Skein modules, character varieties and essential surfaces of 3-manifolds

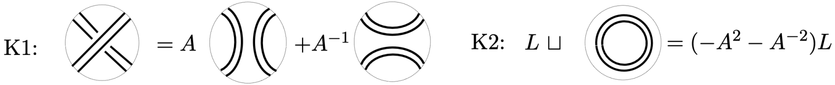
joint w/ R. Detcherry and A. Sikora.

E. Kalfagianni, Michigan State University

Quantum Topology Conference, MPI-Bonn, May, 2025

Kauffman Bracket Skein Module

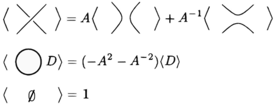
- M= oriented, closed 3-manifold.
- R= commutative ring R with a distinguished invertible element $A \in R$
- $\mathcal{S}(M, R)$=*Kauffman bracket skein module* of M
- $\mathcal{S}(M, R)$= quotient of the free R-module on all framed unoriented links in M, including the empty link, by the relations:
- Left hand side



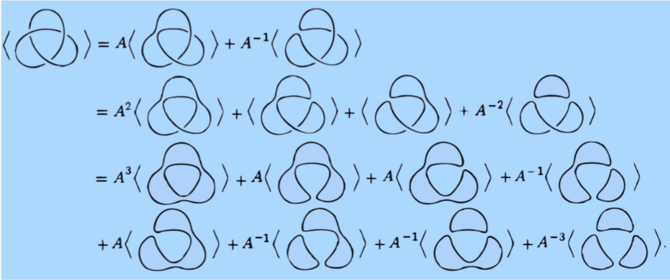
- Defined in the 80's (Przytycki, Turaev).
- *Surface skein algebra*: $\mathcal{S}(M)$, for $M = \text{surface} \times I$.
  - Used in constructions of Witten-Reshetikhin -Turaev TQFT theories.
  - Relations with character varieties, cluster algebras, quantum field theories, hyperbolic geometry.....
- In this talk: M= closed 3-manifold, and $R = \mathbb{Z}[A^{\pm 1}]$ or $R = \mathbb{Q}(A)$.

Examples

- $\mathcal{S}(S^3, R) \cong R$, generated by the *empty link*:
- *Kauffman bracket:* $\langle\ \rangle$: link diagrams $\longrightarrow R$ such that



- For $D = D(K)$ where $K =$ trefoil knot :



- We obtain: $K = J(A, A^{-1}) \cdot \emptyset$, $J(A, A^{-1})$= Jones polyn. of framed K.

Essential surfaces: Why?

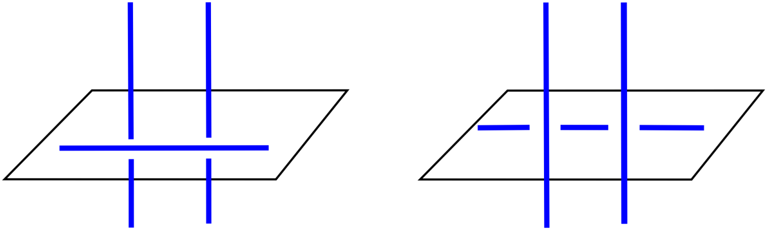
M=closed , orientable, 3-manifold

- **Defin.** *essential surface in M*= embedded, orientable surfaces $S \subset M$ s.t.
  1. S=2-sphere non-trivial in $\pi_2(M)$ or
  2. $S \neq S^2$ and it is π_1-injective.

- Essential surfaces play important roles in 3-manifold theory!.
- For instance,
- By the Geometrization Theorem, M admits a canonical decomposition along essential spheres and tori into pieces that are
  - *hyperbolic*, or
  - *Seifert fibered manifolds*: admit S^1 actions (e.g. S^1=bundles over surfaces).
- Essential surfaces affect the structure of $\mathcal{S}(M, \mathbb{Z}[A^{\pm 1}])$ (more later).

Surfaces and skein modules of 3-manifolds?

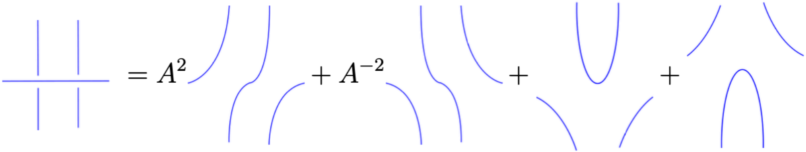
- Embedded surfaces produce relations in $\mathcal{S}(M,\ \mathbb{Z}[A^{\pm 1}])$:
- **Example.** Plane represents part of an embedded separating torus T in M. Horizontal component γ lies on T; vertical stings intersect T exactly twice.



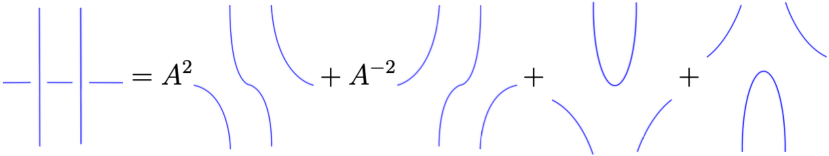
- The two pictures represent isotopic links! (Isotopy defined by annulus $T \setminus n(\gamma)$.)
- The isotopy leads to a relation in $\mathcal{S}(M,\ \mathbb{Z}[A^{\pm 1}])$.

Example con't: Relation

- Apply the skein relations to the crossings shown in last figure:
- Left hand side

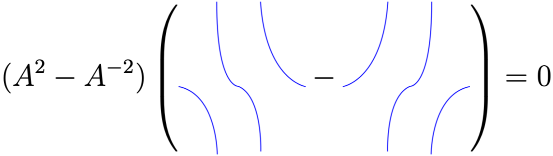


- Right hand side



Example con't: Torsion

- We get a relation (candidate for torsion)



- If the torus T is π_1-injective, this process produces torsion in $\mathcal{S}(M, \mathbb{Z}[A^{\pm 1}])$ (i.e. above relation is not trivial)
- (Przytycki, Veve, '99) Concrete examples where $\mathcal{S}(M, \mathbb{Z}[A^{\pm 1}])$ has torsion coming from π_1-injective tori.
- (Przytycki, 90's) Noted similar phenomena If M contains non-separating 2-spheres. Conjectured that only essential genus surfaces of genus zero and one create torsion.
- (Belletti-Detcherry, 2024) Large families with torsion in $\mathcal{S}(M, \mathbb{Z}[A^{\pm 1}])$ coming any genus π_1-injective surface.

Examples: $\mathcal{S}(M, \mathbb{Z}[A^{\pm 1}])$ can be "wild"

- (Hoste-Przytycki, 90's) $\mathcal{S}(L(p,q),\ \mathbb{Z}[A^{\pm 1}])$ are free, finitely generated.
- (Hoste-Przytycki, 90's) $\mathcal{S}(S^2 \times S^1,\ \mathbb{Z}[A^{\pm 1}])$ is not finitely generated:

$$\mathcal{S}(S^2 \times S^1, \mathbb{Z}[A^{\pm 1}]) = \mathbb{Z}[A^{\pm 1}] \bigoplus \big(\oplus_i \mathbb{Z}[A^{\pm 1}]/(1 - A^{2i+4}) \big).$$

- (Gilmer-Masbaum, Detcherry-Wolf, 2020) Σ_g=genus g surface

$$\dim_{\mathbb{Q}(A)} S(\Sigma_g \times S^1, \mathbb{Q}(A)) = 2^{2g+1} + 2g - 1.$$

- (Mroczkowski, 2011) $S(\mathbb{R}P^3 \sharp \mathbb{R}P^3, \mathbb{Z}[A^{\pm 1}])$ not a direct sum of free and cyclic $\mathbb{Z}[A^{\pm 1}]$-modules, but $\dim_{\mathbb{Q}(A)} S(RP^3 \sharp \mathbb{R}P^3, \mathbb{Q}(A)) = 4$.
- (Kinnear, 2023) Computed $\dim_{\mathbb{Q}(A)} S(M, \mathbb{Q}(A)) < \infty$, for any M that fibers over S^1 with fiber a 2-torus. Finite dimenion.

A finiteness result

- (*The question is attributed to Witten*:) For closed M, is it always

$$\dim_{\mathbb{Q}(A)} S(M, \mathbb{Q}(A)) < \infty?$$

- Indeed,

Theorem (Gunningham, Jordan and Safronov, 2019)

The skein module $S(M, \mathbb{Q}(A))$ is finite dimensional for any closed M.

- **Questions.** What is the invariant $\dim_{\mathbb{Q}(A)} S(M, \mathbb{Q}(A))$?
- How do we compute it?
- (more later)

Character variety connection

- (2000, Przytycki-Sikora, Bullock) $\mathcal{S}(M, \mathbb{Z}[A, A^{-1}])$ is a "deformation" of the $\mathrm{SL}_2(\mathbb{C})$-character variety of M.
- More precisely:
- The $\mathrm{SL}_2(\mathbb{C})$-character variety,

$$\mathcal{X}(M) := \mathrm{Hom}(\pi_1(M), \mathrm{SL}_2(\mathbb{C})) /\!\!/ \mathrm{SL}_2(\mathbb{C})$$

  a scheme over $\mathbb{C}$ ($X(M)$= the algebraic set underlying $\mathcal{X}(M)$).
- $\mathbb{C}[\mathcal{X}(M)]$=*coordinate ring* of $\mathcal{X}(M)$ (i.e. is the algebra of global sections of the structure sheaf of $\mathcal{X}(M)$)
- **Fact 1.** $\mathcal{X}(M)$ can be *non-reduced* (i.e. $\mathbb{C}[\mathcal{X}(M)]$ may contain nilpotents) Then,

$$\mathbb{C}[X(M)] = \mathbb{C}[\mathcal{X}(M)]/\{\mathrm{Nil}-\mathrm{radical}\}.$$

- **Fact 2:** $\rho, \rho' : \pi_1(M) \to SL(2, \mathbb{C})$ are identified in $X(M)$ iff $tr\rho = tr\rho'$.

Character variety connection, con't

- The skein module "at $A = -1$ is the coordinate ring of the character variety".
- Specifically: Let

$$S_{-1}(M) := S(M, \mathbb{Z}[A^{\pm 1}]) \otimes \mathbb{Z}[A^{\pm 1}]\mathbb{C},$$

where the $\mathbb{Z}[A^{\pm 1}]$-module structure of $\mathbb{C}$ is given by sending A to -1.

Theorem (Przytycki-Sikora, 2000)

$S_{-1}(M)$ has the structure of $\mathbb{C}$-algebra that is isomorphic to the coordinate ring $\mathbb{C}[\mathcal{X}(M)]$.

- The isomorphism:

$$\psi : S_{-1}(M) \longrightarrow \mathbb{C}[\mathcal{X}(M)] \text{ sends } K \longrightarrow -t_K,$$

for any knot.

- *Trace function*: $t_K : \mathbb{C}[\mathcal{X}(M)] \longrightarrow \mathbb{C}, \quad t_K([\rho]) = tr\rho([K])$, for all $\rho : \pi_1(M) \longrightarrow \mathrm{SL}_2(\mathbb{C})$.

Two questions:

- Rest of the talk:
- **Question 1:** When is $\mathcal{S}(M)$ finitely generating over $\mathbb{Z}[A^{\pm 1}]$?
- **Question 2:** How does the skein module $\mathcal{S}(M, \mathbb{Q}(A))$ relate to $\mathcal{X}(M)$ and $X(M)$ for generic values of A?
- What we know about Questions 1 and 2: Conjectures and results
- How are the two questions related,
- How existence of *essential* surfaces contained in M affect the answers,
- How progress on them allows to
  1. compute the dimension of $\mathcal{S}(M, \mathbb{Q}(A))$ over $\mathbb{Q}(A)$. Does it relate to known 3-manifold invariants?
  2. begin to establish instances of conjectural relations of $\mathcal{S}(M, \mathbb{Q}(A))$ with "certain" Floer theoretic invariants.

When $\mathcal{S}(M)$ finitely generating over $\mathbb{Z}[A^{\pm 1}]$?

- We have a conjecture (Detcherry-K.-Sikora):

Conjecture (Conjecture A)

The skein module $\mathcal{S}(M, \mathbb{Z}[A^{\pm 1}])$ is finitely generated if and only if M contains no essential surfaces.

- Conjecture A asserts $\mathcal{S}(M, \mathbb{Z}[A^{\pm 1}])$ detects all essential surface.
- **Note!** $\mathrm{SL}_2(\mathbb{C})$-character variety of M detects some but not all essential surfaces!
- (Culler-Shalen, 80's):

  $X(M)$ is infinite $\Rightarrow$ M contains essential surfaces.
- However, (converse is not true)
- there are M containing essential surfaces but $X(M)$ is finite!

What is known:

- Conjecture A is true for M with infinite $X(M)$:

Theorem (Detcherry-K.-Sikora, 2023)

If $X(M)$ is infinite, then $\mathcal{S}(M, \mathbb{Z}[A^{\pm 1}])$ is not finitely generated.

- (Detcherry-K.-Sikora, 2024): Conjecture A for all *Seifert fibered manifolds*

Theorem (*Theorem B*)

A Seifert 3-manifold M contains no essential surfaces if and only if $\mathcal{S}(M, \mathbb{Z}[A^{\pm 1}])$ is finitely generated.

- Examples of M with essential surfaces and finite $X(M)$

Theorem (Mroczkowski, 2011, Belletti-Detcherry 2024)

$\mathcal{S}(M, \mathbb{Z}[A^{\pm 1}])$ is not finitely generated $M := \mathbb{R}P^3 \# L(2p, 1)$, for any $p > 1$.

Applications: What is $\dim_{\mathbb{Q}(A)} \mathcal{S}(M)$?

- Recall: At $A = -1$, the skein module $S_{-1}(M)$ is the coordinate ring the $\mathrm{SL}_2(\mathbb{C})$-character variety of M.
- **Question.** How does the skein module $\mathcal{S}(M) := \mathcal{S}(M, \mathbb{Q}(A))$ relate to $\mathcal{X}(M)$ and $X(M)$ for generic values of A?
- We have an answer if $\mathcal{S}(M, \mathbb{Z}[A^{\pm 1}])$ is finitely generated (Detcherry-K.-Sikora, 2023)

Theorem (*Theorem C*)

If M is a closed 3-manifold with finitely generated $S(M, \mathbb{Z}[A^{\pm 1}])$, then

$$|X(M)| \leq \dim_{\mathbb{Q}(A)} \mathcal{S}(M) \leq \dim_{\mathbb{C}} \mathbb{C}[\mathcal{X}(M)].$$

In particular, if $\mathcal{X}(M)$ is reduced, then $\dim_{\mathbb{Q}(A)} \mathcal{S}(M) = |X(M)|$.

- Hence,
  1. $S(M, \mathbb{Z}[A^{\pm 1}])$ finitely generated $\Rightarrow X(M)$ is finite; or
  2. Conjecture A is true if $X(M)$ is infinite.

Applications cont'

- Computing $\mathcal{X}(M)$ and deciding whether its reduced is not easy in general...
- Using algebraic geometry techniques and Theorem C we computed $\mathcal{S}(M, \mathbb{Q}(A))$ for the first infinite families of hyperbolic 3-manifolds.
- Sample result:

Corollary

For $q \neq 0$, let M_q the 3-manifold obtained by $1/q$-Dehn surgery on the figure-eight knot. Then, $\mathcal{X}(M_q)$ is reduced, and we have

$$\dim_{\mathbb{Q}(A)} S(M_q) = |X(M)| = \frac{1}{2}(|4q+1| + |4q-1|).$$

- **Remark.** $\dim_{\mathbb{Q}(A)} S(M_q) - 1$ is equal to the $\mathrm{SL}_2(\mathbb{C})$-Casson invariant of M_q. (*Invariant defined by Curtis, 1995*).

Applications cont'

- Computed $\mathcal{X}(M)$ for all *Seifert 3-manifolds* with out essential surfaces, determined when it is reduced and computed $\dim_{\mathbb{Q}(A)} \mathcal{S}(M)$.
- For instance: $M := \Sigma(p_1, p_2, p_3)$ is a Brieskorn spheres

$$\dim_{\mathbb{Q}(A)} \mathcal{S}(M) = 1 + \frac{(p_1 - 1)(p_2 - 1)(p_3 - 1)}{4}.$$

- Again, $\dim_{\mathbb{Q}(A)} \mathcal{S}(M) - 1 = \mathrm{SL}_2(\mathbb{C})$-Casson invariant of M.
- in these cases $\dim_{\mathbb{Q}(A)} \mathcal{S}(M)$ is also the dimension of the zero degree part of of a "certain version" (i.e. $HP^{\bullet}_{\#}(M)$) of the $\mathrm{SL}_2(\mathbb{C})$-Floer Homology constructed by Abouzaid-Manolescu (2019)
- It is conjectured that this is always true.
- We verify the conjecture, for example, for $\mathbb{Z}$-homology 3-spheres with finite $X(M) = \mathcal{X}(M)$.

Idea of proof of Theorem C:

- M=closed orientable 3-manifold
- Must prove
- If $\mathcal{S}(M, \mathbb{Z}[A^{\pm 1}])$, is finitely generated, then

$$|X(M)| \leq \dim_{\mathbb{Q}(A)} S(M) \leq \dim_{\mathbb{C}} \mathbb{C}[\mathcal{X}(M)].$$

  In particular, if $\mathcal{X}(M)$ is reduced, then $\dim_{\mathbb{Q}(A)} S(M) = |X(M)|$.
- **Upper Inequality** follows from the Przytycki-Sikora result, that

$$S(M, \mathbb{Z}[A^{\pm 1}]) \underset{A=-1}{\otimes} \mathbb{C} = S_{-1}(M) \simeq \mathbb{C}[\mathcal{X}(M)],$$

  and the fact that

$$\dim_{\mathbb{Q}(A)} S(M) + \dim_{\mathbb{Q}} S^{A+1}(M, \mathbb{Q}[A^{\pm 1}]) = \dim_{\mathbb{C}} \mathbb{C}[\mathcal{X}(M)],$$

- where $S^{A+1}(M, \mathbb{Q}[A^{\pm 1}])$=$(A+1)$-torsion submodule.

Idea of proof of Theorem C, cont':

- **Lower Inequality**: Look at the 3-manifold skein module $S_\zeta(M)$, at roots of unity ζ, and show:

Theorem

For any root of unity of order $2N$ with N odd ζ, we have

$$\dim_{\mathbb{C}}(S_\zeta(M)) \geq |X(M)|.$$

- Having the Theorem, for appropriate ζ

$$S_\zeta(M) = S(M, \mathbb{Z}[A^{\pm 1}]) \underset{A=\zeta}{\otimes} \mathbb{C} \simeq \mathbb{C}^{\dim_{\mathbb{Q}(A)} S(M)}.$$

- Proof of last theorem relies on
  1. Major recent advances on structure of surface skein modules at roots of unity by Bonahon-Wong , Ganev-Jordan-Safronov, Frohman-Kania-Bartoszyńska-Le....
  2. The theory of the $SU(2)$- Reshetikhin-Turaev invariants and the theory of *non-semisimple* sl_2-quantum invariants of 3-manifolds constructed by Constantino, Geer and Patureau-Mirand.

How do we use these ingredients?

- Given a closed 3-manifold M and a $2N$-th root of unity ζ with N odd, $S_\zeta(M)$=Kauffman Bracket Sklein module at $A = \zeta$.
- (Bonahon-Wong, Le) There is a certain action of $\mathbb{C}[\mathcal{X}(M)] = S_{-1}(M)$ on $S_\zeta(M)$.
- also $\mathbb{C}[\mathcal{X}(M)] = S_{-1}(M)$ acts on $\mathbb{C}$ through

$$\mathbb{C}[\mathcal{X}(M)] \to \mathbb{C}[X(M)], \ by \ f \cdot z = f(\chi)z,$$

for any $f \in \mathbb{C}[X(M)]$ and any $z \in \mathbb{C}$.

- We need

Theorem

Given a character $\chi \in X(M)$ that is the trace of a representation ρ, there is a surjective map $S_\zeta(M) \to \mathbb{C}$ that is $\mathbb{C}[\mathcal{X}(M)]$-equivariant with respect to above two actions.

-

How do we get RT_χ ?

- **Irreducible representations.** If χ corresponds an irreducible representation $\rho : \pi_1(M) \longrightarrow SL(2, \mathbb{C})$, we use work of Frohman, Kania-Bartoszyńska and Lê on the sructure of " the reduced skein module of *M*' at ρ. Key point is their determination of relation irreducible reps of surface groups to the Azumaya locus of their skein module at roots of unity!

- **Central characters:** : If χ corresponds to a central representation $\rho : \pi_1(M) \longrightarrow SL(2, \mathbb{C})$, we construct the map RT_χ using Reshetikhin-Turaev $SU(2)$-TQFT properties. We rely on the skein theoretic approach of Balnchet-Habbeger-Masbaum-Vogel.

- **Abelian non central characters:** We use the theory of the so called "non-semisimle" sl_2-quantum TQFT-theory by Constantino, Geer and Patureau-Mirand.

Outline of proof of Theorem B:

- M=Seifert fibered 3-manifold

$\mathcal{S}(M, \mathbb{Z}[A^{\pm 1}])$ is finitely generated $\Leftrightarrow$ M contains no essential surfaces.

- *implication* $\Rightarrow$: From Theorem C

  $\mathcal{S}(M, \mathbb{Z}[A^{\pm 1}])$ finitely generated $\Rightarrow$ $X(M)$ is finite

  This is true for all 3-manifolds!

- *implication* $\Leftarrow$:
  1. Use topological and character variety properties/results of Seifert fibered 3-manifolds to reduce the problem to a special class of Seifert fibered 3-manifolds: *They fiber over S^2 with three exceptional spheres and have non-zero Euler number*.
  2. Use Skein-theoretic techniques to prove that $\mathcal{S}(M, \mathbb{Z}[A^{\pm 1}])$ is finitely generated, for the special class of 3-manifolds.

The manifolds $M = M(S^2; \frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3})$

- Start with $S_{0,3} \times S^1$ ($S_{0,3}$="pair of pants")
- Obtain M by attaching solid V_1, V_2, V_3 tori to ∂N with meridians attached to curves of slopes $\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3}$.
- Euler number $e(M) := \frac{q_1}{p_1} + \frac{q_2}{p_2} + \frac{q_3}{p_3}$.
- The skein module $\mathcal{S}(S_{0,3} \times S^1, \mathbb{Z}[A^{\pm 1}])$ generated by knots that "live" near the boundary.
- Using the Frohman-Gelca basis for skein algebras of tori the skein module $\mathcal{S}(S_{0,3} \times S^1, \mathbb{Z}[A^{\pm 1}])$ corresponds to a subspace of $\mathbb{Z}^6$.
- Adding the solid tori V_i leads to between generators of $\mathcal{S}(S_{0,3} \times S^1, \mathbb{Z}[A^{\pm 1}])$ and a presentation of $\mathcal{S}(M, \mathbb{Z}[A^{\pm 1}])$.
- This perspective allows to obtain an a complexity on $\mathcal{S}(M, \mathbb{Z}[A^{\pm 1}])$ that under the hypothesis that $e(M) \neq 0$, can be used to reduce the set of generators to finitely many.

References

- G. Belletti, R. Detcherry: *On torsion in the Kauffman bracket skein module of 3-manifolds*, **Advances in Math**, vol 427, paper 110167 (2025).
- R. Detcherry, E. Kalfagianni, A. Sikora: *Kauffman bracket skein modules of small 3-manifolds*, math.ArXiv:2305.16188.
- R. Detcherry, E. Kalfagianni, A. Sikora: *Skein modules and character varieties of Seifert manifolds*, math.arXiv:2405.18557.

Happy birthday Stavro!