\mathfrak{sl}_2)$ -representation of dimension 2.

By [5] the number of admissible colorings is independent of the choice of P or Γ and it is equal to the dimension of $V_r(\Sigma)$: Given $\mathbf{c} \in U_r$ the authors in [5] give a construction that

associates a vector $\phi_{\mathbf{c}}$ in $V_r(\Sigma)$, such that the set $\mathcal{B}_P := \mathcal{B}_P^r = \{\phi_{\mathbf{c}} \mid \mathbf{c} \in U_r\}$ is basis of $V_r(\Sigma)$ that is orthonormal with respect to the Hermitian pairing. Throughout the paper we will refer to \mathcal{B}_P as the orthonormal basis corresponding to P (or sometimes to Γ).

Given a multicurve γ , [5] assigns Hermitian operators $T_r^\gamma : V_r(\Sigma) \rightarrow V_r(\Sigma)$, known as *curve operators*: The spaces $V_r(\Sigma)$ are certain quotients of a version of the Kauffman skein module $K(H, \zeta_r)$, where H is a handlebody with $\partial H = \Sigma$. The product on $K(H, \zeta_r)$, where for any $w \in K(H, \zeta_r)$ the skein $\gamma.w$ is the result of stacking γ above w , descends to the Hermitian operator T_r^γ on $V_r(\Sigma)$. We recall two key properties that we need here, and we refer the reader to [15, Section 2] and references there in for more details.

- The map $K(\Sigma, \zeta_r) \rightarrow \text{End}(V_r(\Sigma))$ where $\gamma \rightarrow T_r^\gamma$ is an algebra morphism.
- Let γ is a multicurve such that each curve in it is parallel to some $\alpha_e \in P$, where P is a pants decomposition of Σ . Then, for any vector $\phi_{\mathbf{c}}$ of the basis corresponding to P we have

$$(2) \quad T_r^\gamma(\phi_{\mathbf{c}}) = \prod_{e \in E} \left(-2 \cos \frac{\pi c_e}{r} \right)^{n_e} \phi_{\mathbf{c}},$$

where $n_e \geq 0$ is the number of curves in γ that are parallel to $\alpha_e \in P$.

The large- r behavior of curve operators for closed surfaces of genus at least two was studied by the first author [15]. To recall the results from [15] that we will use in this paper, we need some preparation.

Fix a decomposition P with dual graph Γ and corresponding basis $\mathcal{B}_P = \{\phi_{\mathbf{c}} \mid \mathbf{c} \in U_r\}$. Let \mathbb{R}^E denote the set consisting of all $(3g-3)$ -tuples, defined by functions $\tau : E \rightarrow (0, 1)$. Consider the subset $U \subset \mathbb{R}^E$ consisting of all $(3g-3)$ -tuples, defined by functions such that for any triple of edges e, f, g adjacent to the same vertex of Γ we have

- (i) $\tau_e + \tau_f + \tau_g < 2$; and
- (ii) $\tau_e < \tau_f + \tau_g$, $\tau_f < \tau_e + \tau_g$ and $\tau_g < \tau_e + \tau_f$.

By Definition 2.1, for every $r \geq 3$ and any admissible coloring $\mathbf{c} \in U_r$, we have $\tau := \frac{\mathbf{c}}{r} \in U$, where $\tau_e := \frac{c_e}{r}$, for all $e \in E$.

We recall that given $\mathbf{c} \in U_r$ the article [15] defines a function $\bar{\mathbf{c}} : H_1(\Sigma, \mathbb{Z}/2\mathbb{Z}) \rightarrow \{\pm 1\}$, as follows: Let $L_{\mathbf{c}}$ be a multicurve on Σ consisting of $c_e - 1$ parallel strands along each edge e of Γ , joined at vertices without crossing. As Σ is obtained by gluing together two copies of Γ along corresponding boundary components, let $\tilde{L}_{\mathbf{c}}$ consist of two copies in $L_{\mathbf{c}}$, one in each copy of Γ . For γ a multicurve on Σ , we define $\bar{\mathbf{c}}(\gamma) = (-1)^{I(\tilde{L}_{\mathbf{c}}, \gamma)}$, where $I(\tilde{L}_{\mathbf{c}}, \gamma)$ is the total geometric intersection number of γ and $\tilde{L}_{\mathbf{c}}$.

Theorem 2.3. [15, Theorem 1.1] *Fix P , Γ and U as above. Given a multicurve γ , there is an open subset $V_\gamma \subset U \times [0, 1]$ containing $U \times \{0\}$, and unique analytic functions*

$$G_{\mathbf{k}}^\gamma : V_\gamma \rightarrow \mathbb{C}, \quad \text{where } \mathbf{k} : E \rightarrow \mathbb{Z},$$

such that we have:

- (1) For any $\mathbf{c} \in U_r$ such that $(\frac{\mathbf{c}}{r}, \frac{1}{r}) \in V_\gamma$, we have $G_{\mathbf{k}}^\gamma(\frac{\mathbf{c}}{r}, \frac{1}{r}) = \bar{\mathbf{c}}(\gamma) \langle T_r^\gamma(\phi_{\mathbf{c}}), \phi_{\mathbf{c}+\mathbf{k}} \rangle$ for any $\mathbf{c} \in U_r$.
- (2) We have $G_{\mathbf{k}}^\gamma = 0$ for any \mathbf{k} for which there is $e \in E$ such that $|\mathbf{k}(e)| > I(\alpha_e, \gamma)$ or $\mathbf{k}(e) \not\equiv I(\alpha_e, \gamma) \pmod{2}$. Here, α_e is the curve in P that corresponds to e and $I(\alpha_e, \gamma)$ is the geometric intersection number of α_e, γ .

Remark 2.4. Strictly speaking, in [15] it is only proven that such analytic functions $G_{\mathbf{k}}^\gamma$ exist and uniqueness is not discussed. Note, however, that if an analytic function $G_{\mathbf{k}}^\gamma$ on V_γ satisfies part (1) of Theorem 2.3, then it is determined on all elements $(\frac{\mathbf{c}}{r}, \frac{1}{r}) \in V_\gamma$ where $\mathbf{c} \in U_r$. For $x \in U$, we can choose a sequence of r -admissible colorings \mathbf{c}_r such that $(\frac{\mathbf{c}_r}{r}, \frac{1}{r}) \xrightarrow{r \rightarrow \infty} (x, 0)$. From this we deduce that $G_{\mathbf{k}}^\gamma$ is uniquely determined on $U \times \{0\}$, and also, looking at the expansion of $\langle T_r^\gamma(\phi_{\mathbf{c}_r}), \phi_{\mathbf{c}_r+\mathbf{k}} \rangle$ at any order in $\frac{1}{r}$, that all derivatives of $G_{\mathbf{k}}^\gamma$ with respect to the second variable are determined on $U \times \{0\}$, which implies that $G_{\mathbf{k}}^\gamma$ is uniquely determined on V_γ .

Definition 2.5. Given P a pants decompositions of Σ and γ a multicurve, consider the operators T_r^γ and their expansion in the sense of Theorem 2.3 with respect to the bases corresponding to P . Define $n(\gamma, P)$ to be the number of $\mathbf{k} \neq \mathbf{0}$, such that $G_{\mathbf{k}}^\gamma \neq \mathbf{0}$. We will call $n(\gamma, P)$ the *quantum intersection number* of γ and P .

Also if Q is another pants decomposition, which we can view as a multicurve, define $n(Q, P)$ also as the number of $\mathbf{k} \neq \mathbf{0}$ such that $G_{\mathbf{k}}^Q \neq \mathbf{0}$.

Let us stress that the definition of $n(Q, P)$ is not symmetric in P, Q . The pants decomposition Q plays the role of a multicurve while the pants decomposition P provides a basis of $V_r(\Sigma)$. We also stress that we are restricting ourselves $\mathbf{k} \neq \mathbf{0}$ so we don't count the diagonal terms $G_{\mathbf{0}}^\gamma$ in Definition 2.5.

2.2. Two basic properties. Here we derive two basic properties of the quantum intersection number. We begin with the following lemma.

Lemma 2.6. *Fix a pants decomposition P of Σ . Any multicurve γ , splits into multicurves γ' and γ'' (some of which may be empty), so that each curve in γ'' is parallel to some curve in P . We have $n(\gamma, P) = n(\gamma', P)$.*

Proof. By the properties of curve operators we recalled earlier, we can write T_r^γ as a product $T_r^\gamma = T_r^{\gamma'} T_r^{\gamma''}$, where the matrix representing $T_r^{\gamma''}$ with respect to the basis corresponding to P is diagonal with non-zero eigenvalues (see Equation (2)). Thus, for any $r \geq 3$, the number of non-zero entries in the matrices representing T_r^γ and $T_r^{\gamma'}$ is the same. \square

The mapping class group $\text{Mod}(\Sigma)$ acts on the set of multicurves of Σ and on $C_P^0(\Sigma)$. The next result shows that the the quantum intersection number is invariant under this action.

Proposition 2.7. *For any $\phi \in \text{Mod}(\Sigma)$, we have $n(\phi(\gamma), \phi(P)) = n(\gamma, P)$, for all multicurves γ and $P \in C_P^0(\Sigma)$.*

Proof. We recall from [5] that for any $r > 2$ we have a projective representation

$$\rho_r : \text{Mod}(\Sigma) \rightarrow \mathbb{P}\text{End}(V_r(\Sigma)).$$

It follows from the definitions that for any $\phi \in \text{Mod}(\Sigma)$, and any multicurve γ , we have

$$\rho_r(\phi) T_r^\gamma \rho_r(\phi)^{-1} = T_r^{\phi(\gamma)}.$$

Hence, in particular, if P is a pants decomposition, then $\rho_r(\phi) T_r^P \rho_r(\phi)^{-1} = T_r^{\phi(P)}$. Moreover, $\rho_r(\phi)$ sends the basis \mathcal{B}_P of $V_r(\Sigma)$ associated to P , to the basis $\mathcal{B}_{\phi(P)}$ associated to $\phi(P)$, up to scalars that are roots of unity (up to *phases*). By Equation (2), the basis \mathcal{B}_P is an orthonormal basis of eigenvectors of the curve operators $T_r^{\alpha_e}$, for $\alpha_e \in P$, and the corresponding eigenvectors are non-zero. Moreover, such a basis is unique up to phases. Therefore, for all $r \geq 3$, the coefficients of T_r^γ in \mathcal{B}_P are the same (up to phases) as the coefficients of $T_r^{\phi(\gamma)}$ in the basis $\mathcal{B}_{\phi(P)}$. This implies that $n(\gamma, P) = n(\phi(\gamma), \phi(P))$. \square

2.3. Quantum and geometric intersection number. The main technical result of the paper is the following theorem.

Theorem 2.8. *Given P and γ a pants decomposition and a multicurve on Σ , let $I(\gamma, P)$ denote the total intersection number of γ and P . We have*

$$\frac{I(\gamma, P)}{3g-3} \leq n(\gamma, P) \leq (I(\gamma, P) + 1)^{3g-3} - 1,$$

where $n(\gamma, P)$ denotes the quantum intersection number of γ with respect to P .

The upper inequality in Theorem 2.8, which is a direct consequence of Theorem 2.3, is derived in the next lemma.

Lemma 2.9. *We have $n(\gamma, P) \leq (I(\gamma, P) + 1)^{3g-3} - 1$, for all $P \in C_P^0(\Sigma)$ and multicurves γ on Σ .*

Proof. If $I(\gamma, P) = 0$, then both sides of the inequality are zero and there is nothing to prove.

Suppose that $I(\alpha_e, \gamma) \neq 0$ for some $e \in E$, and that $\mathbf{k} : E \rightarrow \mathbb{Z}$ is a function such that $G_{\mathbf{k}}^\gamma \neq \mathbf{0}$ and $\mathbf{k}(e) \neq 0$.

Then, by part (2) of Theorem 2.3, there are at most $I(\alpha_e, \gamma) + 1$ choices for $\mathbf{k}(e)$. Thus, and since in Definition 2.5 we only count these non-diagonal terms $G_{\mathbf{k}}^\gamma \neq \mathbf{0}$, we have

$$n(\gamma, P) \leq \prod_{e \in E} (I(\alpha_e, P) + 1) - 1 \leq (I(\gamma, P) + 1)^{3g-3} - 1.$$

\square

Note that Theorem 1.3 is a special case for Theorem 2.8 where the multicurve γ is also a pants decomposition Q .

Remark 2.10. Given a pants decomposition $P = \{\alpha_e \mid e \in E\}$ and a multicurve γ , consider the set of maps $\mathbf{m} : E \rightarrow \mathbb{Z}$, such that for any $e \in E$, we have $\mathbf{m}(e) \in \{\pm I(\gamma, \alpha_e)\}$. If $I(\gamma, P) \neq 0$, let $m_\gamma \leq 3g - 3$ denote the number of edges $e \in E$ such that $I(\gamma, \alpha_e) \neq 0$. By [15, Theorem 1.3] and a result of Charles and Marché [12, Theorem 5.1] it follows that for each $\mathbf{m} \neq \mathbf{0}$ we have $G_{\mathbf{m}}^\gamma \neq 0$. This gives that $n(\gamma, P) \geq 2^{m_\gamma}$, which was the only lower bound of the quantum intersection number known before Theorem 2.8.

We now briefly describe our approach to proving the lower inequality of Theorem 2.8: Let P be a pants decomposition of Σ with dual graph Γ . Given a multicurve $\gamma \neq P$, define

$$(3) \quad M(\gamma, P) := \max_{\alpha_e \in P} \{I(\gamma, \alpha_e)\}.$$

We can pick a curve α_{e_0} , dual to an edge $e_0 \in \Gamma$ such that $M(\gamma, P) = I(\gamma, \alpha_{e_0})$. For any integer $|\sigma| \leq I(\gamma, \alpha_{e_0})$ consider the function $\mathbf{k}_\sigma : E \rightarrow \mathbb{Z}$, defined by

$$\mathbf{k}_\sigma(e) = \begin{cases} \sigma & \text{if } e = e_0, \\ I(\gamma, \alpha_e) & \text{if } e \neq e_0. \end{cases},$$

where $\alpha_e \in P$ is the curve dual to the edge $e \in \Gamma$. We will show the following:

Theorem 2.11. *For each integer $\sigma \neq 0$, with $|\sigma| \leq M(\gamma, P)$ and $\sigma \equiv M(\gamma, P) \pmod{2}$, we have $G_{\mathbf{k}_\sigma}^\gamma \neq \mathbf{0}$.*

By Equation (3), $I(\gamma, P) \leq (3g - 3)M(\gamma, P)$ and by Theorem 2.11 and Definition 2.5, $M(\gamma, P) \leq n(\gamma, P)$. Hence, the lower inequality of Theorem 2.8 follows once we have Theorem 2.11. The proof of Theorem 2.11 is given in Section 5.

2.4. Elementary moves and quantum intersection number. Let P be a pants decomposition of a surface Σ of genus $g > 1$. Consider a curve $\alpha_e \in P$ bordering two pairs of pants giving a four-holed 2-sphere subsurface of $\Sigma_{0,4}$ of Σ .

Definition 2.12. An A -move replaces $\alpha_e \in P$ with an essential curve on $\Sigma_{0,4}$ that intersects α_e at two points. The effect of an A -move on the dual graph Γ is *elementary shift*. See top row of Figure 1.

An S -move replaces a curve $\alpha_e \in P$ that lies in the interior of an one-holed torus subsurface $\Sigma_{1,1} \subset \Sigma$, by a curve that lies on $\Sigma_{1,1}$ and intersects α_e once. The S -move leaves the graph Γ unchanged. See bottom row of Figure 1.

Shortly, we will refer to an A or S -move as an *elementary move*.

Note that an elementary move on a pants decomposition P produces another pants decomposition of Σ .

Proposition 2.13. *If P, Q are pants decompositions that differ by an elementary move, then $n(P, Q) = n(Q, P) = 2$.*

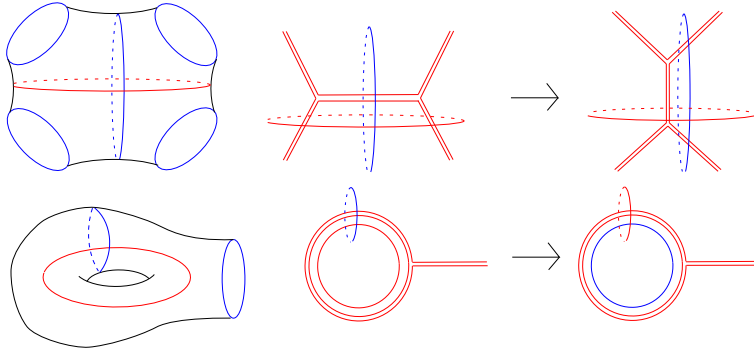


FIGURE 1. Elementary moves of pants decompositions and dual graphs. The portion of the dual graph affected by the elementary move is indicated by the red double lines (grey double lines in black-and-white/grey scale). In both moves the closed curves shown in the interior of the surface shown are exchanged.

Proof. The proof is given in [15, Proposition 4.2]. Alternatively, the proposition follows from [15, Theorem 1.3] and [12, Theorem 5.1] as mentioned in Remark 2.10: Let α_{e_0} be the curve of P that is changed using the A - or S -move. Since $I(Q, \alpha_e) = 0$ for any $e \neq e_0$, and $I(Q, \alpha_{e_0}) \in \{1, 2\}$, Theorem 2.3 and Remark 2.10 imply the result. \square

The proof of Theorem 2.11 will occupy the next three sections. By Lemma 2.6, during the entire course of the proof, we may and will assume that the multicurve γ does not have any component isotopic to a curve in P . That is $I(\gamma, \alpha_e) \neq 0$ for each $\alpha_e \in P$.

With the statement of Theorem 2.8 at hand, the reader who is eager to move to the proofs of Theorems 1.1 and 1.2 could move to Section 7 without loss of continuity.

3. DEHN-THURSTON POSITION AND FUSION RULES

In this section we review the basics of Dehn-Thurston position for curves on surfaces and the fusion theory for computing coefficients of curve operators, both adapted in the setting that will be used for the proof of Theorem 2.11.

3.1. Dehn-Thurston position and swift number of arcs. Let us fix a pants decomposition $P = \{\alpha_e \mid e \in E\}$ of Σ with dual graph Γ , and with set of edges E . We replace each curve α_e in P by two parallel copies which we denote by α_e, α'_e , and we let $P' := \{\alpha'_e \mid e \in E\}$.

Therefore we get a decomposition of Σ into $2g - 2$ pairs of pants and $3g - 3$ annuli.

Definition 3.1. The set of curves $P \cup P'$ will be called a *decomposition system* of Σ . We will refer to the components of $\Sigma \setminus (P \cup P')$ as the pieces of the system $P \cup P'$.

A multicurve γ on Σ is said to be in *Dehn-Thurston position* with respect to $P \cup P'$, if we have the following:

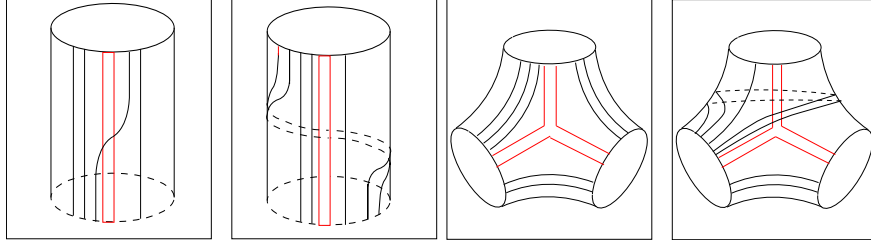


FIGURE 2. Patterns of a multicurve in Dehn-Thurston position on pieces of a decomposition system.

- The intersection of γ with each piece of $P \cup P'$ consists of properly embedded arcs so that the number of intersection points with each curve α_e (or α'_e) is $I(\gamma, \alpha_e)$.
- On the pants pieces of $P \cup P'$ there are two patterns for arcs of γ : Arcs run between different boundary components of a pants piece, or they have both endpoints on the same boundary component of the piece. Arcs of the later kind will be referred as *come-back patterns*. See Figure 2, where the portion of Γ on the decomposition pieces illustrated is shown in red double line (shown as grey in white-and-black/grey scale).
- On the annuli pieces γ is determined, up to *fractional Dehn twists*, by the number of intersection points with the boundary of the annuli. Each arc is assigned the *swift number*, which is defined in Definition 3.2 below. Examples are given in Figure 2.

Given a decomposition system of Σ any multicurve γ can be isotoped into Dehn-Thurston position [20].

To give the precise definition of swift numbers we need some conventions on annuli pieces of $P \cup P'$: We consider such an annulus $A = S^1 \times [0, 1]$ embedded in \mathbb{R}^3 so that, for $i = 0, 1$, $\partial A^i := S^1 \times \{i\}$ is identified with the unit circle centered at the origin on the plane $z = i$. Thus, traveling in the counter clockwise direction we can talk of the first, second, third and fourth quarter of ∂A^i .

Given γ in Dehn-Thurston position, we will assume that the points of $\gamma \cap \partial A^i$ occur in the third and fourth quarter of ∂A^i . Recall that Σ is obtained by gluing Γ to a second copy of itself Γ' . We will assume that $e_S := \Gamma \cap A$ (resp $e_N := \Gamma' \cap A$) is a straight segment on A that runs from the south pole (resp. north pole) of ∂A^1 to the south pole (resp. north pole) of ∂A^0 . In Figure 2, e_S is shown in red double lines (grey in white-and-black/grey scale) while e_N is omitted.

Definition 3.2. With the notation and setting as above, let y be an arc $\gamma \cap A$ that runs from ∂A^1 to ∂A^0 and let $|y \cap (e_N \cup e_S)|$ denote the geometric intersection number of y with the arcs e_N, e_S . The *swift number* of y , denoted by t_y is defined as follows:

- We have $t_y = 0$, if $|y \cap (e_N \cup e_S)| = 0$.
- We have $t_y := |y \cap (e_N \cup e_S)|$, if either

- (i) $y \cap \partial A^1$ is in the third quarter of ∂A^1 and the first point of the intersection of y with $e_N \cup e_S$ is on e_N ; or
 - (ii) $t_y := |y \cap (e_N \cup e_S)|$, if $y \cap \partial A^1$ is in the fourth quarter of ∂A^1 and the first point of the intersection of y with $e_N \cup e_S$ is on e_S .
- We have $t_y := -|y \cap (e_N \cup e_S)|$, if either
- (i) $y \cap \partial A^1$ is in the third quarter of ∂A^1 and the first point of the intersection of y with $e_N \cup e_S$ is on e_S ; or
 - (ii) if $y \cap \partial A^1$ is in the fourth quarter of ∂A^1 and the first point of the intersection of y with $e_N \cup e_S$ is on e_N .

To illustrate Definition 3.2, we discuss the swift numbers of the arcs appearing on the two cylinder pieces of Figure 2: In the first from the left panel of the figure, we have four arcs, three of which have swift number 0 and one has swift number 1. In the second panel of Figure 2, we have three arcs of swift number 0 and two arcs of swift number -1.

3.2. Fusion rules at a limit. Given a multicurve γ in Dehn-Thurston position with respect to a decomposition system $P \cup P'$, we can use a version of the fusion rules of Masbaum-Vogel [26] to compute the coefficient functions G_k^γ of the operators T_r^γ . The adaptation of the fusion rules of [26] that applies to our setting, with the orthonormal bases of the TQFT spaces corresponding to P (described in Section 2), was given in [15]. For the convenience of the reader we recall these fusion rules in Figure 6 in Appendix B of the paper. For more details the reader is referred to [15, Section 4].

In order to simplify the search for non-vanishing coefficients G_k^γ , for the proof of Theorem 2.11, it is convenient for us consider their limit when $r \rightarrow \infty$, $c_e = a$ for all edges e , and $\frac{a}{r} \rightarrow \theta$ where $\theta \in [0, 1]$. This is equivalent to just computing the evaluation $G_k^\gamma(\theta, 0)$, and thus clearly a lower bound for the number of non-vanishing coefficients G_k^γ . The advantage of working with this limit is that we work with a simplified set of fusion rules by replacing the general fusion rules given in Figure 6 by their limit when $r \rightarrow \infty$, we have $c_e = a$ for every edge e of Γ , and $\frac{a}{r} \rightarrow \theta$ where $\theta \in [0, 1]$. These leads to simplified coefficients as shown Figure 3. In the figures, the colors of thick red edges are labelled by letters. The thin black edges are colored by 2. That is they are colored by the fundamental $U_q(\mathfrak{sl}_2)$ -representation of dimension 2, which, as explained in Remark 2.2, corresponds to the color 1 of [5].

Remark 3.3. In the above limiting setting, the fusion coefficients only depend of the type of the fusion rule used (i.e. strand, positive/negative half-twist, positive/negative/mixed triangle, positive/negative bigon) and not on the colors of edges. We also note that the r -admissibility condition (i) of Definition 2.1 implies that a is odd. As a result the signs $(-1)^{a+1}$ for the fusion coefficients of twist rules in the second row of Figure 3 do not appear in our computations.

$$\begin{aligned}
& \left| \begin{array}{c} \text{red double line} \\ a \end{array} \right| = \left| \begin{array}{c} \text{red double line} \\ a+1 \end{array} \right| - \left| \begin{array}{c} \text{red double line} \\ a-1 \end{array} \right| \\
& \left| \begin{array}{c} \text{red double line} \\ a \end{array} \right| \left| \begin{array}{c} \text{red double line} \\ a+1 \end{array} \right| = (-1)^{a+1} e^{i\frac{\pi\theta}{2}} \left| \begin{array}{c} \text{red double line} \\ a+1 \end{array} \right| \\
& \left| \begin{array}{c} \text{red double line} \\ a \end{array} \right| \left| \begin{array}{c} \text{red double line} \\ a-1 \end{array} \right| = -(-1)^{a+1} e^{-i\frac{\pi\theta}{2}} \left| \begin{array}{c} \text{red double line} \\ a-1 \end{array} \right| \\
& \left| \begin{array}{c} \text{red double line} \\ a \end{array} \right| \left| \begin{array}{c} \text{red double line} \\ a+1 \end{array} \right| \left| \begin{array}{c} \text{red double line} \\ a+1 \end{array} \right| = \frac{\langle \frac{3\theta}{2} \rangle \langle \frac{\theta}{2} \rangle^{\frac{1}{2}}}{\langle \theta \rangle} \left| \begin{array}{c} \text{red double line} \\ a \end{array} \right| \left| \begin{array}{c} \text{red double line} \\ a+1 \end{array} \right| \\
& \left| \begin{array}{c} \text{red double line} \\ a \end{array} \right| \left| \begin{array}{c} \text{red double line} \\ a+1 \end{array} \right| \left| \begin{array}{c} \text{red double line} \\ a-1 \end{array} \right| = \frac{\langle \frac{\theta}{2} \rangle}{\langle \theta \rangle} \left| \begin{array}{c} \text{red double line} \\ a \end{array} \right| \left| \begin{array}{c} \text{red double line} \\ a-1 \end{array} \right| \\
& \left| \begin{array}{c} \text{red double line} \\ a \end{array} \right| \left| \begin{array}{c} \text{red double line} \\ a-1 \end{array} \right| \left| \begin{array}{c} \text{red double line} \\ a-1 \end{array} \right| = -\frac{\langle \frac{3\theta}{2} \rangle \langle \frac{\theta}{2} \rangle^{\frac{1}{2}}}{\langle \theta \rangle} \left| \begin{array}{c} \text{red double line} \\ a \end{array} \right| \left| \begin{array}{c} \text{red double line} \\ a-1 \end{array} \right| \\
& \left| \begin{array}{c} \text{red double line} \\ a \pm 1 \end{array} \right| \left| \begin{array}{c} \text{red double line} \\ a \end{array} \right| \left| \begin{array}{c} \text{red double line} \\ a \pm 1 \end{array} \right| = \pm \left| \begin{array}{c} \text{red double line} \\ a \pm 1 \end{array} \right| \\
& \left| \begin{array}{c} \text{red double line} \\ a \mp 1 \end{array} \right| \left| \begin{array}{c} \text{red double line} \\ a \end{array} \right| \left| \begin{array}{c} \text{red double line} \\ a \mp 1 \end{array} \right| = 0
\end{aligned}$$

FIGURE 3. The limit of fusion rules when $c_e = a$, and $r \rightarrow \infty$, $\frac{a}{r} \rightarrow \theta$. Here we let $\langle \theta \rangle = \sin(\pi\theta)$. The red thick lines (shown in grey in black-and-white/grey scale) indicate portions of the graph Γ . The black thinner lines indicate arcs of γ and the color on these portions is 2.

4. FUSION COMPUTATIONS AT A LIMIT

4.1. A state sum formula. With the notation and terminology of Section 3, let us fix a decomposition system $P \cup P' = \{\alpha_e, \alpha'_e \mid e \in E\}$ of Σ , with dual graph Γ . Let γ be a multicurve on Σ that is in Dehn-Thurston position with respect to $P \cup P'$.

Definition 4.1. A *total color shift* on Γ is an E -tuple $\sigma = (\sigma_e | e \in E)$, defined by a function $\sigma : E \rightarrow \mathbb{Z}$, such that $|\sigma_e| \leq I(\gamma, \alpha_e)$.

Recall that we are considering the functions G_k^γ of Theorem 2.3, at the limit when $r \rightarrow \infty$, $c_e = a$ for all edges e , and $\frac{a}{r} \rightarrow \theta$ where $\theta \in [0, 1]$. Let ϕ_a be the TQFT vector corresponding to the coloring that assigns a to each edge of Γ . Given a total color shift σ , we will use $C(\sigma)$ to denote the limit of the coefficients of $T_r^\gamma \phi_a$ along $\phi_{a+\sigma}$. We write

$$(4) \quad T_r^\gamma \phi_a = \sum_{\sigma} C(\sigma) \phi_{a+\sigma} + o(1),$$

where σ ranges over all possible total color shifts, and $o(1)$ denotes a quantity that converges to 0 as $r \rightarrow \infty$.

Our next goal is to give certain state sum expressions for the coefficients $C(\sigma)$. Before we are able to do this we need some preparation: Let X be a pants piece of the decomposition system $P \cup P'$. Note that X contains a Y -shaped portion of Γ , which we will denote by Y . Now X can be constructed by gluing two copies of Y , say Y and Y' along two of their sides. If z is a come-back pattern of γ on X , then z intersects each of Y, Y' exactly once.

Definition 4.2. With the notation and setting as above, we will call the intersection point of z with Y , *the overcrossing* of the come-back pattern. We will call the intersection point of z with Y' , *the undercrossing* of the come-back pattern.

The multicurve γ gives rise to a set of arcs B on Σ , such that for any arc $y \in B$, with endpoints p_1, p_2 , we have one of the following:

- y lies on a pants piece of the decomposition system and p_1, p_2 are on $P \cap \gamma$;
- y lies on a pants piece and p_1 is on $P \cap \gamma$ while p_2 is at an overcrossing or undercrossing of a come-back pattern of γ ;
- y lies on an annulus and p_1 is on $\alpha_e \cap \gamma$, for some $\alpha_e \in P$, while p_2 is on $\alpha'_e \cap \gamma$.

We will refer to B as an *arc system* for γ with respect to $P \cup P'$. Let B_{end} denote the set of all the endpoints of the arcs in B .

Definition 4.3. (*States*) Let $P \cup P', \gamma, B$ and B_{end} be fixed as above. Given a total color shift $\sigma = (\sigma_e | e \in E)$, a *state* on B is a function $s : B_{\text{end}} \mapsto \{\pm 1\}$, such that:

- (a) For any $e \in E$,

$$(5) \quad \sum_{p \in \gamma \cap \alpha_e} s(p) = \sum_{p \in \gamma \cap \alpha'_e} s(p) = \sigma_e.$$

- (b) If p, q are endpoints corresponding to the overcrossing and undercrossing of the same come-back pattern of γ , then $s(p) + s(q) = 0$.

For fixed σ , we will use \mathcal{S}_σ to denote the set of all states on B corresponding to σ . We compute the coefficients $C(\sigma)$ using the fusion rules of Figure 3. For this we need two auxiliary lemmas that we now state. The proofs of the lemmas are given in Section 9.

Lemma 4.4. (Sliding Lemma) *With the conventions of Figure 3 and for any $\varepsilon, \mu \in \{\pm 1\}$, we have*

$$-\left| \begin{array}{c} a + \varepsilon + \mu \\ a + \varepsilon \\ a \end{array} \right| = -\left| \begin{array}{c} a + \varepsilon + \mu \\ a + \mu \\ a \end{array} \right| \quad \text{and} \quad -\left| \begin{array}{c} a + \varepsilon + \mu \\ a + \varepsilon \\ a \end{array} \right| = -\left| \begin{array}{c} a + \varepsilon + \mu \\ a + \mu \\ a \end{array} \right|$$

The second auxiliary lemma shows how to simplify some arcs in annuli pieces of the decomposition system. Those arcs are the type of arcs that remain in annuli pieces after we applied the strand rules.

Lemma 4.5. *With the conventions of Figure 3 and for any $\varepsilon, \mu \in \{\pm 1\}$, we have the following identity*

$$\left| \begin{array}{c} a + \varepsilon \\ a \\ a + \mu \end{array} \right| = (-1)^{(a+1)t} \varepsilon^{t+1} z^{\varepsilon t} \delta_{\varepsilon, \mu} \left| \begin{array}{c} a + \varepsilon \\ a \\ a \end{array} \right|$$

where the arc on the left hand side has swift number t , and $\delta_{\varepsilon, \mu}$ is the Kronecker symbol.

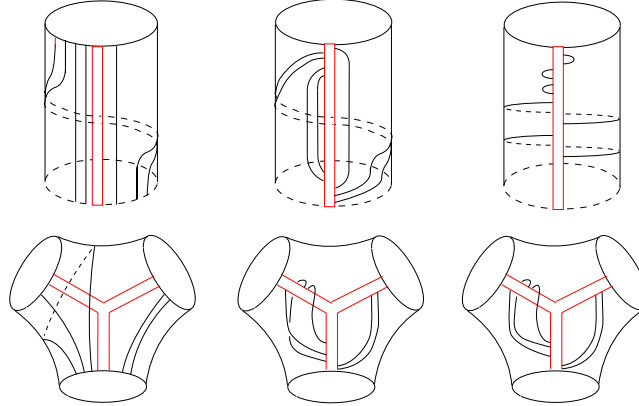


FIGURE 4. Patterns of γ on pieces of a decomposition system before (left) and after Step 1 (middle) and after Step 2 (right).

The strategy for computing the coefficients $C(\sigma)$ is as follows:

- Step 1: For each $e \in E$, we start by applying the strand rule at each point of $\gamma \cap \alpha_e$ and $\gamma \cap \alpha'_e$ as well as at each come-back patterns of γ . The left column of Figure 4 shows patterns of γ on an annulus and pants piece, while the middle column shows the resulting configurations at the end of Step 1.
- Step 2: Apply the Sliding Lemma to simplify the configurations resulting from Step 1. This lemma allows to slide and permute endpoints of remnants of γ on the edges of Γ and to remove crossings between arcs of γ resulting from Step 1. The right column

of Figure 4 shows the results of applying this step to the configurations resulting from Step 1.

- *Step 3:* Use the half-twist, triangle and bigon fusion rules, and Lemma 4.5 on the configurations resulting from Step 2, and collect terms corresponding to the same vectors ϕ_c of the fixed basis \mathcal{B}_P to compute $C(\sigma)$.

Next we assign a *weight* to each arc in B that captures the “cost” of applying Steps 1-3 above. Recall that

$$(6) \quad \Delta_{++} = \frac{\left(\langle \frac{3\theta}{2} \rangle \langle \frac{\theta}{2} \rangle\right)^{\frac{1}{2}}}{\langle \theta \rangle}, \quad \Delta_{--} = -\frac{\left(\langle \frac{3\theta}{2} \rangle \langle \frac{\theta}{2} \rangle\right)^{\frac{1}{2}}}{\langle \theta \rangle} \quad \text{and} \quad \Delta_{+-} = \Delta_{-+} = \frac{\langle \frac{\theta}{2} \rangle}{\langle \theta \rangle}.$$

Given $y \in B$ with endpoints p_1, p_2 , and a state $\sigma \in \mathcal{S}_\sigma$, recall that the values $s(p_1), s(p_2)$ are in $\{+1, -1\}$. Slightly abusing notation, we will use $\Delta_{s(p_1), s(p_2)}$ to denote the quantity from Equation (6) for which the choice of the labels \pm agrees with the signs of $s(p_1), s(p_2)$. For instance, if $s(p_1) = +1$ and $s(p_2) = -1$, then $\Delta_{s(p_1), s(p_2)} = \Delta_{+-}$.

Definition 4.6. (*Weights on arcs*) Given a state s and an arc $y \in B$ with endpoints p_1, p_2 , we define the *weight* of y with respect to s , denoted by $w(y, s)$, as follows:

- If y lies on a pants piece of $P \cup P'$, and p_1, p_2 are on $\gamma \cap P$, then

$$w(y, s) := s(p_1)s(p_2)\Delta_{s(p_1), s(p_2)}.$$

- If y lies on a pants piece and p_1 is on $P \cap \gamma$ while p_2 is at an undercrossing of a come-back pattern, then

$$w(y, s) := s(p_1)\Delta_{s(p_1), s(p_2)}.$$

- If y lies on a pants piece and p_1 is on $P \cap \gamma$ while p_2 is at an overcrossing of a come-back pattern, then

$$w(y, s) = s(p_1)s(p_2)z^{2s(p_2)}\Delta_{s(p_1), s(p_2)}, \quad \text{where } z = e^{i\frac{\pi\theta}{2}}.$$

- If y lies on an annulus piece, then

$$w(y, s) := \begin{cases} 0, & \text{if } s(p_1) \neq s(p_2), \\ s(p_1)^{t+1}z^{ts(p_1)}, & \text{otherwise.} \end{cases}$$

where $t = t_y$ is the swift number of the arc y .

Definition 4.7. (*Total weight of a state*) With the notation and setting as above, we define the *total weight* of a state $s \in \mathcal{S}_\sigma$ by $W(s) := \prod_{y \in B} w(y, s)$.

We are now ready to give the state sum expressions for $C(\sigma)$ promised in the beginning of the subsection.

Proposition 4.8. *Let $P \cup P'$, γ and B be fixed as above. Let ϕ_a be the TQFT vector corresponding to P with all edges colored by a , and let us consider the limit $r \rightarrow \infty$, $\frac{a}{r} \rightarrow \theta$ where θ is some fixed real number. For a total color shift $\sigma = (\sigma_e | e \in E)$, let $C(\sigma)$ denote the limit of the coefficients of $T_r^\gamma \phi_a$ along $\phi_{a+\sigma}$. Then,*

$$C(\sigma) = \varepsilon(\gamma) \sum_{s \in \mathcal{S}_\sigma} W(s) = \varepsilon(\gamma) \sum_{s \in \mathcal{S}_\sigma} \prod_{y \in B} w(y, s).$$

where $\varepsilon(\gamma) \in \{\pm 1\}$ depends only on the curve γ .

Proof. Following [15], we will view $T_r^\gamma \phi_a$ as γ layered over Γ with each edge colored by a and we will apply the three step process outlined earlier. Recall that Step 1 is to apply the strand rule at each intersection point of γ with the decomposition system $P \cup P'$, plus once for each come-back pattern, separating the overcrossing and undercrossing of each come-back pattern into two different arcs. In Steps 2 and 3 we will apply Lemmas 4.4 and 4.5 and fusion rules to delete black arcs in the remnants of Step 1. Because the Sliding Lemma allows to permute the endpoints of the configurations resulting from Step 1 along the edges of Γ , the order with which we apply the strand rule at come-back arcs and for intersection points $\gamma \cap \alpha_e$ or $\gamma \cap \alpha'_e$ is not important. See Figure 4.

Let us first give an interpretation of states explaining, in particular, how conditions (a)-(b) of Definition 4.3 arise in our setting. For each endpoint p of an arc in B that is an intersection point of $\gamma \cap (P \cup P')$ the sign $s(p)$ corresponds to the choice of ± 1 color shift we used while applying the strand rule at p . If p is an overcrossing (resp. undercrossing) of a come-back arc, $s(p)$ is the color shift (resp. -1 times the color shift) we used applying the strand rule at that come-back pattern. This produces a function $s : B_{\text{end}} \mapsto \{\pm 1\}$ that satisfies condition (b) of Definition 4.3.

Note that for each $s \in \mathcal{S}_\sigma$, the remaining pattern after the initial strand rules is a scalar multiple of the vector $\phi_{a+\sigma}$. To compute this scalar (up to sign), we have associated a weight to each arc, representing loosely the cost of removing that arc using fusion rules. Note that the strand rules themselves produce signs, which we need to distribute to the remaining arcs. Our convention is, that the sign of a strand rule corresponding to a come-back pattern, will be distributed to the overcrossing end, while the sign for a strand rule at a point $\gamma \cap (P \cup P')$ will be distributed to the arc that is in a pants piece.

In Step 2, we use Lemma 4.4 to *separate* all remaining arcs. By that we mean that we want to put all arcs in annuli pieces in different slices of the annuli, and for arcs in pants pieces we want no crossing between two different black arcs (with respect to the projection induced by the dual graph). This process is illustrated in Figure 4.

Although the use of Lemma 4.4 to separate arcs may create extra signs, we note that the total sign depends only on the number of times we had to use the second part of Lemma 4.4, which depends only on γ and not the state s .

Finally, we computed the coefficients needed to erase the remaining annuli arcs in Lemma 4.5; they match the weights of Proposition 4.8. Now we explain how condition (a) of Definition 4.3 is used in this process: By above discussion, after applying Steps 1-3 we have a function $s : B_{\text{end}} \mapsto \{\pm 1\}$. Note that if s does not satisfy condition (a) of Definition 4.3, then there is an edge e of Γ , lying on an annulus piece of $P \cup P'$ with boundary consisting of curves α_e, α'_e , such that

$$\sum_{p \in \gamma \cap \alpha_e} s(p) \neq \sum_{p \in \gamma \cap \alpha'_e} s(p).$$

Then, when applying Lemma 4.5 to e , we will have instances where, and with the notation of the figure in the statement of Lemma 4.5, we will have $\epsilon \neq \mu$ yielding zero coefficients and zero contribution to $C(\sigma)$. Hence only functions s that satisfy condition (a) of Definition 4.3 contribute to $C(\sigma)$.

To erase an arc in a pants piece with two endpoints in $\gamma \cap (P \cup P')$, a single triangle rule is needed, which, together with the strand rule signs, gives exactly the weight $w(y, s)$ of Definition 4.6. Finally, to erase an arc from an overcrossing or undercrossing to an intersection point $\in \gamma \cap (P \cup P')$, we need to apply a half-twist rule, then a triangle rule. The overcrossing and undercrossing of a come-back arc yield exactly the same half-twist coefficient. It will be convenient for us to move both coefficients into the weight of the arc of the overcrossing. This gives the second and third rule of Definition 4.6, remembering that we also assign the strand rule sign to overcrossings. \square

4.2. A Laurent polynomial. The second main result of the section is the following proposition the proof of which uses Proposition 4.8.

Proposition 4.9. *Let the notation and setting be as in Proposition 4.8. For any total color shift σ , there exists $n, m \in \mathbb{Z}$ and a Laurent polynomial $P_\sigma(z) \in \mathbb{Q}[z^{\pm 1}]$ such that*

$$C(\delta) = \Delta_{++}^n \Delta_{+-}^m P_\sigma(z),$$

at $z = e^{i\frac{\pi\theta}{2}}$. Here, Δ_{++}, Δ_{+-} are as defined in Equation (6).

Proof. We will deduce this proposition from Proposition 4.8. Note that by Definition 4.6, for any $s \in \mathcal{S}_\sigma$, the total weight is of the form

$$W(s) = \pm \Delta_{++}^{n_1(s)} \Delta_{+-}^{n_2(s)} \Delta_{--}^{n_3(s)} z^{m(s)},$$

where Δ_{++}, Δ_{+-} and Δ_{--} are as defined in Equation (6). Moreover, we claim that $n_1(s) + n_2(s) + n_3(s) = |B_{\text{pants}}|$, where B_{pants} is the set of arcs of B that belong to a pants piece of the decomposition system. Indeed, each triangle coefficient $\Delta_{\varepsilon\mu}$ is created when erasing an arc of B_{pants} .

Next we claim that

$$(7) \quad n_2(s) + 2n_3(s) = K_\gamma + \sum_{e \in E} (I(\alpha_e, \gamma) - \sigma_e),$$

where K_γ is the number of come-back patterns of γ . Hence, in particular $n_2(s) + 2n_3(s)$ is independent of the state $s \in \mathcal{S}_\sigma$.

To prove the claim, first note that by Definition 4.3, the triangle rules of Figure 3 and by Equation (6), it follows that the number $n_2(s) + 2n_3(s)$ is the same as the number of endpoints p of arcs in B_{pants} such that $s(p) = -1$. Thus it is enough to argue that this later number is equal to the quantity in the right hand side of Equation (7).

Recall that pants pieces contain two types of arcs; come-back patterns and arcs that have their endpoints on two different boundary components of a pair of pants. Consider any state $s \in \mathcal{S}_\sigma$. Because s satisfies condition (b) of Definition 4.3, exactly one of the two endpoints of any come-back arc has a $-$ sign. Moreover, for any edge $e \in E$, with total shift σ_e at e , we have

$$\sigma_e = \sum_{p \in \alpha_e \cap \gamma} s(p).$$

The total number of terms in the right hand side of above equation, is the intersection number $I(\alpha_e, \gamma)$. Hence, s takes the value $+1$ exactly $\frac{1}{2}(I(\alpha_e, \gamma) + \sigma_e)$ times on $\alpha_e \cap \gamma$, and the value -1 exactly $\frac{1}{2}(I(\alpha_e, \gamma) - \sigma_e)$ times on $\alpha_e \cap \gamma$. Finally, s also takes the value -1 exactly $\frac{1}{2}(I(\alpha_e, \gamma) - \sigma_e)$ times on points of $\alpha'_e \cap \gamma$ by the same reasoning and by condition (a) of Definition 4.3.

Let us now pick a state $s_0 \in \mathcal{S}_\sigma$ such that $n_2(s_0)$ is maximal, and let us set $N_i := n_i(s_0)$, for $i = 1, 2, 3$. From the above discussion and some elementary linear algebra, we deduce that one can write for any $s \in \mathcal{S}_\sigma$:

$$n_1(s), n_2(s), n_3(s) = N_1 + k(s), N_2 - 2k(s), N_3 + k(s)$$

where $k(s)$ is a non-negative integer. Thus for any $s \in \mathcal{S}_\sigma$, we have that $\frac{W(s)}{\Delta_{++}^{N_1+N_3} \Delta_{+-}^{N_2}}$ is of the form $\pm z^l \left(\frac{\Delta_{++}^2}{\Delta_{+-}^2} \right)^k$ where $l \in \mathbb{Z}$ and $k \geq 0$. We conclude noting that

$$\frac{\Delta_{++}^2}{\Delta_{+-}^2} = \frac{\langle \frac{3\theta}{2} \rangle}{\langle \frac{\theta}{2} \rangle} = \frac{e^{\frac{3i\pi\theta}{2}} - e^{-\frac{3i\pi\theta}{2}}}{e^{\frac{i\pi\theta}{2}} - e^{-\frac{i\pi\theta}{2}}} = e^{i\pi\theta} + 1 + e^{-i\pi\theta} = z^2 + 1 + z^{-2}.$$

□

5. PRODUCING NON VANISHING COEFFICIENTS

In this section we apply the settings and results of Section 4 to prove Theorem 2.11.

5.1. Proof outline. Fix a decomposition system $P \cup P' = \{\alpha_e, \alpha'_e \mid e \in E\}$ of Σ , and suppose that γ is a multicurve in Dehn-Thurston position with respect to it. As before we have an arc system $B = B(\gamma)$ for γ . For any color shift σ on E we can define the set of states \mathcal{S}_σ on B and, as in Proposition 4.8, the limits $C(\sigma)$ of the coefficients T_r^γ .

Let $M(\gamma, P)$ be as in Equation (3) and let $e_0 \in E$ such that $M(\gamma, P) = I(\gamma, \alpha_{e_0})$.

For any $|\delta| \leq M(\gamma, P)$ consider the total color shift $\sigma_\delta = (\sigma_e \mid e \in E)$ given by

$$(8) \quad \sigma_\delta = \begin{cases} \delta & \text{if } e = e_0, \\ I(\gamma, \alpha_e) & \text{if } e \neq e_0. \end{cases},$$

where $\alpha_e \in P$ is the curve dual to the edge $e \in \Gamma$. Since, with the items fixed earlier, σ_δ varies only as δ does, we will use the notation $C(\delta) := C(\sigma_\delta)$ and $\mathcal{S}_\delta := \mathcal{S}_{\sigma_\delta}$.

Theorem 2.11 follows immediately from the following.

Theorem 5.1. *For each $|\delta| \leq M(\gamma, P)$ with $\delta \equiv M(\gamma, P) \pmod{2}$, we have $C(\delta) \neq 0$.*

We begin by noting that the fact that $M(\gamma, P) = I(\gamma, \alpha_{e_0})$ puts restrictions on the types of arcs of γ allowed on the pieces of $P \cup P'$ bordering α_{e_0} or the parallel copy α'_{e_0} .

Lemma 5.2. *Suppose that γ is in Dehn-Thurston position with respect to $P \cup P'$ and that $M(\gamma, P) = I(\gamma, \alpha_{e_0})$, for some $e_0 \in E$. Then, we have the following:*

- (a) *Suppose α_{e_0} and α'_{e_0} belong on two different pants pieces of the decomposition. Then, any come-back pattern of γ on the pants containing α_{e_0} (or α'_{e_0}) must have their endpoints on α_{e_0} (or α'_{e_0}).*
- (b) *Suppose α_{e_0} and α'_{e_0} belong on a single pants piece of the decomposition. Then there is no come-back pattern of γ on this pants piece.*

Proof. We observe that if a pants piece of $P \cup P'$ has boundary curves $\alpha_e, \alpha_f, \alpha_g$, then there is one (or more) come-back arcs with two endpoints on α_e if and only if

$$I(\alpha_e, \gamma) > I(\alpha_f, \gamma) + I(\alpha_g, \gamma).$$

This inequality is an immediate consequence of the fact that γ is into Dehn-Thurston position and it implies the first part (a) of the lemma.

Suppose now that α_{e_0} and α'_{e_0} are boundary curves on a single pants piece and let α_f denote the third boundary curve of the pants. The hypothesis $M(\gamma, P) = I(\gamma, \alpha_{e_0})$, implies that there are no come-back arc in that pants with two endpoints on α_f . Since, by the discussion above,

$$I(\alpha_{e_0}, \gamma) \leq I(\alpha_{e_0}, \gamma) + I(\alpha_f, \gamma),$$

there is also no come-back arc in that pants with two endpoints on α_{e_0} or α'_{e_0} . \square

Given $C(\delta)$ as in the statement of Theorem 5.1, by Proposition 4.9 there is a Laurent polynomial $P_\delta(z) \in \mathbb{Q}[z^{\pm 1}]$ so that

$$C(\delta) = \Delta_{++}^n \Delta_{+-}^m P_\delta(e^{i\frac{\pi\theta}{2}}),$$

for some $m, n \in \mathbb{Z}$ and $z = e^{i\frac{\pi\theta}{2}}$. We will show the following:

Proposition 5.3. *For each $|\delta| \leq M(\gamma, P)$ with $\delta \equiv M(\gamma, P) \pmod{2}$, we have $P_\delta(z) \neq 0$ in $\mathbb{Q}[z^{\pm 1}]$.*

Having Proposition 5.3 at hand, Theorem 5.1 is derived as follows: If $C(\delta)$ is zero then $P_\delta(e^{i\frac{\pi\theta}{2}}) = 0$. Since we can choose θ so that $e^{i\frac{\pi\theta}{2}}$ is a transcendental number we arrive at contradiction.

The remaining of the section will be devoted to the proof of Proposition 5.3.

5.2. Two lemmas and set up. We will use $\deg_z(P_\delta(z))$ to denote the highest power of z in $P_\delta(z)$. By Proposition 4.8 we have

$$C(\delta) = \varepsilon(\gamma) \sum_{s \in \mathcal{S}_\delta} W(s),$$

where $W(s) = \prod_{y \in B} w(y, s)$, $\varepsilon(\gamma) \in \{\pm 1\}$ and the arc weights are given in Definition 4.6.

Note that for any state the total weight $W(s)$ is of the form

$$W(s) = \pm \Delta_{++}^{n_1(s)} \Delta_{+-}^{n_2(s)} z^{m_s},$$

where $m_s \in \mathbb{Z}$. The proof of Proposition 4.9 shows that after multiplying all weights by a common non-zero factor $\Delta_{++}^N \Delta_{+-}^M$, we get monomials $P(s)$ with $\deg_z(P(s)) = m_s + n_1(s)$ up to a constant that does not depend on s . Therefore we will set

$$\deg_z(W(s)) := m_s + n_1(s)$$

or equivalently

$$(9) \quad \deg_z(\Delta_{\varepsilon, \mu}) = \delta_{\varepsilon, \mu} = \frac{1 + \varepsilon\mu}{2}$$

where δ is the Kronecker symbol and the second equality holds since we consider $\varepsilon, \mu \in \{\pm 1\}$. Also, considering the sign of the coefficient of highest degree in z after renormalization, we will set:

$$(10) \quad \text{sign}(\Delta_{++}) = \text{sign}(\Delta_{+-}) = +1, \text{ and } \text{sign}(\Delta_{--}) = -1.$$

To prove Proposition 5.3 for each $\delta \leq |M(\gamma, P)|$, we will determine all the states in \mathcal{S}_δ that contribute to $\deg_z(P_\delta(z))$ and show that there is no cancellation between the total weights corresponding to these states. From the above, we can look for states with $\deg_z(W(s))$ maximal, and show that the corresponding terms do not cancel out.

We begin with two lemmas that describe some properties of the states in \mathcal{S}_δ .

Lemma 5.4. *Suppose that $|\gamma \cap \alpha_{e_0}| > 1$. Let s, s' be two states in \mathcal{S}_δ such that there are $p_2, q_2 \in B_{\text{end}}$ corresponding to the overcrossing and undercrossing of the same come-back arc of γ , respectively, so that $s(p_2) = +1$ and $s(q_2) = -1$ while $s'(p_2) = -1$ and $s'(q_2) = +1$. Suppose, moreover, that for any $q \neq p_2, q_2$, we have $s(q) = s'(q)$. Then, we have $\deg_z(W(s)) < \deg_z(W(s'))$.*

Proof. Suppose that p_2 and q_2 belong to arcs y and y' in B , respectively. Let $p_1, q_1 \in B_{\text{end}}$ denote the second endpoints of y and y' , respectively.

Suppose that $s(p_1) = s'(p_1) = \mu$ and $s(q_1) = s'(q_1) = \epsilon$, where $\epsilon, \mu \in \{\pm 1\}$.

By assumption, the only difference between $W(s)$ and $W(s')$ is the product of weights on the arcs y, y' , where we have

$$w(y, s) w(y', s) = \epsilon \mu z^2 \Delta_{+\mu} \Delta_{-\epsilon}, \quad \text{while} \quad w(y, s') w(y', s') = -\epsilon \mu z^{-2} \Delta_{-\mu} \Delta_{+\epsilon}.$$

By Equation (9), we get:

$$\deg_z(w(y, s) w(y', s)) = 3 + \frac{\mu - \epsilon}{2}, \quad \text{while} \quad \deg_z(w(y, s') w(y', s')) = -1 - \left(\frac{\mu - \epsilon}{2} \right)$$

Since $\mu, \epsilon \in \{\pm 1\}$, we get:

$$\deg_z(w(y, s') w(y', s')) \leq \deg_z(w(y, s) w(y', s)) - 2.$$

and the lemma follows. \square

By Lemma 5.4, while looking to identify the states that contribute to the maximum degree of $P_\delta(z)$, it is enough to look at states that assign the value $+1$ to all points in B_{end} that correspond to overcrossings of some come-back pattern of γ .

With this in mind, we define \mathcal{S}_δ^+ to be the set of all $s \in \mathcal{S}_\delta$ such that we have $s(p) = +1$ for any point $p \in B_{\text{end}}$ that is an overcrossing of a come-back pattern of γ . Hence, by Part (2) of Definition 4.3, for any q that is an undercrossing of a come-back pattern of γ we have $s(q) = -1$.

Lemma 5.5. *We have the following:*

- (a) *Let $s \in \mathcal{S}_\delta$. For any edge $e \neq e_0$ of Γ and any $p \in (\gamma \cap \alpha_e)$, we have $s(p) = 1$.*
- (b) *For any two states $s_1, s_2 \in \mathcal{S}_\delta^+$ and any arc $y \in B$ that has none of its endpoints on $\alpha_{e_0} \cup \alpha'_{e_0}$, we have $w(y, s_1) = w(y, s_2)$.*

Proof. Part (a) follows from Equation (5) in Definition 4.3, and the fact that, by the Equation (8) we have $\sigma_e = I(\gamma, \alpha_e)$. Part (b) follows from (a) and Definition 4.6. \square

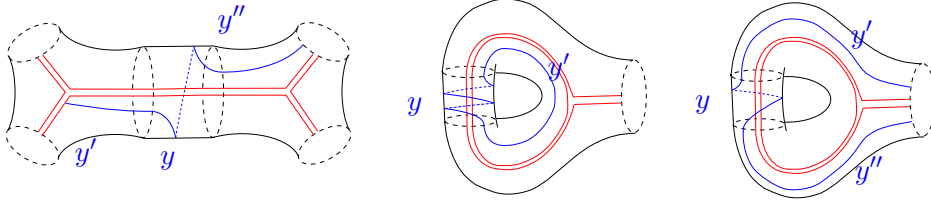


FIGURE 5. Arcs of γ and the edge e_0 in Σ .

To continue let B_{e_0} denote the subset of the arcs system B consisting of arcs that have at least one endpoint on $\alpha_{e_0} \cup \alpha'_{e_0}$. An arc $y \in B_{e_0}$ can have both endpoints on $\alpha_{e_0} \cup \alpha'_{e_0}$ and it lies on the annulus piece of the decomposition system that is dual to the edge e_0 , or can have only one endpoint on $\alpha_{e_0} \cup \alpha'_{e_0}$ and it lies on a pants piece adjacent to the annulus.

The arcs $y \in B_{e_0}$ with both endpoints on $\alpha_{e_0} \cup \alpha'_{e_0}$ are exactly the arcs for which we have $y \cap \alpha_{e_0} = y \cap \alpha'_{e_0} \neq \emptyset$.

Next we will summarize the properties of these arcs that are useful to us. There are two cases to consider according to whether the edge e_0 connects two vertices of Γ or not. See Figure 5.

- (I) The edge e_0 is a loop in Γ and lies on an one-holed torus of Σ . The intersection of γ with the one-holed torus component consists of either simple closed curves included in the one-holed torus, or arcs that start and end on the boundary component α_f of the one-holed torus. In the first case, we may decompose the curve into arcs $y_1, z_1, \dots, y_p, z_p$, where y_i lies inside the pants delimited by $\alpha_{e_0}, \alpha'_{e_0}$ and α_f , while z_i lies in the annulus piece bounded by α_{e_0} and α'_{e_0} . In the second case we decompose the arc into arcs $y_0, z_1, y_1, \dots, y_{p-1}, z_p, y_p$ similarly defined except y_0 and y_p have an extremity on α_f .
- (II) The edge e_0 connects two distinct vertices of Γ corresponding to two distinct pants pieces of the decomposition system. The endpoints of an arc y that lies on the annulus piece dual to e_0 , are connected to two arcs $y', y'' \in B_{e_0}$ that lie on distinct pants pieces of the decomposition. In this case, we define u_y to be the number of endpoints of y, y'' that are undercrossings of a come-back pattern of γ and we define o_y to be the number of endpoints of y, y'' that are overcrossings of a come-back pattern of γ . We also define $v_y := 0$.

In the case where e_0 is not an 1-edge loop, there is a concise description of states of maximum degree, that can be defined using the following notion of complexity:

Definition 5.6. Assume that e_0 is not an 1-edge loop. Given $y \in B_{e_0}$ that lies on the annulus piece of the decomposition dual to e_0 , define the complexity

$$\tau(y \cap \alpha_{e_0}) := t_y - u_y + v_y$$

where t_y is the swift number of y (see Definition 3.2), and u_y, v_y are as defined above.

We will partition the set of points in $\gamma \cap \alpha_{e_0} \neq \emptyset$ into disjoint sets T_1, T_2, \dots, T_k such that, for $1 \leq i \leq k-1$,

- (i) all the points T_i have the same complexity τ_i ; and
- (ii) the points in T_i have complexity strictly smaller than the points in T_j , for any $i < j \leq k-1$. That is $\tau_i < \tau_j$.

5.3. Determining the leading term. Next we will determine all the states in \mathcal{S}_δ that contribute to the maximum degree $\deg_z(P_\delta(z))$. By definition for any $s \in \mathcal{S}_\delta$, the only negative values of s on points in $\gamma \cap P$ occur on points in $\gamma \cap \alpha_{e_0}$ and the number of these negative values is $l = \frac{1}{2}(I(\gamma, \alpha_{e_0}) - \delta)$.

Definition 5.7. Let $\mathcal{S}_{\max} \subset \mathcal{S}_\delta^+$ denote the set of all states such that for any $p, q \in \gamma \cap \alpha_{e_0}$, we have: If $s(p) = -1$ and $s(q) = 1$ then $\tau(p) < \tau(q)$.

Remark 5.8. We can construct the states of Definition 5.7 as follows: Begin by assigning values -1 to points in the set T_1 of minimum complexity. If $l \leq |T_1|$ then we have $\binom{|T_1|}{l}$ states in \mathcal{S}_{\max} . In general, if l is less or equal to the cardinality, say κ , of the set $\cup_{i=1}^j T_i$ and $l - \kappa \leq |T_{j+1}|$, then we have $\binom{|T_{j+1}|}{l-\kappa}$ states in \mathcal{S}_{\max} .

We have the following lemma which implies Proposition 5.3 and hence Theorem 5.1, in the case where e_0 is not a 1-edge loop.

Lemma 5.9. *Assume that e_0 is not a 1-edge loop and $|\gamma \cap \alpha_{e_0}| \geq 1$. Then, we have*

$$\sum_{s \in \mathcal{S}} W(s) \neq 0.$$

Furthermore, for any $s \in \mathcal{S}_{\max}$ and any $s' \notin \mathcal{S}_{\max}$, we have $\deg_z(W(s')) < \deg_z(W(s))$.

Proof. To prove the lemma it is enough to prove the statements (a)-(b) below.

- (a) For any state $s \in \mathcal{S}_{\max}$ and any $s' \notin \mathcal{S}_{\max}$ we have $\deg_z(W(s')) < \deg_z(W(s))$.
- (b) The terms in the sum $\sum_{s \in \mathcal{S}_{\max}} W(s)$ have the same sign; so there is no cancellation.

Next we claim that item (a) will follow from Definition 5.7 and the following statement:

- (c) There is a constant B_γ depending only on γ such that for all $s \in \mathcal{S}_\delta^+$, we have

$$\deg_z(W(s)) = B_\gamma + \sum_{p \in \gamma \cap \alpha_{e_0}} s(p)\tau(p).$$

Indeed, lets us first note that by Lemma 5.4, it is enough to consider states in \mathcal{S}_δ^+ , since for $s \notin \mathcal{S}_\delta^+$, there is $s' \in \mathcal{S}_\delta^+$ such that $\deg_z(W(s)) < \deg_z(W(s'))$. Namely, we can take $s' \in \mathcal{S}_\delta^+$ that differs from s only at overcrossings and undercrossings.

Now, assuming that (c) is true, then for a state s to maximize $\deg_z(W(s))$ one needs to maximize the quantity

$$\sum_{p \in \gamma \cap \alpha_{e_0}} s(p)\tau(p).$$

But for any $p \in \gamma \cap \alpha_{e_0}$, we have $s(p) \in \{\pm 1\}$. Furthermore, for any $s \in \mathcal{S}_\delta^+$, there are the same number of $p \in \gamma \cap \alpha_{e_0}$ such that $s(p) = +1$. Hence to maximize the above sum, one needs to distribute the p 's for which $s(p) = +1$ in decreasing order of the $\tau(p)$, which is, by Definition 5.7, is the same as requesting that $s \in \mathcal{S}_{\max}$. This finishes the proof of the claim that statement (c) implies statement (a).

Let us now turn to proving (b) and (c): For any $s \in \mathcal{S}_\delta^+$ set

$$W_{e_0}(s) := \prod_{y \in B_{e_0}} w(y, s).$$

By Lemma 5.5 and for every state $s \in \mathcal{S}_\delta^+$, we can write $W(s)$ as a product of $W_{e_0}(s)$ and a factor that doesn't depend on s . Thus to finish the proof of the lemma it is enough to prove item (b) and (c) using $W_{e_0}(s)$ instead of $W(s)$.

Note that the type of arcs in the pants with α_{e_0} have the form described in the first case of Figure 5 thanks to Lemma 5.2.

Then we can partition the arcs in B_{e_0} into triples y, y', y'' as described in case (II) of Definition 5.6 and we write

$$W_{e_0}(s) := \prod_{y, y', y'' \in B_{e_0}} (w(y, s) w(y', s) w(y'', s)).$$

Given such a triple of arcs and $s \in \mathcal{S}_\delta^+$, we let $\epsilon_y := s(y \cap \alpha_{e_0}) = s(y \cap \alpha'_{e_0})$, which is the value of s for both end points of y and for one endpoint of each of y', y'' . Let μ and ν denote the values that s assigns to the second endpoint of y' and y'' respectively.

By Definition 4.6, and since s assigns $+1$ to all overcrossing points, we have

$$w(y, s) w(y', s) w(y'', s) = (\epsilon_y)^{t_y+1} z^{t_y \epsilon_y} z^{2o_y} \Delta_{\mu, \epsilon_y}, \Delta_{\epsilon_y, \nu}.$$

By Equation (9), we get:

$$\begin{aligned} \deg_z(w(y, s) w(y', s) w(y'', s)) &= t_y \epsilon_y + 2o_y + \frac{1 + \epsilon_y \mu}{2} + \frac{1 + \epsilon_y \nu}{2} \\ (11) \quad \deg_z(w(y, s) w(y', s) w(y'', s)) &= t_y \epsilon_y + 2o_y + 1 + \epsilon_y(1 - u_y) \end{aligned}$$

where to get the second equality we used that $\mu, \nu = -1$ if the second endpoint of y', y'' is an undercrossing, and $\mu, \nu = +1$ else. This implies that $\frac{\mu + \nu}{2} = 1 - u_y$, for any $\mu, \nu \in \{\pm 1\}$ and $u_y \in \{0, 1, 2, \}$. Moreover, by Equation (10), we get

$$(12) \quad \text{sign}(w(y, s) w(y', s) w(y'', s)) = \epsilon_y^{t_y+1+u_y} = \epsilon_y^{t_y+1-u_y}$$

Combining the contribution of all arcs in B_{e_0} we get:

$$\begin{aligned} \deg_z(W_{e_0}(s)) &= \sum_{y \in B_{e_0}} (t_y \epsilon_y - \epsilon_y u_y + 2o_y + 1 + \epsilon_y) \\ &= 2O_{e_0} + M(\gamma, P) + \delta + \sum_{y \in B_{e_0}} \epsilon_y \tau_y \end{aligned}$$

where O_{e_0} is the total number of overcrossing points in B_{e_0} , $M(\gamma, P)$ is given by Equation (3), and we have set $\tau_y := \tau(y \cap \alpha_{e_0})$. This proves item (c) and thus only the states in \mathcal{S}_{\max} contribute to the maximum degree of $P_\delta(z)$.

Finally, by Equation (12) we have

$$\begin{aligned} \text{sign}(W_{e_0}(s)) &= \prod_{y \in B_{e_0}} (\epsilon_y)^{t_y+1-u_y} \\ &= \prod_{y, \epsilon_y = -1} (-1)^{\tau_y+1}. \end{aligned}$$

This quantity is the same for all $s \in \mathcal{S}_{\max}$, by the definitions, finishing the proof of (b). \square

5.4. Loop-edge. The case where e_0 is a one-edge loop is similar, except we cannot give an explicit description of maximal states. However, it is still possible to show there is no cancellation among maximal states and prove:

Lemma 5.10. *Assume that e_0 is a 1-edge loop and $|\gamma \cap \alpha_{e_0}| \geq 1$. Then we have*

$$\sum_{s \in \mathcal{S}} W(s) \neq 0.$$

Note that thanks to Lemma 5.2, the arcs in the pants containing α_{e_0} are of the form described in the second and third drawing of Figure 5. We should remark that although it is not represented in the picture, they can have arbitrarily large intersection with α_{e_0} .

Proof. We will prove that states of maximal degree all appear with the same sign. Since maximal states are all in \mathcal{S}_δ , it is sufficient to compute the degree and sign of terms $W(s)$ with s in \mathcal{S}_δ . Let us consider the collections Y_1 and Y_2 of components of the intersection γ with the one-holed torus component that is the connected component of α_{e_0} in $\Sigma \setminus (P \setminus \alpha_{e_0})$, where Y_1 corresponds to simple closed curves and Y_2 to arcs that start and end on α_f the boundary component of the one-holed torus.

Let us consider a closed curve y in Y_1 , decomposed as $y_1 z_1 \dots y_p z_p$ where y_i are in the pants piece and z_i are in the annulus piece. Let us call ε_i the values of the state on the intersection points in $z_i \cap (\alpha_{e_0} \cup \alpha'_{e_0})$, remembering that these values are equal at start and end of any arc in an annulus piece. We have

$$w(y, s) := \prod_{i=1}^p w(y_i, s) w(z_i, s) = \prod_{i=1}^p (\varepsilon_i)^{t_i+1} z^{\varepsilon_i t_i} \Delta_{\varepsilon_{i-1}, \varepsilon_i}$$

where $\varepsilon_0 = \varepsilon_p$ by definition and t_i is the swift number of the annulus arc z_i , therefore

$$\deg(w(y, s)) = \sum_{i=1}^p \left(t_i \varepsilon_i + \frac{1 + \varepsilon_{i-1} \varepsilon_i}{2} \right)$$

and

$$\text{sign}(w(y, s)) = (-1)^{\sum_{i=1}^p \left(\frac{(t_i+1)(\varepsilon_i-1)}{2} + \frac{(\varepsilon_i-1)(\varepsilon_{i-1}-1)}{4} \right)} = (-1)^{\frac{1}{2} \sum_{i=1}^p \left(t_i \varepsilon_i + \frac{1 + \varepsilon_{i-1} \varepsilon_i}{2} \right) - \frac{p}{2} - \frac{1}{2} \sum_{i=1}^p t_i},$$

where we used that for $\varepsilon \in \{\pm 1\}$ we have $\varepsilon = (-1)^{\frac{\varepsilon-1}{2}}$, and moreover $\text{sign}(\Delta_{\varepsilon, \mu}) = (-1)^{\frac{(\varepsilon-1)(\mu-1)}{4}}$. Similarly, for an arc y in Y_2 decomposed as $y_0 z_1 \dots z_p y_p$ we have

$$w(y, s) := w(y_0, s) \prod_{i=1}^p w(y_i, s) w(z_i, s) = \Delta_{\varepsilon_0, \varepsilon_1} \prod_{i=1}^p (\varepsilon_i)^{t_i+1} z^{\varepsilon_i t_i} \Delta_{\varepsilon_i, \varepsilon_{i+1}}$$

where by definition $\varepsilon_0 = \varepsilon_p = +1$ since those shifts correspond to intersection points on α_f and we also get that

$$\text{sign}(w(y, s)) = (-1)^{\deg(w(y, s)) + B},$$

where B is a constant which does not depend on $s \in \mathcal{S}_\delta$. Taking the product over arcs in Y_1 and Y_2 , we get that

$$\text{sign}(W(s)) = (-1)^{\deg(W(s))+B'},$$

for all $s \in \mathcal{S}_\delta$, where B' is a constant which does not depend on s . Therefore the maximal degree terms do not cancel out and $\sum_{s \in \mathcal{S}} W(s) \neq 0$. □

We are now ready to complete the proof of Proposition 5.3 which as explained in the beginning of the section also proves Theorem 2.11. Recall from Section 2 that the completion of the proof of Theorem 2.11 also completes the proof of Theorem 2.8.

Proof of Proposition 5.3. By Proposition 4.8 and Lemma 5.9 we have

$$C(\delta) = \varepsilon(\gamma) \sum_{s \in \mathcal{S}_\delta} W(s) \neq 0,$$

and by Proposition 4.9 we have $P_\delta(z) \neq 0$ in $\mathbb{Q}[z^{\pm 1}]$. □

6. A METRIC FROM TQFT ON THE PANTS GRAPH

Let Σ be a closed surface of genus $g > 1$. The *pants graph* $C_P^1(\Sigma)$ is an abstract graph whose set of vertices $C_P^0(\Sigma)$ are in one-to-one correspondence with pants decompositions of Σ . Two vertices are connected by an edge if they are related by an A -move or an S -move shown in Figure 1. It is known that the graph $C_P^1(\Sigma)$ is connected [22].

The path metric $d_\pi : C_P^0(\Sigma) \times C_P^0(\Sigma) \rightarrow [0, \infty)$, assigns to a pair of pants decompositions P, Q the minimum number of edges over all the paths in $C_P^1(\Sigma)$ from P to Q . In this metric the length of each edge of $C_P^0(\Sigma)$ has length 1. It is known that the mapping class group acts by isometries on $(C_P^1(\Sigma), d_\pi)$.

In the setting of Proposition A.2 of Appendix A, d_π is the metrification of the function

$$\pi : C_P^0(\Sigma) \times C_P^0(\Sigma) \rightarrow [0, \infty],$$

that is defined by $\pi(P, Q) = 1$ if P and Q are connected by an edge in $C_P^1(\Sigma)$ and $\pi(P, Q) = \infty$ otherwise.

6.1. The TQFT metric. We consider $d_{qt} : C_P^0(\Sigma) \times C_P^0(\Sigma) \rightarrow [0, \infty)$, the metrification of $n : C_P^0(\Sigma) \times C_P^0(\Sigma) \rightarrow [0, \infty)$, where $n(Q, P)$ is the quantum intersection number of Q with respect to P . More specifically, given $P, Q \in C_P^0(\Sigma)$, set $\nu(P, Q) := \min(n(P, Q), n(Q, P))$ and let $S(P, Q)$ denote the set of all sequences $x_n := (P_1, \dots, P_n)$ of points $C_P^0(\Sigma)$, where $n \geq 0$ is an integer, and $P_1 = P$ and $P_n = Q$. Now we define

$$(13) \quad d_{qt}(P, Q) = \inf_{x_n \in S(P, Q)} (\nu(P_1, P_2) + \dots + \nu(P_{n-1}, P_n)).$$

We stress that in the above, we do not require P_i and P_{i+1} to be related by an elementary move.

The following lemma follows immediately from the definition.

Lemma 6.1. *We have $\nu(P, Q) \geq d_{qt}(P, Q)$.*

Next we prove that d_{qt} is a metric on the 0-skeleton of the pants graph.

Lemma 6.2. *The function d_{qt} is a metric and the mapping class group $\text{Mod}(\Sigma)$ acts by isometries on $(C_P^0(\Sigma), d_{qt})$. That is,*

- (i) *we have $d_{qt}(P, Q) = 0$ if and only if $P = Q$; and*
- (ii) *for any $P, Q \in C_P^0(\Sigma)$ and $\phi \in \text{Mod}(\Sigma)$, $d_{qt}(\phi(P), \phi(Q)) = d_{qt}(P, Q)$.*

Proof. The first statement follows from Proposition A.2 and the fact that $n(P, Q) \geq 2$ whenever $P \neq Q$ which follows from Proposition 2.13. The second statement follows from the fact that $n(\phi(P), \phi(Q)) = n(P, Q)$ for $\phi \in \text{Mod}(\Sigma)$ and $P, Q \in C_P^0(\Sigma)$, which was shown in Proposition 2.7. \square

Definition 6.3. A function between metric spaces $f : (X_1, d_1) \rightarrow (X_2, d_2)$ is called a *quasi-isometric embedding* if there are constants A, B such that

$$\frac{1}{A}d_1(x, y) - B \leq d_2(f(x), f(y)) \leq Ad_1(x, y) + B,$$

for any $x, y \in X_1$.

If f has uniformly dense image we will say that f is a *quasi-isometry*, and that the spaces (X_1, d_1) and (X_2, d_2) are *quasi-isometric*.

In the case where $X_1 = X_2$ we will say that the metrics d_1, d_2 are quasi-isometric if the identity is a quasi-isometric embedding.

6.2. Comparing to the path metric. The main result in this section is the following theorem that shows that the metrics $d_{qt}(P, Q)$ and $d_\pi(P, Q)$ are quasi-isometric. Theorem 6.4, combined with work of Brock [8, 9], allows us to relate d_{qt} to the Weil-Petersson metric on the Teichmüller space and to hyperbolic 3-dimensional geometry.

Theorem 6.4. *The identity map $(C_P^0(\Sigma), d_\pi) \rightarrow (C_P^0(\Sigma), d_{qt})$ is a quasi-isometry. More specifically there is a constant $A > 0$, only depending on the topology of Σ , such that*

$$\frac{A}{3g-3}d_\pi(P, Q) \leq d_{qt}(P, Q) \leq 2d_\pi(P, Q),$$

for any $P, Q \in C_P^0(\Sigma)$

The upper bound in Theorem 6.4 is shown in the next lemma.

Lemma 6.5. *For any $P, Q \in C_P^0(\Sigma)$, we have $d_{qt}(P, Q) \leq 2d_\pi(P, Q)$.*

Proof. By Proposition 2.13, if two pairs of pants $P, P' \in C_P^0(\Sigma)$ are related an S -move or an A -move then $n(P, P') = 2$. Thus, $d_{qt}(P, P') = \nu(P, P') := \min(n(P, P'), n(P', P)) \leq 2$. Thus if $d_\pi(P, P') = 1$, then $d_{qt}(P, P') \leq 2$ and we have

$$(14) \quad d_{qt}(P, P') = 2 d_\pi(P, P').$$

For arbitrary $P, Q \in C_P^0(\Sigma)$, pick a minimum length path from P to Q in $C_P^1(\Sigma)$ and write $d_\pi(P, Q) = \sum_{i=1}^r d_\pi(P_i, P_{i+1})$, with $d_\pi(P_i, P_{i+1}) = 1$. Now by (14)

$$2d_\pi(P, Q) = \sum_{i=1}^r 2d_\pi(P_i, P_{i+1}) = \sum_{i=1}^r d_{qt}(P_i, P_{i+1}) \geq d_{qt}(P, Q),$$

which finishes the proof. \square

6.3. The lower bound. To prove the lower inequality of 6.4 we need the following.

Proposition 6.6. *Given Σ a closed surface of genus at least two, there is a constant $B > 0$, only depending on Σ , so that for any $P, Q \in C_P^0(\Sigma)$ we have $d_\pi(P, Q) \leq B I(P, Q)$.*

Proposition 6.6 is well known to experts on curve complexes of surfaces and their relations to hyperbolic geometry. A proof is implicit in Brock's work [8]: The proof of [8, Lemma 3.3] implies that there is $B > 0$ so that the statement of Proposition 6.6 holds. The proof that B only depends on Σ is given in the proofs of [8, Lemma 4.2] using a result of Mazur and Minsky [27, Theorem 6.12] implying that the path distance in the pants complex of a surface is bounded above by a sum of distances in the curve complexes of (non-annular) subsurfaces. Aougab, Taylor and Webb [2] have made some of the estimates of [8, 27] effective and [2, Lemma 7.1] states an effective version of Proposition 6.6 making the constant B explicit. Finally, Proposition 6.6 can also be derived by [27, Theorem 6.12] and a result of Choi and Rafi [14, Corollary D].

We are now ready to give the proof of Theorem 6.4.

Proof. By Lemma 6.5 we only need to deduce the lower inequality.

By definition of d_{qt} , given $P, Q \in C_P^0(\Sigma)$ we can find a sequence of points $\{P_i\}_{i=1}^r$ in $C_P^0(\Sigma)$ such that we have

- (i) $P = P_1$ and $Q = P_r$,
- (ii) $d_{qt}(P_i, P_{i+1}) = \nu(P_i, P_{i+1}) := \min\{n(P_i, P_{i+1}), n(P_{i+1}, P_i)\}$, and
- (iii) $d_{qt}(P, Q) = \sum_{i=1}^r d_{qt}(P_i, P_{i+1})$.

By Theorem 2.8, for $1 \leq i \leq r$, we have

$$d_{qt}(P_i, P_{i+1}) = \nu(P_i, P_{i+1}) \geq \frac{1}{(3g-3)} I(P_i, P_{i+1}),$$

which by Proposition 6.6 gives

$$d_{qt}(P_i, P_{i+1}) \geq \frac{1}{B(3g-3)} d_\pi(P_i, P_{i+1}).$$

for $i = 1, \dots, r$. Thus we obtain

$$d_{qt}(P, Q) = \sum_{i=1}^r d_{qt}(P_i, P_{i+1}) \geq \frac{1}{B(3g-3)} \sum_{i=1}^r d_\pi(P_i, P_{i+1}) \geq \frac{A}{(3g-3)} d_\pi(P, Q),$$

where $A := \frac{1}{B}$, and the desired result follows.

□

7. RELATIONS WITH TEICHMÜLLER SPACE AND HYPERBOLIC GEOMETRY

7.1. Weil-Petersson metric and TQFT. The *Teichmüller space* $\mathcal{T}(\Sigma)$ parametrizes finite area hyperbolic structures on Σ : Points in $\mathcal{T}(\Sigma)$ are equivalence classes of pairs (X, ϕ) of a finite area hyperbolic surface and a homomorphism $\phi : X \rightarrow \Sigma$, where (X, ϕ) is equivalent to (Y, ψ) if there is an isometry $f : X \rightarrow Y$ so that $\psi \circ f$ and ϕ are isotopic.

Brock [8] proved that the 0-skeleton $C_P^0(\Sigma)$ of the pants graph is quasi-isometric to $\mathcal{T}(\Sigma)$ with the Weil-Petersson metric d_{WP} . We will use this result and Theorem 6.4 to derive Theorem 1.1. We will not need a detailed definition of the Weil-Petersson metric and we will not work with its geometric properties. Nevertheless, to convey the nature of the quasi-isometry of Theorem 1.1 we will briefly discuss some features of the metric d_{WP} . For details the reader is referred to [8] and references provided therein.

Let $\mathbb{H}^2 = \{z \in \mathbb{C}, \text{Im}(z) > 0\}$ denote the upper-half complex plane with the *Poincaré metric*. By uniformization, every $X \in \mathcal{T}(\Sigma)$ becomes an 1-dimensional complex manifold (a Riemann surface) as quotient of \mathbb{H}^2 by a Fuchsian group. The space $\mathcal{T}(\Sigma)$ has the structure of a $3g - 3$ complex manifold.

The space of holomorphic quadratic differential forms of type $\phi(z)dz^2$ on X , denoted by $Q(X)$, has a natural identification with the holomorphic co-tangent space of $\mathcal{T}(\Sigma)$ at X . One can define a Hermitian inner product on $Q(X)$ by integrating over X , using the hyperbolic metric of X which induces a Hermitian inner product on the tangent space at X . Since we have such an inner product for each $X \in \mathcal{T}(\Sigma)$ we obtain a Riemannian metric, the Weil-Petersson. Let $d_{WP} : \mathcal{T}(\Sigma) \times \mathcal{T}(\Sigma) \rightarrow [0, \infty)$ denote the associated distance of this metric. It is known that $\text{Mod}(\Sigma)$ acts by isometries on $(\mathcal{T}(\Sigma), d_{WP})$.

By a result of Bers there is a constant $L > 0$ (Ber's constant) only depending on the surface Σ , so that given $X \in \mathcal{T}(\Sigma)$ there is a pants decomposition $P_X \in C_P^0(\Sigma)$ such that for each curve $\gamma \in P_X$ the length of γ in the hyperbolic metric of X , say $l_X(\gamma)$, is less than L . This implies that the sets $\{V(P)\}_{P \in C_P^0(\Sigma)}$, where

$$(15) \quad V(P) := \{X \in \mathcal{T}(\Sigma) \mid \max_{\gamma \in P} \{l_X(\gamma)\} < L\},$$

cover the Teichmüller space $\mathcal{T}(\Sigma)$.

Theorem 7.1. ([8, Theorem 3.2]) *An embedding $g : C_P^0(\Sigma) \rightarrow \mathcal{T}(\Sigma)$, with $g(P) \in V(P)$, for all $P \in C_P^0(\Sigma)$, is a quasi-isometry of $(C_P^0(\Sigma), d_\pi)$ to $(\mathcal{T}(\Sigma), d_{WP})$.*

Theorem 6.4, implies that any embedding $g : C_P^0(\Sigma) \rightarrow \mathcal{T}(\Sigma)$ as in Theorem 7.1, is also quasi-isometry of $C_P^0(\Sigma)$ with the quantum metric d_{qt} to $\mathcal{T}(\Sigma)$ with d_{WP} . Hence we have:

Theorem 7.2. *There are constants $A_1, A_2 > 0$, only depending on Σ , such that*

$$\frac{1}{A_1} d_{qt}(P_X, P_Y) - A_2 \leq d_{WP}(X, Y) \leq A_1 d_{qt}(P_X, P_Y) + A_2,$$

for any $X, Y \in \mathcal{T}(\Sigma)$.

7.2. Relations to volumes of fibered 3-manifolds. Here we discuss relations between the quantum metric d_{qt} and hyperbolic volumes of 3-manifolds. To state our result we need the following definition.

Definition 7.3. Given a metric (X, d) and an isometry $\phi : X \rightarrow X$, the *translation length* is defined by

$$L_d(\phi) = \inf\{d(\phi(x), x) \mid x \in X\}.$$

Since $\text{Mod}(\Sigma)$ acts by isometries on $(C_P^1(\Sigma), d_{qt})$, given $\phi \in \text{Mod}(\Sigma)$ we can define the translation length $L^{d_{qt}}(\phi)$. Similarly, we can define $L^{d_\pi}(\phi)$. Also since $\text{Mod}(\Sigma)$ is known to act by isometries on (\mathcal{T}, d_{WP}) , and we can define $L^{d_{WP}}(\phi)$.

Given $\phi \in \text{Mod}(\Sigma)$, let $M_\phi = F \times [0, 1]/(x, 1) \sim (\phi(x), 0)$ be the mapping torus of ϕ . By work of Thurston [31], a mapping class $\phi \in \text{Mod}(\Sigma)$ is pseudo-Anosov if and only if the mapping torus M_ϕ is a hyperbolic 3-manifold. By Mostow rigidity, the hyperbolic metric is essentially unique and the hyperbolic volume $\text{vol}(M_\phi)$ is a topological invariant of M_ϕ .

Theorem 1.2. *There exist a positive constant N , depending only on the topology of the surface Σ , so that for any pseudo-Anosov mapping class $\phi \in \text{Mod}(\Sigma)$ we have*

$$\frac{1}{N} L^{d_{qt}}(\phi) \leq \text{vol}(M_\phi) \leq N L^{d_{qt}},$$

Proof. A result of Brock [9, Theorem 1.1] states that there is a constant $k > 0$ only depending on the topology of Σ so that

$$(16) \quad \frac{1}{k} L^{d_{WP}}(\phi) \leq \text{vol}(M_\phi) \leq k L^{d_{WP}}(\phi).$$

By Theorem 7.2 the space $(C_P^0(\Sigma), d_{qt})$ is quasi-isometric to (\mathcal{T}, d_{WP}) . Thus, after re-adjusting the constants of the quasi-isometry and Definition 7.3, we have $\frac{1}{M} L^{d_{qt}}(\phi) \leq L^{d_{WP}}(\phi) \leq M L^{d_{qt}}(\phi)$, which combined with (16) gives the result. \square

To conclude this section, we remark that there exist several open conjectures predicting strong relations of asymptotic aspects of the $SU(2)$ -Witten-Reshetikhin-Turaev TQFT with hyperbolic geometry, and partial verifications of these conjectures [11, 35].

7.3. Nielsen-Thurston type detection. By the Nielsen-Thurston classification mapping classes are divided into three types: *periodic*, *reducible* and *pseudo-Anosov*. Furthermore, the type of f determines the geometric structure, in the sense of Thurston, of the 3-manifold obtained as mapping torus of f . The AMU conjecture, due to Andersen, Masbaum and Ueno [1] asserts that the large- r asymptotic behavior of $SU(2)$ -quantum mapping class group representations $\{\rho_r\}_{r \geq 3}$ detects pseudo-Anosov elements. For example, it states that for any pseudo-Anosov mapping class $\phi \in \text{Mod}(\Sigma)$, the image $\rho_r(\phi)$ has infinite order for all but finitely many r . Furthermore, in the case of the 4-holed sphere [1] (or the punctured torus in [30]), it has been observed that the spectral radii of the images $\{\rho_r\}_{r \geq 3}$ also determine the *stretch factor* of ϕ .

Our results in this subsection are, in the broad sense of things, similar in spirit to the AMU conjecture. The first result, which is Corollary 1.4 stated in the introduction, sheds some light on the geometric content of the complexity $n(P, \phi(P))$ for pseudo-Anosov $\phi \in \text{Mod}(\Sigma)$.

Corollary 1.4. *There is a constant $N > 0$, only depending on the topology of Σ , so that for any pseudo-Anosov mapping class $\phi \in \text{Mod}(\Sigma)$ and any $P \in C_P^0(\Sigma)$ we have*

$$n(P, \phi(P)) \geq N \text{vol}(M_\phi).$$

Proof. By Theorem 1.2 and the definition of $L^{d_{qt}}(\phi)$, there is a positive constant N only depending on the topology of Σ , such that

$$d_{qt}(P, \phi(P)) \geq N L^{d_{qt}}(\phi),$$

for any $P \in C_P^0(\Sigma)$. On the other hand, by Lemma 6.1, we have

$$\nu(P, \phi(P)) \geq d_{qt}(P, \phi(P)),$$

where $\nu(P, \phi(P)) = \min(n(P, \phi(P)), n(\phi(P), P))$, and the desired result follows. \square

The next result uses Theorem 2.8 to derive a characterization of pseudo-Anosov mapping classes in terms of the behavior of quantum intersection numbers under iteration in $\text{Mod}(\Sigma)$.

Corollary 7.4. *A mapping class $\phi \in \text{Mod}(\Sigma)$ is pseudo-Anosov if and only if for any multicurve γ , we have $\lim_{k \rightarrow \infty} n(\phi^k(\gamma), P) = \infty$, for all $P \in C_P^0(\Sigma)$.*

Proof. By Theorem 2.8, for any γ we have $n(\phi^k(\gamma), P) \geq \frac{I(\phi^k(\gamma), P)}{3g-3}$, for all $P \in C_P^0(\Sigma)$. If ϕ is pseudo-Anosov, it is known that $\lim_{k \rightarrow \infty} I(\phi^k(\gamma), P) = \infty$ by [20, Theorem 12.2].

For the converse, we argue that if ϕ is not pseudo-Anosov, there is a multicurve γ such that, for any $P \in C_P^0(\Sigma)$, the sequence $\{n(\phi^k(\gamma), P)\}_{k \geq 1}$ is bounded. To that end, recall that, by definition, if ϕ is not pseudo-Anosov, then there is a multicurve γ , and an integer $k_1 > 0$, such that ϕ^{k_1} leaves γ fixed pointwise. It follows that, for any $0 < k \in \mathbb{N}$, $n(\phi^k(\gamma), P) \leq B < \infty$, where $B := \max_{1 \leq k \leq k_1} n(\phi^k(\gamma), P)$. \square

Remark 7.5. In fact, it is known that if $\phi \in \text{Mod}(\Sigma)$ is not periodic then we can find a simple closed curve $\gamma \subset \Sigma$ so that $I(\phi^k(\gamma), \gamma)$ increases with k . Completing γ to a pants decomposition and applying Theorem 2.8, $\lim_{k \rightarrow \infty} n(\phi^k(\gamma), P) = \infty$. Thus, we can characterize periodic mapping classes as follows: ϕ is periodic if and only if for any multicurve γ , we have $\lim_{k \rightarrow \infty} n(\phi^k(\gamma), P) < \infty$, for all $P \in C_P^0(\Sigma)$.

We continue with the following corollary:

Corollary 7.6. *Let $\phi \in \text{Mod}(\Sigma)$ be a pseudo-Anosov mapping class with stretch factor λ_ϕ . Then, for any $P \in C_P^0(\Sigma)$ and any simple closed curve γ , we have*

$$\lambda_\phi \leq \liminf_{k \rightarrow \infty} \sqrt[k]{n(\phi^k(\gamma), P)} \leq \limsup_{k \rightarrow \infty} \sqrt[k]{n(\phi^k(\gamma), P)} \leq \lambda_\phi^{3g-3}.$$

Proof. By [20, Theorem 12.2], for any simple closed curve β we have

$$(17) \quad \lim_{k \rightarrow \infty} \sqrt[k]{I(\phi^k(\gamma), \beta)} = \lambda_\phi.$$

Given γ and P , for any $k > 0$, pick $\alpha(k) \in P$ so that $\max_{\alpha \in P} \{I(\phi^k(\gamma), \alpha)\} = I(\phi^k(\gamma), \alpha(k))$. Since $I(\phi^k(\gamma), \alpha(k)) \leq I(\phi^k(\gamma), P) \leq (3g - 3)I(\phi^k(\gamma), \alpha(k))$, Theorem 2.8 gives

$$(18) \quad \frac{I(\phi^k(\gamma), \alpha(k))}{3g - 3} \leq n(\phi^k(\gamma), P) \leq (6g - 6)^{3g-3} I(\phi^k(\gamma), \alpha(k))^{3g-3}.$$

We can split \mathbb{N} into finitely many subsequences so that for each of them, say \mathbf{s} , for all $k \in \mathbf{s}$ the curve $\alpha(k)$ of (18) is the same curve $\beta \in P$. Then, using (17) and (18), for each of the subsequences the result follows. \square

We close the section with the proof of Corollary 1.5 which we restate here.

Corollary 1.5. *Let ϕ be a pseudo-Anosov mapping class. If there is a multicurve γ and $P \in C_P^0(\Sigma)$ so that $\limsup_{m \rightarrow \infty} [r_0(\phi^m(\gamma), P)] < \infty$, then, ϕ satisfies the AMU conjecture.*

Proof. Suppose for a contradiction that there is a pseudo-Anosov $\phi \in \text{Mod}(\Sigma)$ as in the statement of the corollary that fails the AMU conjecture: That is, there is an infinite sequence of positive integers $\{m_r\}_{r \rightarrow \infty}$ such that $\rho_r((\phi)^{m_r}) = 1$. As in the proof of Proposition 2.7, $T_r^{\phi^{m_r}(\gamma)} = T_r^\gamma$. This implies that as $r \rightarrow \infty$, the matrix $A(\phi, m_r)$, representing $T_r^{\phi^{m_r}(\gamma)}$ in the basis \mathcal{B}_P , is equal to the matrix $A(\gamma, r)$ representing T_r^γ . For r large enough, the matrix $A(\gamma, r)$ has $n(\gamma, P)$ non main diagonals containing non-zero entries. Since ϕ is pseudo-Anosov, by Corollary 7.4, the quantum intersection number $n(\phi^{m_r}(\gamma), P)$ increases as m_r does. Thus, for r large enough $A(\phi, m_r)$ can equal $A(\gamma, r)$ only if $r < r_0(\phi^{m_r}(\gamma), P)$, which implies that $\limsup_{r \rightarrow \infty} [r_0(\phi^{m_r}(\gamma), P)] = \infty$, contradicting our hypothesis. \square

8. NORMS OF CURVE OPERATORS AND QUANTUM INTERSECTION NUMBER

Let the notation and setting be as in Section 2. Let γ be a multicurve and P a pants decomposition of Σ . Consider the curve operators $\{T_r^\gamma\}_{r \geq 3}$, and the orthonormal basis $\mathcal{B}_P = \{\phi_{\mathbf{c}} \mid \mathbf{c} \in U_r\}$ of the Hermitian spaces $V_r(\Sigma)$ corresponding to P . Let $m_r(\gamma, P)$ denote the number of non-zero entries in the matrix representing T_r^γ with respect to the basis \mathcal{B}_P . Recall that the quantum intersection number $n(\gamma, P)$ denotes the number of non-main diagonals containing non-zero entries and this is captured by the number of non-zero functions $G_{\mathbf{k}}^P(\frac{\epsilon}{r}, \frac{1}{r})$ in Equation (1). We have the following.

Proposition 8.1. *For any multicurve γ and pants decomposition P , for r large enough,*

$$(19) \quad O(v_r n(\gamma, P)) \leq m_r(\gamma, P) \leq v_r (n(\gamma, P) + 1),$$

where $v_r := \dim V_r(\Sigma)$ is given by the Verlinde formula [5].

Proof. For any $r \geq 3$, the upper inequality follows by the definitions, where the term $+1$ accounts for the main diagonal, which was discarded in the count for $n(\gamma, P)$.

There are $n(\gamma, P)$ non-zero functions $G_{\mathbf{k}}^P$, and furthermore those functions are analytic on a subset $V_\gamma \subset U \times [0, 1]$, where U is the open set defined in Section 2. Since they are non-zero and analytic, we can pick an open subset of V_γ of the form $V \times [0, h]$ on which all of the functions $G_{\mathbf{k}}^P$ are non-vanishing. Then we have that $m_r(\gamma, P) \geq |\frac{1}{r}U_r \cap V|$, which grows like r^{3g-3} , the same as v_r , since it amounts to counting a number of lattice points in the open set rV . \square

It follows that for r large enough, the quantity $m_r(\gamma, P)$ admits two-sided bounds in terms of the geometric intersection number $I(\gamma, P)$ and v_r .

Next we consider the l^1 and l^2 -norms of T_r^γ , with respect to \mathcal{B}_P , defined by

$$\|T_r^\gamma\|_{\{l^s, P\}} = \left(\sum_{\mathbf{c}, \mathbf{d} \in U_r} |\langle T_r^\gamma \phi_{\mathbf{c}}, \phi_{\mathbf{d}} \rangle|^2 \right)^{\frac{1}{s}},$$

for $s = 1, 2$.

Lemma 8.2. *The norm $\|T_r^\gamma\|_{\{l^2, P\}}$ is independent of the pants decomposition P and depends only on the orbit of γ under the action of $\text{Mod}(\Sigma)$.*

Proof. Recall that the l^2 -norm can also be computed as $\|T_r^\gamma\|_{\{l^2, P\}} = \text{Tr}(T_r^\gamma (T_r^\gamma)^*)^{\frac{1}{2}}$, where $(T_r^\gamma)^*$ is the adjoint for the natural Hermitian norm on $V_r(\Sigma)$. However, T_r^γ is an Hermitian operator, hence

$$\text{Tr}(T_r^\gamma (T_r^\gamma)^*) = \text{Tr}((T_r^\gamma)^2) = \sum_{\lambda \in \sigma(T_r^\gamma)} |\lambda|^2,$$

where $\sigma(T_r^\gamma)$ is the set of eigenvalues of T_r^γ . This expression shows that the l^2 -norm is independent of P , since a trace is independent of basis, and the l^2 -norm depends only on the orbit of γ under $\text{Mod}(\Sigma)$ since $T_r^{\phi(\gamma)} = \rho_r(\phi) \circ T_r^\gamma \circ \rho_r(\phi)^{-1}$ for $\phi \in \text{Mod}(\Sigma)$. \square

Turning to the l^1 -norm, an easy application of the Cauchy-Schwarz inequality gives $\|T_r^\gamma\|_{\{l^1, P\}} \leq (m(\gamma, P))^{1/2} \|T_r^\gamma\|_{l^2}$, which combined with the last inequality and (19), gives

$$\frac{\|T_r^\gamma\|_{\{l^1, P\}}}{\|T_r^\gamma\|_{l^2}} \leq (n(\gamma, P) + 1)^{1/2} v_r^{1/2},$$

and

$$(20) \quad T(\gamma, P) := \limsup_{r \rightarrow \infty} \frac{\|T_r^\gamma\|_{\{l^1, P\}}}{\|T_r^\gamma\|_{l^2} v_r^{1/2}} \leq (n(\gamma, P) + 1)^{1/2}.$$

Hence, information about the geometric content of $n(\gamma, P)$ translates into information on the geometric content of the $T(\gamma, P)$ that encodes the asymptotic behavior of the curve operator norms. For example, combining (20) with Corollary 7.6 we get,

Corollary 8.3. *Let $\phi \in \text{Mod}(\Sigma)$ be a pseudo-Anosov mapping class with stretch factor λ_ϕ . Then, for any $P \in C_P^0(\Sigma)$ and any simple closed curve γ , we have*

$$\limsup_{k \rightarrow \infty} \sqrt[k]{T(\phi^k(\gamma), P)} \leq \lambda_\phi^{\frac{3g-3}{2}}.$$

As another example, in the case that γ is also a pants decomposition, one can directly compare the function T to the metric d_{qt} by means of Proposition A.2.

Building on our work in Sections 5 and 4, in a subsequent paper, we will carry out a more detailed study of the asymptotics of the coefficients $G_{\mathbf{k}}^\gamma(\frac{\varepsilon}{r}, \frac{1}{r})$ to derive a lower bound of $T(\gamma, P)$ in terms of $n(P, Q)$.

9. PROOFS OF AUXILIARY LEMMAS

We give the proofs of Lemmas 4.4 and 4.5. In the figures, thin black edges are colored by 2, while the colors of red edges are specified and the fusion rules applied are the ones of Figure 3.

Lemma 4.4. (Sliding Lemma) *With the conventions of Figure 3, and for any $\varepsilon, \mu \in \{\pm 1\}$, we have the following identities.*

$$\begin{array}{c} \text{---} \\ \parallel \\ a + \varepsilon + \mu \\ \parallel \\ a + \varepsilon \\ \parallel \\ a \end{array} = \text{---} \begin{array}{c} \parallel \\ a + \varepsilon + \mu \\ \parallel \\ a + \mu \\ \parallel \\ a \end{array} \quad \text{and} \quad \begin{array}{c} \text{---} \\ \parallel \\ a + \varepsilon + \mu \\ \parallel \\ a + \varepsilon \\ \parallel \\ a \end{array} = - \begin{array}{c} \diagup \\ \parallel \\ a + \varepsilon + \mu \\ \parallel \\ a + \mu \\ \parallel \\ a \end{array}$$

Proof. Using the strand rule, then the triangle rule, we get

$$\begin{array}{c} \text{---} \\ \parallel \\ a + \varepsilon + \mu \\ \parallel \\ a + \varepsilon \\ \parallel \\ a \end{array} = \begin{array}{c} \text{---} \\ \parallel \\ a + \varepsilon + \mu \\ \parallel \\ a + \varepsilon + \mu + 1 \\ \parallel \\ a + \varepsilon + \mu \\ \parallel \\ a + \varepsilon \\ \parallel \\ a \end{array} - \begin{array}{c} \text{---} \\ \parallel \\ a + \varepsilon + \mu \\ \parallel \\ a + \varepsilon + \mu - 1 \\ \parallel \\ a + \varepsilon + \mu \\ \parallel \\ a + \varepsilon \\ \parallel \\ a \end{array} \\ = A \begin{array}{c} \parallel \\ a + \varepsilon + \mu \\ \parallel \\ a + \varepsilon + \mu + 1 \\ \parallel \\ a \end{array} - B \begin{array}{c} \parallel \\ a + \varepsilon + \mu \\ \parallel \\ a + \varepsilon + \mu - 1 \\ \parallel \\ a \end{array}$$

where A and B are constants that now we explain how to compute: Note that in the identity above, passing from the first to the second row, we have used triangle moves to eliminate a triangle from each of the second and the third term of the first row. However, the triangle rules we need here are not the ones depicted in Figure 3 since, for instance, the red sides of the triangles eliminated are not colored by the same color. Hence to compute A, B we need

to first to apply the general triangle rules of Figure 6 to compute the coefficient function, and then take the limit $r \rightarrow \infty$, $\frac{a}{r} \rightarrow \theta$.

If $\varepsilon = +1$, we get

$$A = \lim_{r \rightarrow \infty, \frac{a}{r} \rightarrow \theta} \frac{\langle \frac{2+a+\varepsilon+\mu+1-(a+\varepsilon)}{2} \rangle \langle \frac{2-(a+\varepsilon+\mu+1)+a+\varepsilon}{2} \rangle}{\langle a + \varepsilon + \mu + 1 \rangle \langle a \rangle} = 0,$$

and

$$B = \lim_{r \rightarrow \infty, \frac{a}{r} \rightarrow \theta} - \frac{\langle \frac{2+a+\varepsilon+\mu+a+\varepsilon-1}{2} \rangle \langle \frac{a+\varepsilon+\mu+a+\varepsilon-2-1}{2} \rangle}{\langle a + \varepsilon + \mu - 1 \rangle \langle a \rangle} = -1.$$

Since $a + \varepsilon + \mu - 1 = a + \mu$, this agrees with the first identity in the statement of the lemma.. Similarly, in the case $\varepsilon = -1$, we get $A = 1$ and $B = 0$, which, since $a + \varepsilon + \mu + 1 = a + \mu$ in that case, agrees with first identity in the statement of the lemma.

Now we explain how to prove the second identity in the statement of the lemma: Using the Kauffman bracket skein rule and keeping in mind that $\lim_{r \rightarrow \infty, \frac{a}{r} \rightarrow \theta} \zeta_r = -1$, we get:

$$\begin{array}{c} \diagdown \quad \parallel \begin{array}{l} a + \varepsilon + \mu \\ a + \mu \\ a \end{array} \quad \diagup \end{array} = - \left(\begin{array}{c} \parallel \begin{array}{l} a + \varepsilon + \mu \\ a + \mu \\ a \end{array} \quad \text{---} \quad \begin{array}{c} \parallel \begin{array}{l} a + \varepsilon + \mu \\ a + \mu \\ a \end{array} \end{array} \right)$$

Next we can simplify the first figure of the right hand side with the bigon rule. If $\varepsilon = \mu$, then the first figure vanishes and we get the lemma. On the other hand, if $\varepsilon = -\mu$ we have

$$\left(\begin{array}{c} \parallel \begin{array}{l} a \\ a - \varepsilon \\ a \end{array} \end{array} \right) = \varepsilon \left(\begin{array}{c} \parallel \begin{array}{l} a \\ a \\ a \end{array} \end{array} \right) = \varepsilon \left(\begin{array}{c} \parallel \begin{array}{l} a \\ a + 1 \\ a \end{array} \end{array} \right) - \varepsilon \left(\begin{array}{c} \parallel \begin{array}{l} a \\ a \\ a - 1 \end{array} \end{array} \right)$$

Combining the last two equations, we get the desired conclusion. \square

Lemma 4.5. *With the conventions of Figure 3, and for any $\varepsilon, \mu \in \{\pm 1\}$, we have the following identity*

$$\begin{array}{c} \parallel \begin{array}{l} a + \varepsilon \\ a \\ a \\ a + \mu \end{array} \end{array} = (-1)^{(a+1)t} \varepsilon^{t+1} z^{\varepsilon t} \delta_{\varepsilon, \mu} \begin{array}{c} \parallel \begin{array}{l} a + \varepsilon \end{array} \end{array}$$

where the arc on the left hand side has swift number t , and $\delta_{\varepsilon, \mu}$ is the Kronecker symbol.

Proof. When the swift number t is equal to zero, the identity in the lemma reduces to the bigon rule. Next we treat the case $t = \pm 1$. Using the half-twist rule, for the following swift number 1 arc, we check that

$$\left. \begin{array}{c} a + \varepsilon \\ a \\ a + \mu \end{array} \right\| = (-1)^{a+1} \varepsilon z^\varepsilon \left. \begin{array}{c} a + \varepsilon \\ a \\ a + \mu \end{array} \right\| = (-1)^{a+1} z^\varepsilon \delta_{\varepsilon, \mu} \left. \begin{array}{c} a + \varepsilon \\ a + \mu \end{array} \right\|$$

The computation is similar for the other 3 swift number ± 1 patterns. Indeed, the other swift number 1 is obtained from the previous one by rotation by π along a vertical axis, hence we will get the same coefficient, while the swift number -1 patterns are obtained by mirror image from those ones, hence the coefficient we will get is the complex conjugate, which matches the lemma. For a pattern with swift number $\pm t$, we apply the strand rule $t - 1$ times, to get:

$$\left. \begin{array}{c} a + \varepsilon \\ a \\ a + \mu \end{array} \right\| = \varepsilon^{t-1} \left. \begin{array}{c} a + \varepsilon \\ a + \varepsilon \\ a + \varepsilon \\ a + \varepsilon \\ a + \mu \end{array} \right\|$$

For each strand rule applied, we only get one non zero contribution since from the previous computation, a single arc swift number ± 1 pattern with opposite color shift on top and bottom vanishes. Now, simplifying each swift number ± 1 pattern, we get the lemma. \square

APPENDIX A. METRIFICATION OF A TWO-VARIABLE FUNCTION

In this section, we introduce a general operation to produce a metric starting from a non-negative two variable function on a set. We use the process to define the metric d_{qt} in Section 6.

Let X be a set. Recall that $d : X \times X \rightarrow [0, \infty)$ is called a metric on X if it satisfies:

- (Symmetry) For any $x, y \in X$, we have $d(x, y) = d(y, x)$.
- (Triangular inequality) For any $x, y, z \in X$, we have $d(x, z) \leq d(x, y) + d(y, z)$.
- (Identity of indiscernibles) For any $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$.

We will call a function $d : X \times X \rightarrow [0, \infty]$ a *semi-metric* if it satisfies only the first two axioms and $d(x, x) = 0$ for all $x \in X$. Notice we allow semi-metrics to sometimes take the value ∞ .

Definition A.1. Let X be a set and $f : X \times X \rightarrow [0, \infty]$ be a function. For $x, y \in X$, let $P(x, y)$ be the set of all sequences $\mathbf{x} := (x_0, x_1, \dots, x_n)$ where $n \geq 0$ is an integer and $x_0 = x, x_n = y$. Then for any $x, y \in X$, let

$$d_f(x, y) = \inf_{\mathbf{x} \in P(x, y)} (\min(f(x_0, x_1), f(x_1, x_0)) + \dots + \min(f(x_{n-1}, x_n), f(x_n, x_{n-1}))).$$

We call d_f the metrification of f .

The name *metrification* is motivated by the following proposition:

Proposition A.2. *Let X be a set and $f : X \times X \rightarrow [0, \infty]$ be a function. Then:*

- (a) d_f is a semi-metric on X and satisfies $d_f(x, y) \leq f(x, y)$ for any $x, y \in X$.
- (b) For any semi-metric d on X such that $d \leq f$ on $X \times X$, we have $d \leq d_f$.
- (c) Suppose that for any $x, y \in X$ there is a finite sequence $\mathbf{x} := (x_0, x_1, \dots, x_n)$ in $P(x, y)$, such that for all $0 \leq i \leq n - 1$, one has $f(x_i, x_{i+1}) < \infty$. Then d_f takes value in $[0, \infty)$.
- (d) Suppose that the hypothesis of (c) holds and furthermore that there is a constant $B > 0$ so that for any $x \neq y$, $f(x, y) \geq B$. Then d_f is a metric on X .
- (e) Assume that there is a group G acting on X so that f is G -invariant. Then d_f is G -invariant.
- (f) If $f, g : X \times X \rightarrow [0, \infty]$ are functions such that $f \leq g$ on then $d_f \leq d_g$.

Proof. (a) For any $x, y \in X$, we have $(x, y) \in P(x, y)$, so

$$d_f(x, y) \leq \min(f(x, y), f(y, x)) \leq f(x, y).$$

Moreover, for any $x \in X$, the 1-tuple $(x) \in P(x, x)$, therefore $d_f(x, x) = 0$.

Next we show that d_f is symmetric. For any $(x_0, x_1, \dots, x_n) \in P(x, y)$, we have that $(x_n, x_{n-1}, \dots, x_0) \in P(y, x)$. Therefore

$$\begin{aligned} d_f(y, x) &\leq \min(f(x_n, x_{n-1}), f(x_{n-1}, x_n)) + \dots + \min(f(x_1, x_0), f(x_0, x_1)) \\ &= \min(f(x_0, x_1), f(x_1, x_0)) + \dots + \min(f(x_{n-1}, x_n), f(x_n, x_{n-1})). \end{aligned}$$

Taking the infimum over $P(x, y)$ we get $d_f(y, x) \leq d_f(x, y)$. By symmetry, $d_f(y, x) = d_f(x, y)$.

Finally, d_f satisfies the triangle inequality. Indeed, for any $x, y, z \in X$, for any $(x = x_0, \dots, x_n = y) \in P(x, y)$ and $(y_0 = y, \dots, y_m = z) \in P(y, z)$, we have that $(x = x_0, x_1, \dots, x_n = y_0, y_1, \dots, y_m = z) \in P(x, z)$. Thus

$$\begin{aligned} d_f(x, z) &\leq (\min(f(x_0, x_1), f(x_1, x_0)) + \dots + \min(f(x_{n-1}, x_n), f(x_n, x_{n-1}))) \\ &\quad + (\min(f(y_0, y_1), f(y_1, y_0)) + \dots + \min(f(y_{m-1}, y_m), f(y_m, y_{m-1}))). \end{aligned}$$

Taking infimum over $(x_0, \dots, x_n) \in P(x, y)$ and $(y_0, \dots, y_m) \in P(y, z)$, we get $d_f(x, z) \leq d_f(x, y) + d_f(y, z)$.

(b) Assume that d is a semi-metric and $d \leq f$ on $X \times X$. For any x, y , as d is symmetric we have $d(x, y) = \min(d(x, y), d(y, x)) \leq \min(f(x, y), f(y, x))$. Moreover, by the triangle inequality, for any $(x_0, \dots, x_n) \in P(x, y)$, we have

$$\begin{aligned} d(x, y) &\leq d(x, x_1) + \dots + d(x_{n-1}, y) \\ &\leq \min(f(x, x_1), f(x_1, x)) + \dots + \min(f(x_{n-1}, y), f(y, x_{n-1})). \end{aligned}$$

Taking the infimum over $P(x, y)$, we get $d(x, y) \leq d_f(x, y)$.

(c) Let $x, y \in X$ and let x_0, \dots, x_n be the sequence provided by the hypothesis. By definition of d_f , we get that $d_f(x, y) \leq f(x_0, x_1) + \dots + f(x_{n-1}, x_n)$, which is finite, which proves the claim.

(d) First remark that d_f takes value in $[0, \infty)$ by (c). Moreover, if $x \neq y$, any element $(x_0 = x, x_1, \dots, x_n = y)$ of $P(x, y)$ contains two consecutive points x_i, x_{i+1} that are distinct. Then we have

$$\begin{aligned} d_f(x, y) &= \min(f(x_0, x_1), f(x_1, x_0)) + \dots + \min(f(x_{n-1}, x_n), f(x_n, x_{n-1})) \geq \\ &\geq \min(f(x_i, x_{i+1}), f(x_{i+1}, x_i)) \geq B. \end{aligned}$$

(d) For any $g \in G$, for any $x, y \in X$ and $(x_0 = x, \dots, x_n = y) \in P(x, y)$, we have that $(g \cdot x_0, \dots, g \cdot x_n) \in P(g \cdot x, g \cdot y)$. Moreover, as f is G -invariant, we have

$$\begin{aligned} \min(f(g \cdot x_0, g \cdot x_1), f(g \cdot x_1, g \cdot x_0)) + \dots + \min(f(g \cdot x_{n-1}, g \cdot x_n), f(g \cdot x_n, g \cdot x_{n-1})) \\ = \min(f(x_0, x_1), f(x_1, x_0)) + \dots + \min(f(x_{n-1}, x_n), f(x_n, x_{n-1})). \end{aligned}$$

Taking infimum over $P(x, y)$, we get $d_f(g \cdot x, g \cdot y) \leq d_f(x, y)$, and by symmetry, $d_f(g \cdot x, g \cdot y) = d_f(x, y)$. Finally, (e) is clear from the definition of d_f, d_g . \square

APPENDIX B. GENERAL FUSION RULES

Fix a decomposition system $P \cup P'$ on Σ with dual graph Γ and let γ be a multicurve in Dehn-Thurston position with respect to $P \cup P'$. By [15], the coefficients $G_{\mathbf{k}}^\gamma$ of the operators T_r^γ can be computed using fusion rules of Figure 6, which are an adaptation of the Masbaum-Vogel [26] $SU(2)$ -fusion rules in our setting. Note that, as in [15], and in contrast with [26, 5] we are using the *normalized fusion rules*, since, as explained in Section 2, the bases \mathcal{B}_P of the TQFT spaces are normalized so that they are orthonormal (i.e. the vectors in them have Hermitian norm 1). The difference between the two versions of the fusion rules is explained in detail in [15, Section 4.1]; page 3072.

As we explained in Section 2 surface Σ is viewed as being constructed by glueing together two copies of Γ , say Γ, Γ' . The red thick lines in Figure 6 (shown in grey in black-and-white/grey scale) indicate portions of the graph Γ while the copy Γ' is omitted. We consider $\Gamma \cup \gamma$ on Σ and the intersections of γ with Γ' are recorded as over/under crossings of γ and Γ . The labels indicate admissible colorings of Γ which correspond to vectors in the basis of the TQFT spaces $V_r(\Sigma)$ obtained from P .

Note that all the relations occur between portions of colored copies of $\Gamma \cup \gamma$ that lie on a single pants or annulus piece of the fixed decomposition of Σ . Reading Figure 6 from top to bottom, and rows from left to right, we refer to these rules as follows: Strand rule, positive/negative half-twist rules, positive/negative/mixed triangle rules and bigon rule.

The black thinner lines indicate arcs of γ and the color on these portions is 2. As we remarked in Section 2 the colors of all arcs have been shifted by 1 from these of [5]. Thus, in particular, the color 2 corresponds to the fundamental $U_q(\mathfrak{sl}_2)$ -representation of dimension 2. Finally, for $x \in \mathbb{Z}$, we have $\langle x \rangle := \sin(\frac{\pi x}{r})$.

$$\begin{aligned}
 \left| \begin{array}{c} \text{---} \\ | \\ \text{---} \\ n \end{array} \right| &= \left(\frac{\langle n+1 \rangle}{\langle n \rangle} \right)^{\frac{1}{2}} \begin{array}{c} \text{---} \\ \diagup \\ \text{---} \\ n+1 \end{array} - \left(\frac{\langle n-1 \rangle}{\langle n \rangle} \right)^{\frac{1}{2}} \begin{array}{c} \text{---} \\ \diagdown \\ \text{---} \\ n-1 \end{array} \\
 \begin{array}{c} \text{---} \\ \diagup \\ \text{---} \\ n \\ \text{---} \\ n+1 \end{array} &= (-1)^{n+1} e^{i \frac{\pi(n-1)}{2r}} \begin{array}{c} \text{---} \\ \diagup \\ \text{---} \\ n \\ \text{---} \\ n+1 \end{array} \\
 \begin{array}{c} \text{---} \\ \diagdown \\ \text{---} \\ n \\ \text{---} \\ n-1 \end{array} &= (-1)^n e^{-i \frac{\pi(n+1)}{2r}} \begin{array}{c} \text{---} \\ \diagdown \\ \text{---} \\ n \\ \text{---} \\ n-1 \end{array} \\
 \begin{array}{c} \text{---} \\ \diagup \\ \text{---} \\ a \\ \text{---} \\ b+1 \end{array} &= \left(\frac{\langle \frac{a+b+c+1}{2} \rangle \langle \frac{b+c-a+1}{2} \rangle}{\langle b+1 \rangle \langle c+1 \rangle} \right)^{\frac{1}{2}} \begin{array}{c} \text{---} \\ \diagup \\ \text{---} \\ a \\ \text{---} \\ b+1 \end{array} \\
 \begin{array}{c} \text{---} \\ \diagdown \\ \text{---} \\ a \\ \text{---} \\ b+1 \end{array} &= \left(\frac{\langle \frac{a-b+c-1}{2} \rangle \langle \frac{a+b-c+1}{2} \rangle}{\langle b+1 \rangle \langle c-1 \rangle} \right)^{\frac{1}{2}} \begin{array}{c} \text{---} \\ \diagdown \\ \text{---} \\ a \\ \text{---} \\ b+1 \end{array} \\
 \begin{array}{c} \text{---} \\ \diagup \\ \text{---} \\ a \\ \text{---} \\ b-1 \end{array} &= - \left(\frac{\langle \frac{a+b+c-1}{2} \rangle \langle \frac{b+c-a-1}{2} \rangle}{\langle b-1 \rangle \langle c-1 \rangle} \right)^{\frac{1}{2}} \begin{array}{c} \text{---} \\ \diagup \\ \text{---} \\ a \\ \text{---} \\ b-1 \end{array} \\
 \begin{array}{c} \text{---} \\ | \\ \text{---} \\ c \pm 1 \end{array} &= \pm \left(\frac{\langle c \rangle}{\langle c \pm 1 \rangle} \right)^{\frac{1}{2}} \left| \begin{array}{c} \text{---} \\ | \\ \text{---} \\ c \pm 1 \end{array} \right| \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \\ c \mp 1 \end{array} = 0
 \end{aligned}$$

FIGURE 6. The fusion rules.

REFERENCES

- [1] J. E. Andersen, G. Masbaum, and K. Ueno, *Topological quantum field theory and the Nielsen-Thurston classification of $M(0, 4)$* , Math. Proc. Cambridge Philos. Soc. **141** (2006), no. 3, 477–488.
- [2] T. Aougab, S. Taylor, and Webb R., *Effective masur–minsky distance formulas and applications to hyperbolic 3-manifolds*, available at <https://math.temple.edu/~tue38263/>.
- [3] Giulio Belletti, Renaud Detcherry, Efstratia Kalfagianni, and Tian Yang, *Growth of quantum 6j-symbols and applications to the volume conjecture*, J. Differential Geom. **120** (2022), no. 2, 199–229.
- [4] C. Blanchet, N. Habegger, G. Masbaum, and P. Vogel, *Three-manifold invariants derived from the Kauffman bracket*, Topology **31** (1992), no. 4, 685–699.

- [5] ———, *Topological quantum field theories derived from the Kauffman bracket*, *Topology* **34** (1995), no. 4, 883–927.
- [6] F. Bonahon and H. Wong, *Representations of the Kauffman bracket skein algebra I: invariants and miraculous cancellations*, *Invent. Math.* **204** (2016), no. 1, 195–243.
- [7] F. Bonahon, H. Wong, and T. Yang, *Asymptotics of quantum invariants of surface diffeomorphisms I: conjecture and algebraic computations*, arXiv:2112.12852.
- [8] J. F. Brock, *The Weil-Petersson metric and volumes of 3-dimensional hyperbolic convex cores*, *J. Amer. Math. Soc.* **16** (2003), no. 3, 495–535.
- [9] ———, *Weil-Petersson translation distance and volumes of mapping tori*, *Comm. Anal. Geom.* **11** (2003), no. 5, 987–999.
- [10] D. Bullock, C. Frohman, and J. Kania-Bartoszyńska, *Understanding the Kauffman bracket skein module*, *J. Knot Theory Ramifications* **8** (1999), no. 3, 265–277.
- [11] A. Champanerkar, O. Dasbach, E. Kalfagianni, I. Kofman, W. Neumann, and N_i Stoltzfus (eds.), *Interactions between hyperbolic geometry, quantum topology and number theory*, *Contemporary Mathematics*, vol. 541, American Mathematical Society, Providence, RI, 2011.
- [12] L. Charles and J. Marché, *Multicurves and regular functions on the representation variety of a surface in $SU(2)$* , *Comment. Math. Helv.* **87** (2012), no. 2, 409–431.
- [13] Q. Chen and T. Yang, *Volume conjectures for the Reshetikhin-Turaev and the Turaev-Viro invariants*, *Quantum Topol.* **9** (2018), no. 3, 419–460.
- [14] Young-Eun Choi and Kasra Rafi, *Comparison between Teichmüller and Lipschitz metrics*, *J. Lond. Math. Soc. (2)* **76** (2007), no. 3, 739–756.
- [15] R. Detcherry, *Asymptotic formulae for curve operators in TQFT*, *Geom. Topol.* **20** (2016), no. 6, 3057–3096.
- [16] R. Detcherry and E. Kalfagianni, *Quantum representations and monodromies of fibered links*, *Adv. Math.* **351** (2019), 676–701.
- [17] ———, *Gromov norm and Turaev-Viro invariants of 3-manifolds*, *Ann. Sci. Éc. Norm. Supér. (4)* **53** (2020), no. 6, 1363–1391.
- [18] ———, *Cosets of monodromies and quantum representations*, *Indiana Univ. Math. J.* **71** (2022), no. 3, 1101–1129.
- [19] R. Detcherry, E. Kalfagianni, and T. Yang, *Turaev-Viro invariants, colored Jones polynomials, and volume*, *Quantum Topol.* **9** (2018), no. 4, 775–813.
- [20] A. Fathi, F. Laudenbach, and V. Poénaru, *Thurston’s work on surfaces*, *Mathematical Notes*, vol. 48, Princeton University Press, Princeton, NJ, 2012, Translated from the 1979 French original by Djun M. Kim and Dan Margalit.
- [21] W. M. Goldman, *Invariant functions on Lie groups and Hamiltonian flows of surface group representations*, *Invent. Math.* **85** (1986), no. 2, 263–302.
- [22] A. Hatcher and W. Thurston, *A presentation for the mapping class group of a closed orientable surface*, *Topology* **19** (1980), no. 3, 221–237.
- [23] E. Kalfagianni and J.M. Melby, *Constructions of q -hyperbolic knots*, *Annales l’Institut Fourier*, to appear.
- [24] Sanjay Kumar and Joseph M. Melby, *Asymptotic additivity of the Turaev-Viro invariants for a family of 3-manifolds*, *J. Lond. Math. Soc. (2)* **106** (2022), no. 4, 3043–3068.
- [25] J. Marché and T. Paul, *Toeplitz operators in TQFT via skein theory*, *Trans. Amer. Math. Soc.* **367** (2015), no. 5, 3669–3704.
- [26] G. Masbaum and P. Vogel, *3-valent graphs and the Kauffman bracket*, *Pacific J. Math.* **164** (1994), no. 2, 361–381.
- [27] H. A. Masur and Y. N. Minsky, *Geometry of the complex of curves. II. Hierarchical structure*, *Geom. Funct. Anal.* **10** (2000), no. 4, 902–974.

- [28] J. H. Przytycki and A. S. Sikora, *On skein algebras and $Sl_2(\mathbf{C})$ -character varieties*, *Topology* **39** (2000), no. 1, 115–148.
- [29] N. Reshetikhin and V. G. Turaev, *Invariants of 3-manifolds via link polynomials and quantum groups*, *Invent. Math.* **103** (1991), no. 3, 547–597.
- [30] Ramanujan Santharoubane, *Limits of the quantum $SO(3)$ representations for the one-holed torus*, *J. Knot Theory Ramifications* **21** (2012), no. 11, 1250109, 13.
- [31] William P. Thurston, *Hyperbolic structures on 3-manifolds, ii: Surface groups and 3-manifolds which fiber over the circle*, arXiv:math/9801045.
- [32] V. G. Turaev, *Skein quantization of Poisson algebras of loops on surfaces*, *Ann. Sci. École Norm. Sup. (4)* **24** (1991), no. 6, 635–704.
- [33] ———, *Quantum invariants of knots and 3-manifolds*, de Gruyter Studies in Mathematics, vol. 18, Walter de Gruyter & Co., Berlin, 1994.
- [34] Witten, E., *Quantum field theory and the Jones polynomial*, *Comm. Math. Phys.* **121** (1989), no. 3, 351–399.
- [35] T. Yang, *Recent progresses on the volume conjectures for Reshetikhin-Turaev and Turaev-Viro invariants*, *Acta Math. Vietnam.* **46** (2021), no. 2, 389–398.

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