Asymptotic behavior of quantum representations

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Outline

$\mathrm{Mod}(\Sigma)$ =mapping class group of surface Σ (closed or with boundary)

- **Quantum Representations.** Given odd integer $r \geqslant 3$, and a primitive $2r$-th root of unity there is a (projective) representation

$$\rho_r : \mathrm{Mod}(\Sigma) \to \mathbb{P}\mathrm{Aut}(RT_r(\Sigma)).$$

- "Large -r behavior of ρ_r and Nielsen-Thurston Classification : Know facts and and open conjectures (AMU Conjecture).
- Recall basics about TQFT underlying the quantum representations.: In particular Turaev-Viro $TV_r(M_f)$ invariants of a mapping torus M_f are obtained from traces of ρ_r.
- *Chen-Yang Conjecture* $\Rightarrow$ *AMU conjecture:* Exponential r-growth for $TV_r(M_f)$ implies f satisfies *AMU*.
- How do we "check" exponential r-growth for for $TV_r(M_f)$? Do we need to know C-Y conjecture?
- TV invariants and geometric decompositions of mapping tori: Integer vs non integer values of TV invariants.

Nielsen–Thurston classification

Convention. $\Sigma = \Sigma_{g,n}$= surface of genus g and n-bdry components.
Assume $3g - 3 + n > 0$.

Given a mapping class $\in \mathrm{Mod}(\Sigma)$ there is a representative $g : \Sigma \longrightarrow \Sigma$ such that at least one of the following holds:

1. g is *periodic*, i.e. some power of g is the identity;
2. g is *reducible*, i.e. preserves some finite union of disjoint simple closed curves Γ on Σ; or
3. g is pseudo-Anosov (never periodic or reducible)

- If $g : \Sigma \longrightarrow \Sigma$ reducible, then a power of g acts on each component of Σ cut along Γ.
- If at least one of the "pieces" is is pseudo-Anosov, we say g *has non-trivial pseudo-Anosov pieces.*

Mapping tori and Nielsen–Thurston classification

For $f \in \mathrm{Mod}(\Sigma)$ a mapping class let

$$M_f = F \times [0,1]/_{(x,0)\cong(f(x),1)}$$

be the mapping torus of f.

We have:

- f is reducible iff M_f has incompressible tori. In that case M_f can be cut along a canonical collection of such tori into geometric pieces (JSJ decomposition-geometric decomposition).
- In fact, by the Geometrization Theorem, each piece of the decomposition will be either Seifert fibered manifold or a hyperbolic.
- *Gromov norm of M_f:* $||M_f|| = v_{\mathrm{tet}} \mathrm{Vol}\,(H)$, $\mathrm{Vol}\,(H)$ is the sum of the hyperbolic volumes of components of the geometric decomposition.
- f is periodic iff M_f is a Seifert fibered manifold ($||M_f|| = 0$).
- f is pseudo-Anosov, iff M_f has hyperbolic structure.

- **Summary:** $f \in \mathrm{Mod}(\Sigma)$ has non-trivial pseudo-Anosov pieces iff $||M_f|| > 0$.

Quantum representations

- Witten-Reshetikin-Turaev, $SO(3)$-representations:
- For each odd integer $r \geqslant 3$, let $U_r = \{0, 2, 4, \ldots, r-3\}$.
- Given a primitive $2r$-th root of unity ζ_{2r}, a compact oriented surface Σ, and a coloring c of the components of $\partial\Sigma$ by elements of U_r,
- there is a finite dimensional $\mathbb{C}$-vector space, $RT_r(\Sigma, c)$ and a projective representation:

$$\rho_{r,c} : \mathrm{Mod}(\Sigma) \to \mathrm{End}(RT_r(\Sigma, c)).$$

- We have $\dim(RT_r(\Sigma_{g,n}, c)) \leq r^{3g-3+n}$. (dimensions given by Verlinde formula)
- **Note.** For different root of unity $\rho_{r,c}$ are related by Galois group actions: Say, e.g. if $\rho_{r,c}(f)$ has finite order at some $f \in \mathrm{Mod}(\Sigma)$, for some root of unity then, $\rho_{r,c}(f)$ has finite order for all roots of unity.
- We will work with $\zeta_{2r} = e^{\frac{i\pi}{r}}$.

Context:

- **Question.** What geometric information of $\mathrm{Mod}(\Sigma)$ do the representations $\rho_{r,c}$ detect? Do they detect the Nielsen-Thurston classification of mapping classes?
- The representations $\rho_{r,c}$ are not faithful! The images of Dehn twists have finite order! However, $\rho_{r,c}$ are asymptotically faithful:
- *(Freedman-Walker-Wang, Andersen)* Let $f \in \mathrm{Mod}(\Sigma)$. If $\rho_{r,c}(f) = 1$, for all r, c, then $f = 1$.
  [except in the few cases when $\mathrm{Mod}(\Sigma)$ has center and f is an involution.]
- **Corollary.** There is n, such that

$$(\rho_{r,c}(f))^n = \lambda Id \text{ for all } r, c, \text{ iff } f^n = 1.$$

  (i.e f is periodic) [*again some exceptions*].
- *Andersen-Masbaum-Ueno* conjectured (2002).
- **Conjecture.** *(AMU)* $f \in \mathrm{Mod}(\Sigma)$ has PA pieces iff for ever $r >> 0$ there is r a choice of colors c such that $\rho_{r,c}(\phi)$ has infinite order.

What is known:

- Andersen, Masbaum and Ueno (2004) proved their conjecture when $\Sigma = \Sigma_{0,3}$ or $\Sigma_{0,4}$; the three or four-holed sphere.
- Santharoubane proved the conjecture for the one-holed torus.
- Egsgaard and Jorgensen (2012) and Santharoubane (2015) proved the conjecture for families for mapping classes in $\Sigma = \Sigma_{0,n}$, for all $n > 4$.
- In all above cases the quantum representations turn out to be related to previously studied braid group representations: (specializations of Burau representations, McMullen's representations related to actions on homology of branched covers of $\Sigma_{0,n}$.)
- For surfaces of genus $g > 1$ no examples known till 2016!
- Using Birman exact sequences of mapping class groups, one extracts representations on $\pi_1(\Sigma)$ from the representations $\rho_{r,c}$.
- Marché and Santharoubane used these representations to obtain examples of pseudo-Anosov mappings classes satisfying the AMU conjecture by exhibiting "appropriate" elements in $\pi_1(\Sigma)$. Gave explicit curves on genus 2 surfaces (more next).

Quantum representations of surface groups

- $\chi(\Sigma) < 0$ and x_0 a marked point in the interior of Σ and $\mathrm{Mod}(\Sigma, x_0)$ group of classes preserving x_0
- **Birman Exact Sequence.**

$$0 \longrightarrow \pi_1(\Sigma, x_0) \longrightarrow \mathrm{Mod}(\Sigma, x_0) \longrightarrow \mathrm{Mod}(\Sigma) \longrightarrow 0.$$

- **Kra's criterion.** $\gamma \in \pi_1(\Sigma, x_0)$ represents a pseudo-Anosov mapping class iff γ *fills* Σ.
- The quantum representations give projective representation:

$$\rho_{r,c} : \pi_1(\Sigma) \to \mathrm{End}(RT_r(\Sigma, c)).$$

- (*Koberda-Satharoubane*) used $\rho_{r,c}$ to answer an open question (asked by several people independently Kent, Kisin, Marché, McMullen, ...):
- Constructed a linear a representation of $\pi_1(S)$, that has infinite image, but the image of every simple closed curve has finite order!
- Their work led to (another) algorithm that decides whether or not $\gamma \in \pi_1(\Sigma, x_0)$ is freely homotopic to a simple loop!

The examples of Marché-Satharoubane

- Gave first examples of pseudo-Anosov mapping classes, for surfaces of genus > 1, that satisfy the
- **AMU Conjecture for surface groups.** If a non-trivial element $\gamma \in \pi_1(\Sigma, x_0)$ is not a power of a class represented by a simple loop, then $\rho_{r,c}(\gamma)$ has infinite order for $r >> 0$ and a choice of c.
- Their examples are realized by immersed curves that *fill* Σ and satisfy an additional technical condition they called *Euler incompressibility*.
- They use skein theoretic methods in $S^1 \times \Sigma$ to construct a polynomial invariants links Σ. Roughly speaking, non-triviality of of some invariant for $\gamma \in \pi_1(\Sigma, x_0)$, implies that γ satisfies the AMU Conjecture for surface groups. Euler incompressibility of γ assures desired non-triviallity.
- Their criterion is hard to apply and, for fixed genus, it leads to finitely many (up to conjugation and powers) pseudo-Anosov mapping classes that satisfy the AMU Conjecture
- Gave explicit examples in genus two. The first evidence for AMU conjecture for genus > 1.

Another approach: Growth of TV invariants and AMU

- M compact, orientable 3-manifold with empty or toroidal boundary.
- For $r = odd$ and $q = e^{\frac{2\pi i}{r}}$ we have the Turaev-Viro invariant $TV_r(M) := TV_r(M, e^{\frac{2\pi i}{r}})$. Let

$$LTV(M) = \limsup_{r\to\infty} \frac{2\pi}{r} \log|TV_r(M)|, \quad ITV(M) = \liminf_{r\to\infty} \frac{2\pi}{r} \log|TV_r(M)|.$$

- **Remark.** (Generalized) Chen-Yang Conjecture would assert $ITV(M) = LTV(M) = v_{\mathrm{tet}}||M||$.
- **Weaker statement.** $ITV(M) > 0$; exponential growth with respect to r: For $r >> 0$, we have $\log|TVr(M)| \geq Br$, for some $B > 0$.
- **Theorem A.** (*Detcherry-K., 2017*) Let $f \in \mathrm{Mod}(\Sigma)$ mapping class and let M_f be the mapping torus of f. If $ITV(M_f) > 0$, then f satisfies the AMU conjecture.
- **Note.** $ITV(M_f) > 0$ also implies that the mapping class f has PA part (Next).

$ITV(M) > 0$ implies $||M|| > 0$

- **Theorem B.** (*Detcherry-K., 2017*) There exists a universal constant $C > 0$ such that for any compact orientable 3-manifold M with empty or toroidal boundary we have

$$ITV(M) \leq LTV(M) \leq C||M||.$$

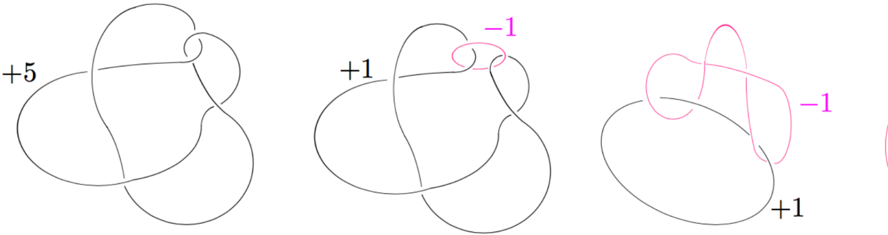
- Computing $ITV(M)$ is hard!
- We don't always have to compute $ITV(M)$ to decide exponential growth!
- **Limits do not increase under Dehn filling.** *(Detcherry-K)* If M is obtained by Dehn filling from M' then

$$ITV(M) \leq ITV(M') \quad \text{and} \quad LTV(M) \leq LTV(M').$$

- **Example.** Adding components to a link preserves exponential growth of TV invariants of link complement.

An example: Knot 5_2 and parents

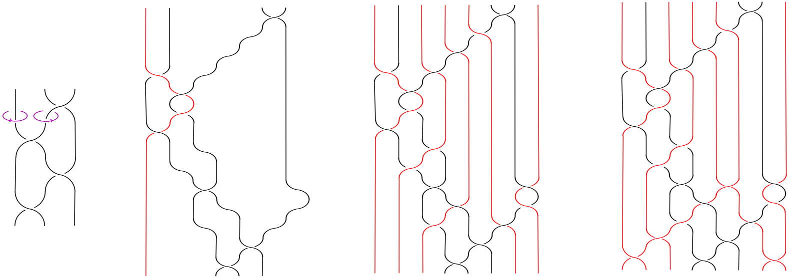
- $K(p)$= 3-manifold obtained by p-surgery on M.
- $LTV(4_1(-5)) = \mathrm{Vol}\,(4_1(-5)) \simeq 0.9813688 > 0$ [Ohtsuki, 2017]
- Observe $5_2(5)$ is homeomorphic to $4_1(-5)$.



- Dehn filling result implies $ITV(S^3 \setminus 5_2) \geqslant ITV(5_2(5)) = ITV(4_1(-5)) > 0$
- But Dehn filling result also implies that for any link containing 5_2 as a component we have exponential growth

Constructions and examples

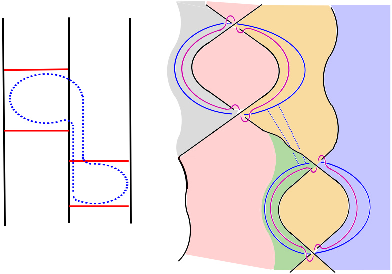
- Start with $L \subset S^3$ be a link with $ITV(S^3 \setminus L) > 0$.
- (*Stallings*) We can add a component K so that $K \cup L$ is a fibered link.
- In fact, $K \cup L$ will be a closed *homogeneous braid* and fiber is a Seifert surface obtained from closed braid projection.



- Refined Stallings process so that $K \cup L$ is a hyperbolic and monodromy of any fibration satisfies AMU Conjecture.
- There are only finitely many f. m. link types in homogeneous closed braids of fixed genus! Need to modify by appropriate Stallings twists.

Stallings twists

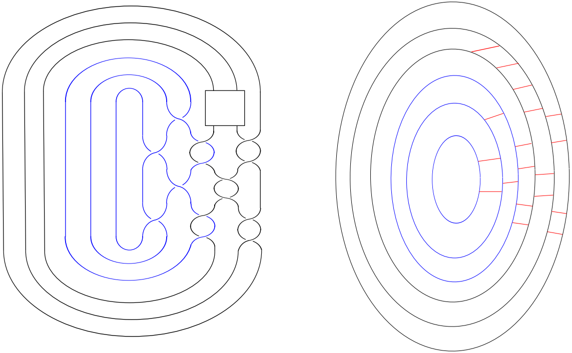
- c= a non-trivial s.c.c on the fiber with $lk(c, c^+) = 0$, c^+ is the curve c pushed along the positive normal of F. Assume c not parallel l to ∂F and bound a disc in $D \subset S^3$ that intersects F transversally:



- **A Stallings twist of order** m**:** A full twist of order m along D.
- The complement of the link L_m , obtained from L , fibers over S^1 with fiber F and the monodromy is $f \circ \tau_c^m$, where τ_c is the Dehn-twist on F along c.
- If f pseudo-Anosov , then the family $\{f \circ \tau_c^m\}_m$ contains infinitely many pseudo-Anosov homeomorphisms.

Concrete examples

- $K_1 = 4_1$=closure of the alternating braid $\sigma_2^{-1}\sigma_1\sigma_2^{-1}\sigma_1$. Have $ITV(S^3 \setminus 4_1) > 0$.
- Let L_m= 4 components shown below; with $2m$ crossings in box. $ITV(S^3 \setminus L) > ITV(S^3 \setminus 4_1) > 0$.



- hyperbolic link (alternating) and fibered (homogeneous closed braid)
- Fiber supports non-trivial Stallings twists. Genus of the fiber $g = 3 + m$. Monodromies elements in $\mathrm{Mod}(\Sigma_{g,4})$.

How many examples

- We have many constructions of examples now. Recall
- *fundamental shadow links:*
  1. Universal: they produce all 3-manifolds with empty or toroidal boundary by Dehn filling.
  2. TV invariants of their complements have exponential growth ($LTV > 0$).
- Let $\mathcal{M}$ denote the set of all 3-manifolds N that are complements of fundamental shadow links in orientable 3-manifolds with empty or toroidal boundary and their doubles DN.
- All 3-manifolds contain fibered links. We have:
- **Theorem D.** (*Belletti-Decherry-K- Yang*) Given $M \in \mathcal{M}$ and a (possibly empty) link $L \subset M$, there is a knot $K \subset M$ such that the link $K \cup L$ is fibered in M. Furthermore, the monodromy of any fibration of $M \setminus (K \cup L)$ is a mapping class that satisfies the AMU Conjecture.
- **(Vague) Question.** What mapping classes are realized by Theorem D? How "big" is the set of mpc?

TV Invariants as State sum on triangulations.

Quantum integer: $r \geqslant 3$ odd integer and $q = e^{\frac{2i\pi}{r}}$.

$$\{n\} = q^n - q^{-n} = 2\sin(\frac{2n\pi}{r}) = 2\sin(\frac{2\pi}{r})[n], \text{ where } [n] = \frac{q^n - q^{-n}}{q - q^{-1}} = \frac{2\sin(\frac{2n\pi}{r})}{2\sin(\frac{2\pi}{r})}.$$

Quantum factorial: $\{n\}! = \prod_{i=1}^{n} \{i\}$.

Set of colors: $I_r = \{0, 2, 4 \ldots, r-3\}$ even integers less than $r-2$.

Admissible Triple: (a_i, a_j, a_k) of elements in I_r,

$$a_i + a_j + a_k \leqslant 2(r-2), \quad \textit{and}$$

$$a_i \leqslant a_j + a_k, \ a_j \leqslant a_i + a_k, \ \ a_k \leqslant a_i + a_j.$$

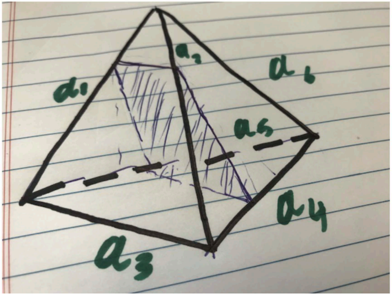
$$\Delta(a_i, a_j, a_k) = \zeta_r^{\frac{1}{2}} \left(\frac{\{\frac{a_i+a_j-a_k}{2}\}!\{\frac{a_j+a_k-a_i}{2}\}!\{\frac{a_i+a_k-a_j}{2}\}!}{\{\frac{a_i+a_j+a_k}{2}+1\}!} \right)^{\frac{1}{2}}$$

where $\zeta_r = 2\sin(\frac{2\pi}{r})$.

Admissible 6-tuple: $(a_1, a_2, a_3, a_4, a_5, a_6) \in I_r^6$ each triple is dmisibble

$$F_1 = (a_1, a_2, a_3), \quad F_2 = (a_2, a_4, a_6), \quad F_3 = (a_1, a_5, a_6) \text{ and } F_4 = (a_3, a_4, a_5).$$

Tetrahedron colorings: Given an admissible 6-tuple:



Faces : $T_1 = \dfrac{a_1 + a_2 + a_3}{2}, \quad T_2 = \dfrac{a_1 + a_5 + a_6}{2}, \quad T_3 = ... \text{ and } T_4 = ...$

Quadrilaterals:

$$Q_1 = \frac{a_1 + a_2 + a_4 + a_5}{2}, \quad Q_2 = \frac{a_1 + a_3 + a_4 + a_6}{2} \text{ and } Q_3 = \frac{a_2 + a_3 + a_5 + a_6}{2}.$$

Quantum 6j-symbol: Given admissible 6-tuple $\alpha := (a_1, a_2, a_3, a_4, a_5, a_6) \in I_r^6$,

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{vmatrix} = \Delta(\alpha) \times \sum_{z=\max\{T_1,T_2,T_3,T_4\}}^{\min\{Q_1,Q_2,Q_3\}} \frac{(-1)^z\{z+1\}!}{\prod_{j=1}^4\{z-T_j\}!\prod_{k=1}^3\{Q_k-z\}!} \qquad (1)$$

where

$$\Delta(\alpha) := (\zeta_r)^{-1}(\sqrt{-1})^{\lambda}\prod_{i=1}^{4}\Delta(F_i),$$

and

$$\lambda = \sum_{i=1}^{6} a_i,$$

and

$$\zeta_r = 2\sin(\frac{2\pi}{r}).$$

Colorings of Triangulations

Given a compact orientable 3-manifold M consider a triangulation τ of M. If $\partial M \neq \emptyset$ allow τ to be a (partially) *ideal triangulation*: some vertices of the tetrahedra are truncated and the truncated faces triangulate ∂M.

- V=set of vertices of τ which do not lie on ∂M.
- E= set of interior edges (thus excluding edges coming from the truncation of vertices).
- *Admissible coloring at level r:* An assignment

$$c : E \longrightarrow I_r$$

  so that edges of each tetrahedron get an *admissible 6-tuple*.
- Given a coloring c and an edge $e \in E$ let

$$|e|_c = (-1)^{c(e)}[c(e) + 1].$$

- For Δ a tetrahedron in τ let $|\Delta|_c$ be the quantum 6j-symbol corresponding to the admissible 6-tuple assigned to Δ by c.

The invariant

- $A_r(\tau)$= the set of r-admissible colorings of τ
- $\eta_r = \dfrac{2\sin(\frac{2\pi}{r})}{\sqrt{r}}$.
- Turaev-Viro invariants as a state-sum over $A_r(\tau)$.

Theorem (Turaev-Viro 1990)

Let M be a compact, connected, orientable manifold closed or with boundary. Let b_2 denote the second $\mathbb{Z}_2$-Betti number of M. Then the state sum

$$TV_r(M) = 2^{b_2 - 1}\ \eta_r^{2|V|} \sum_{c \in A_r(\tau)} \prod_{e \in E} |e|_c \prod_{\Delta \in \tau} |\Delta|_c, \qquad (2)$$

is independent of the partially ideal triangulation τ of M, and thus defines a topological invariant of M.

- $6j$-sympols satisfy identities (Biedenharn-Elliot identity, Orthogonality relation). These identities are used to show that state sum in 2 is invariant under Pachner moves of triangulations of M. Thus invariant of M.

TV invariants as part of a TQFT

- *Witten-Reshetikhin-Turaev* TQFT/ *Blanchet-Habegger-Masbaum-Vogel.*
- For $r \geq 3$ and $\zeta_r = e^{\frac{i\pi}{r}}$, we have a TQFT functor RT_r:
- M=closed, oriented 3-manifold $RT_r(M)$=$\mathbb{C}$-valued invariant.
- Σ=compact, oriented surface, w. coloring c of $\partial\Sigma$,

$$RT_r(\Sigma, c) = f.d.\ \mathbb{C} - \text{vector space.}$$

- M=cobordism with $\partial M = -\Sigma_0 \cup \Sigma_1$, there is a map

$$RT_r(M) \in \mathrm{End}(RT_r(\Sigma_0), RT_r(\Sigma_1)).$$

- RT_r takes composition of cobordisms to composition of linear maps.
- We have a f.d. projective representation:

$$\rho_r : \mathrm{Mod}(\Sigma) \to \mathrm{End}(RT_r(\Sigma, c)).$$

- If $\partial\Sigma = \emptyset$, and C_f=mapping cylinder of f,

$$\rho_r(f) = RT_r(C_f).$$

- If $\partial\Sigma \neq \emptyset$ we color $\partial\Sigma$ with elements of U_r. To define $\rho_{r,c}$ need RT_r for cobordisms w. colored tangles.

Proof of Theorem A

- Using work of Roberts, Walker, Benediti-Pertronio and TQFT properties we get
$$TV_r(M_f) = \sum_c |\mathrm{Tr}\rho_{r,c}(f)|^2,$$
where the sum ranges over all colorings of the boundary components of M_f by elements of U_r.
- Since $ITV(M_f) > 0$, the sequence $\{TV_r(M_f)\}_r$ is bounded below by a sequence that is exponentially growing in r as $r \to \infty$.
- The sequence $\sum_c \dim(RT_r(\Sigma, c))$ only grows polynomially in r. In fact, $\dim(RT_r(\Sigma_{g,n}, c)) \leq r^{3g-3+n}$.
- r, there will be at least one c such that $|\mathrm{Tr}\rho_{r,c}(f)| > \dim(RT_r(\Sigma, c))$.
- Then $\rho_{r,c}(f)$ must have an eigenvalue of modulus 1. Thus it has infinite order.

Mapping Tori: Integer and non integer values of TV_r

- *(D-K)* Let M_f be the mapping torus of a periodic mapping class $f \in \mathrm{Mod}(\Sigma)$ of order N. Then, for any odd integer $r \geqslant 3$, with $\gcd(r, N) = 1$, we have $TV_r(M_f) \in \mathbb{Z}$, for any choice of root of unity.
- In particular: $TV_r(M_f) \in \mathbb{Z}$, for infinitely many r.
- If $ITV(M_f) > 0$ at some root of unity, then there can be at most finitely many values r for which $TV_r(M_f) \in \mathbb{Z}$.
- **Conjecture.** Suppose that $f \in \mathrm{Mod}(\Sigma)$ contains a PA part. Then, there can be at most finitely many odd integers r such that $TV_r(M_f) \in \mathbb{Z}$.

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THANK YOU!