Quantum representations and geometry of mapping class groups

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Mod(\(\Sigma\)) = mapping class group of surface \(\Sigma\) (closed or with boundary)

- **Quantum Representations.** Given odd integer \(r \geq 3\), and a primitive \(2r\)-th root of unity there is a (projective) representation

\[
\rho_r : \text{Mod}(\Sigma) \rightarrow \mathbb{P}\text{Aut}(RT_r(\Sigma)).
\]

- “Large-\(r\)” behavior of \(\rho_r\) and Nielsen-Thurston Classification: Know facts and and open conjectures (*Andersen-Masbaum-Ueno Conjecture*).

Recall basics about TQFT underlying the quantum representations: In particular *Turaev-Viro* invariants of a mapping torus \(M_f\), denoted \(TV_r(M_f)\), are obtained from traces of \(\rho_r\).

- **Key point.** Exponential \(r\)-growth for \(TV_r(M_f)\) implies \(f\) satisfies the *AMU* conjecture.

- Manifolds with exponential \(r\)-growth for \(TV_r\)—Context/Setting (volume conjectures).

- Constructions of mapping tori with exponential \(r\)-growth of \(TV\) invariants using properties of fibered links (open book decompositions) in 3-manifolds.
Convention. $\Sigma = \Sigma_{g,n}$ is a surface of genus $g$ and $n$-bdry components. Assume $3g - 3 + n > 0$.

Given a mapping class $f \in \text{Mod}(\Sigma)$ there is a representative $g : \Sigma \to \Sigma$ such that at least one holds:

1. $g$ is periodic, i.e. some power of $g$ is the identity;
2. $g$ is reducible, i.e. preserves some finite union of disjoint simple closed curves $\Gamma$ on $\Sigma$; or
3. $g$ is pseudo-Anosov (never periodic or reducible)

- If $g : \Sigma \to \Sigma$ is reducible, then a power of $g$ acts on each component of $\Sigma$ cut along $\Gamma$.
- If at least one of the “pieces” is pseudo-Anosov, we say $g$ has non-trivial pseudo-Anosov pieces.
For $f \in \text{Mod}(\Sigma)$ a mapping class let

$$M_f = \Sigma \times [0, 1]/(x,0) \sim (f(x),1)$$

be the mapping torus of $f$. We have:

- $f$ is **reducible** iff $M_f$ has **incompressible** tori. In that case $M_f$ can be cut along a canonical collection of such tori into geometric pieces (JSJ decomposition-geometric decomposition).

- Each piece of the decomposition will be either **Seifert fibered manifold** or a **hyperbolic**.

- **Gromov norm of $M_f$:** $\nu_{\text{tet}} \|M_f\| = \text{Vol}(H)$, $\text{Vol}(H)$ is the sum of the hyperbolic volumes of components of the geometric decomposition.

- $f$ is **periodic** iff $M_f$ is a **Seifert fibered** manifold ($\|M_f\| = 0$).

- $f$ is **pseudo-Anosov**, iff $M_f$ has **hyperbolic structure**.

**Summary:** $f \in \text{Mod}(\Sigma)$ has non-trivial pseudo-Anosov pieces iff $\|M_f\| > 0$. 
Quantum representations

- **Witten-Reshetikin-Turaev, SO(3)-representations:**
  - For each odd integer $r \geq 3$, let $U_r = \{0, 2, 4, \ldots, r - 3\}$.
  - Given a primitive $2r$-th root of unity $\zeta_r$, a compact oriented surface $\Sigma$, and a coloring $c$ of the components of $\partial \Sigma$ by elements of $U_r$,
  - there is a finite dimensional $\mathbb{C}$-vector space, $RT_r(\Sigma, c)$ and representations:
    $$\rho_{r,c} : \text{Mod}(\Sigma) \to \mathbb{P}\text{Aut}(RT_r(\Sigma, c)).$$

- We have $\dim(RT_r(\Sigma_{g,n}, c) \cong r^{3g-3+n}$. (dimensions grow polynomially in $r$; Verlinde formula.)
- We will work with $\zeta_r = e^{i\pi/r}$. (*TQFT is not unitary*)
**Question.** What geometric information of $\text{Mod}(\Sigma)$ do the representations $\rho_{r,c}$ detect?

The representations $\rho_{r,c}$ are not faithful! The images of Dehn twists have finite order! However, $\rho_{r,c}$ are asymptotically faithful:

**Theorem**

*(Andersen, Freedman-Walker-Wang, Marché-Narimannejad)* Let $f \in \text{Mod}(\Sigma)$. If $\rho_{r,c}(f) = 1$, for all $r, c$, then $f = 1$. *[except in the few cases when $\text{Mod}(\Sigma)$ has center and $f$ is an involution.]*

Hence: There is $n$, such that

\[(\rho_{r,c}(f))^n = \lambda I_d \text{ for all } r, c, \text{ iff } f^n = 1.\]

(i.e $f$ is periodic) *[again some exceptions].*

**Conjecture.** *(AMU, 2004)* $f \in \text{Mod}(\Sigma)$ has PA pieces iff for every $r >> 0$ there a choice of colors $c$ such that $\rho_{r,c}(f)$ has infinite order.

**Remark.** $f \in \text{Mod}(\Sigma)$ satisfies the AMU iff at least of its PA pieces does.
What is known:

- **Andersen, Masbaum and Ueno (2004)** proved their conjecture when $\Sigma = \Sigma_{0,4}$ = the four-holed sphere.

- **Santharoubane** proved the conjecture for the one-holed torus.

- **Egsgaard and Jorgensen (2012) and Santharoubane (2015)** proved the conjecture for families for mapping classes in $\Sigma = \Sigma_{0,n}$, for all $n > 4$.

In all above cases the quantum representations turn out to be related to previously studied braid group representations: (specializations of Burau representations, McMullen’s representations related to actions on homology of branched covers of $\Sigma_{0,n}$.)

- For surfaces of genus $g > 1$ no examples known till 2016.

- Using **Birman exact sequences** of mapping class groups, one extracts representations of $\pi_1(\Sigma)$ from the representations $\rho_{r,c}$.

- **Marché and Santharoubane** used these representations to obtain examples of pseudo-Anosov mappings classes satisfying the AMU conjecture by exhibiting “appropriate” elements in $\pi_1(\Sigma)$. Gave explicit curves on genus 2 surfaces (more next).
Quantum representations of surface groups

- $\chi(\Sigma) < 0$ and $x_0$ a marked point in the interior of $\Sigma$ and $\text{Mod}(\Sigma, x_0)$ group of classes preserving $x_0$.

- **Birman Exact Sequence.**
  
  $0 \longrightarrow \pi_1(\Sigma, x_0) \longrightarrow \text{Mod}(\Sigma, x_0) \longrightarrow \text{Mod}(\Sigma) \longrightarrow 0$.

- **Kra’s criterion.** $\gamma \in \pi_1(\Sigma, x_0)$ represents a pseudo-Anosov mapping class iff $\gamma$ *fills* $\Sigma$.

- The quantum representations give representations:
  
  $\rho_{r,c} : \pi_1(\Sigma) \rightarrow \mathbb{P}\text{Aut}(RT_r(\Sigma, c))$.

- *(Koberda-Satharoubane)* used $\rho_{r,c}$ to answer an open question (asked by several people independently Kent, Kisin, Marché, McMullen, ...):

  - Constructed a linear representation of $\pi_1(\Sigma, x_0)$, that has infinite image, but the image of every simple closed curve has finite order!
  - Their work led to (another) algorithm that decides whether or not $\gamma \in \pi_1(\Sigma, x_0)$ is freely homotopic to a simple loop!
The examples of Marché-Satharoubane

- Gave first examples of pseudo-Anosov mapping classes, for surfaces of genus \( > 1 \), that satisfy the following \((\text{implied by AMU})\).

- **AMU Conjecture for surface groups.** If a non-trivial element \( \gamma \in \pi_1(\Sigma, x_0) \) is not a power of a class represented by a simple loop, then \( \rho_{r,c}(\gamma) \) has infinite order for \( r \gg 0 \) and a choice of \( c \).

- Their examples are realized by immersed curves that \textit{fill} \( \Sigma \) and satisfy an additional technical condition they called \textit{Euler incompressibility}.

- They use WRT-TQFT (at “usual” root of unity) to construct a \((\text{Jones-type})\) polynomial invariant for links in \( S^1 \times \Sigma \). Roughly speaking, non-triviality of the invariant for \( \gamma \in \pi_1(\Sigma, x_0) \), viewed as link in \( S^1 \times \Sigma \), implies that \( \gamma \) satisfies the AMU Conjecture for surface groups. \textit{Euler incompressibility} of \( \gamma \) is used to derive non-triviality.

- For fixed genus, their criterion, leads to finitely many (up to conjugation and powers) pseudo-Anosov mapping classes that satisfy the AMU Conjecture.

- Gave explicit examples in genus two. The first evidence for AMU conjecture for genus \( > 1 \).
Another approach: Growth of TV invariants and AMU

- \( r = \text{odd}, \ TV_r(M) := TV_r(M, e^{\frac{2\pi i}{r}}) = \) Turaev-Viro invariant at level \( r \),

\[
ITV(M) := \liminf_{r \to \infty} \frac{2\pi}{r} \log |TV_r(M)|.
\]

- (Generalized) Q. Chen- T. Yang volume conjecture:

\[
ITV(M) = \nu_{\text{tet}} ||M|| > 0
\]

- (Weaker) Exponential Growth Conjecture:

\[
ITV(M) > 0 \iff ||M|| > 0
\]

Relevance to AMU Conjecture: EGC imples AMU:

Proposition. (Detcherry-K., 2017) Let \( f \in \text{Mod}(\Sigma) \) mapping class and let \( M_f \) be the mapping torus of \( f \). If \( ITV(M_f) > 0 \), then \( f \) satisfies AMU.

Key point: View \( TV_r(M) \) as part of the \( SO(3) \)-TQFT theory rather than a combinatorial state sum defined on triangulations of \( M \).
TV invariants as part of a TQFT

- For $r \geq 3$ and $\zeta_r = e^{\frac{i\pi}{r}}$, we have a TQFT functor $RT_r$:
- $M=\text{closed, oriented 3-manifold } RT_r(M) = \mathbb{C}$-valued invariant.
- $\Sigma=\text{compact, oriented surface, w. } U_r$-coloring $c$ of $\partial \Sigma$, 
  $$RT_r(\Sigma, c) = f.d. \mathbb{C} - \text{vector space}.$$ 

- $M=\text{cobordism with } \partial M = -\Sigma_0 \cup \Sigma_1$, there is a map 
  $$RT_r(M) \in \text{End}(RT_r(\Sigma_0), RT_r(\Sigma_1)).$$

- $RT_r$ takes composition of cobordisms to composition of linear maps.
- We get 
  $$\rho_{r,c} : \text{Mod}(\Sigma) \rightarrow \mathbb{P}\text{Aut}(RT_r(\Sigma, c)).$$

- If $\partial \Sigma = \emptyset$, and $C_f=\text{mapping cylinder of } f$, $\rho_r(f) = RT_r(C_f)$.
- If $\partial \Sigma \neq \emptyset$ we color $\partial \Sigma$ with elements of $U_r$. To define $\rho_{r,c}$ need $RT_r$ for cobordisms w. colored tangles.
Proof of Proposition:

- By Beliakova, Roberts, Turaev, Walker, (Benediti-Pertronio) and TQFT structure

\[
TV_r(M_f) = \sum_c (\text{Tr} \rho_{r,c}(f))^2.
\]

where the sum ranges over all colorings of the boundary components of \( M_f \) by elements of \( U_r \).

- Since \( \text{ITV}(M_f) > 0 \), the sequence \( \{TV_r(M_f)\}_r \) is bounded below by a sequence that is exponentially growing in \( r \) as \( r \to \infty \).

- The sequence \( \sum_c \text{dim}(RT_r(\Sigma, c)) \) only grows polynomially in \( r \).

- So, there will be at least one \( c \) such that \( |\text{Tr} \rho_{r,c}(f)| > \text{dim}(RT_r(\Sigma, c)) \).

- Then \( \rho_{r,c}(f) \) must have an eigenvalue of modulus bigger than 1. Thus it has infinite order.
More detail: Torus orthonormal basis

- RHS evaluated at $\zeta_r = e^{i\pi r}$, and LHS at $\zeta_r^2$, and $\langle .. \rangle =$Hermitian pairing of $RT_r(\Sigma, c)$.

$$TV_r(M_f) = ||RT_r(M_f)||^2 = \langle RT_r(M_f), RT_r(M_f) \rangle.$$

- $RT_r(\partial M_f)$ has orthonormal basis $e_c$, where $c$ runs over all $n$-tuples; one for each boundary component.

- $e_c$ is also the $RT_r$-vector of the cobordism of $n$ solid tori, with the $i$-th solid torus containing the core colored by $c_i$.

- Write $RT_r(M_f) = \sum_c \lambda_c e_c$. Thus

$$TV_r(M_f) = \sum_c |\lambda_c|^2 = \sum_c |\langle RT_r(M_f), e_c \rangle|^2.$$

- to get $\langle RT_r(M_f), e_c \rangle$: fill $\partial$-components of $M_f$; add link $L = \text{union cores colored by } c_i$. Thus

$$\langle RT_r(M_f), e_c \rangle = RT_r(M_\tilde{f}, (L, c)) = \text{Tr}(\rho_{r,c}(f)).$$
Theorem

(Detcherry-K., 2017) There exists a universal constant $C > 0$ such that for any compact orientable 3-manifold $M$ with empty or toroidal boundary we have

$$ITV(M) \leq C ||M||.$$ 

- **Remark.** if $ITV(M_f) > 0$, for some mapping class $f$, then $f$ satisfies AMU.
- **Computing $ITV$ is hard!** But we don’t always have to compute it to decide exponential growth!
- **Limits do not increase under Dehn filling.** (Detcherry-K) If $M$ is obtained by Dehn filling from $M'$ then

$$ITV(M) \leq ITV(M').$$

- **Example.** Adding components to a link preserves exponential growth of TV invariants of link complement.
An example: Knot $5_2$ and parents

- $K(p)$ is a 3-manifold obtained by $p$-surgery on $M$.
- $ITV(4_1(-5)) = Vol(4_1(-5)) \approx 0.9813688 > 0$ [Ohtsuki, 2017]
- Observe $5_2(5)$ is homeomorphic to $4_1(-5)$.

Dehn filling result implies $ITV(S^3 \setminus 5_2) \geq ITV(5_2(5)) = ITV(4_1(-5)) > 0$

But Dehn filling result also implies that for any link containing $5_2$ as a component we have **exponential growth**

$$ITV(S^3 \setminus L) \geq ITV(S^3 \setminus 5_2) > 0.$$
Manifolds with $ITV(M) = v_3 ||M|| > 0$

- (Detcherry-K.- Yang, 2016) Figure-8 knot and Borromean rings complements.
- (Ohtsuki, 2017) Infinite family of closed hyperbolic 3-manifolds: Manifolds obtained by integral integer fillings of $S^3$ along Figure-8 knot complement.
- (Belletti-Detcherry-K.- Yang, 2018) Infinite family of cusped hyperbolic 3-manifolds. These are the complements of Fundamental Shadow Links in connected sums of copies of $S^1 \times S^2$.
- (Constantino- D. Thurston, 2005) Every orientable 3-manifold $M$ with empty or toroidal boundary contains a complement of a FSL: $M$ contains links $L \subset M$ with $ITV(M \setminus L) > 0$. Doubles of link complements give closed 3-manifolds with $ITV > 0$.
- Kumar, Belletti, 2019: More octahedral link complements.
- For the AMU conjecture we need: mapping tori $M_f$ with $ITV(M_f) > 0$.
- There exist many fibered links in all (closed) 3-manifolds. Look at fibered links in closed manifolds.
(Vague) Question. For $n > 0$. How large is the class of $f \in \text{Mod}(\Sigma_{g,n})$ realized as monodromies of fibered links in closed 3-manifolds we know to have $\mathcal{ITV} > 0$? All of them?

Theorem

(Detcherry-K, 2019) For $g \gg 0$, there is $f \in \text{Mod}(\Sigma_{g,1})$ and a rank $\left\lfloor \frac{g}{2} \right\rfloor$ free abelian subgroup

$$H < \text{Mod}(\Sigma_{g,1}),$$

such that any class in the coset $fH$ is PA and satisfies the AMU conjecture.

• Similarly there is $g \in \text{Mod}(\Sigma_{g,1})$ there is a rank two free subgroup

$$F < \text{Mod}(\Sigma_{g,1}),$$

such that any class in the coset $gF$ is PA and satisfies the AMU.

Note. No examples of PA mappings for closed surfaces of $g > 2$ that satisfy the AMU are known.
Constructions of PAs: Links in $S^3$

- Start with $L \subset S^3$ be a link with $ITV(S^3 \setminus L) > 0$.
- \textit{(Stallings, 60's)} We can add a component $K$ so that $K \cup L$ is a fibered.
- In fact, $K \cup L$ will be a closed \textit{homogeneous braid} and fiber is a Seifert surface obtained from closed braid projection.

- Refine process so that $K \cup L$ is a hyperbolic and $ITV(S^3 \setminus (L \cup K)) > 0$.
- There are only finitely many f. m. link types in homogeneous closed braids of fixed genus! \textit{No problem:} Use Stallings twists....“wisely”.
Stallings twists

- $L$ fibered link with fiber $F$ and monodromy $f$.
- $c$ a non-trivial s.c.c on the fiber with $lk(c, c^+) = 0$, $c^+$ is the curve $c$ pushed along the positive normal of $F$. Need $c$ not parallel to $\partial F$ that bound a disc in $D \subset S^3$:

A Stallings twist of order $m$: A full twist of order $m$ along $D$.

- Gives fibered links $L_m$ with fiber $F$ and monodromy $f \circ \tau_c^m$, where $\tau_c =$ Dehn-twist on $F$ along $c$.

(Long-Morton, Fathi) If $f$ pseudo-Anosov, for all $m \gg 0$, $f \circ \tau_c^m$ is pseudo-Anosov.
Concrete examples: start with $\text{ITV}(S^3 \setminus 4_1) > 0$.

- $4_1$ = closure of the alternating braid $\sigma_2^{-1}\sigma_1\sigma_2^{-1}\sigma_1$.
- Fibered, hyperbolic, monodromies in $\text{Mod}(\Sigma_{7+2k,2})$, for $m = 4$, $l = 1$.
- Fiber supports $k$-Stallings twists.
Concrete examples Cont’n

- Take a monodromy \( f \) for \( L(4, 1, k) \). Dehn twists on Stallings curves generate Rank \( k \) abelian subgroup of \( H < \text{Mod}(\Sigma_{7+2k}, 2) \).
- Elements in coset \( fH \) satisfy AMU. For large powers of Dehn twists maps are all pseudo-Anosov.
- Dehn \((-5)\)-surgery on the figure-8 component with produces examples in \( N = 4_1(-5) \). This manifold is hyperbolic.
- The result of \( L(4, 1, k) \) in the closed manifold is a knot, \( K(k) \) that fibers with fiber the fiber of \( L(4, 1, k) \) with one boundary component capped-off. Monodromies are in \( \text{Mod}(\Sigma_{7+2k}, 1) \).
- The Stallings twists will survive the surgery along \( 4_1 \). We get abelian subgroup of rank \( k \) in \( H_k < \text{Mod}(\Sigma_{7+2k}, 1) \).
- We have \( ITV(N) > 0 \) by Ohtsuki’s result! Hence, all link complements in \( N \) have same property.
- For \( k \) take \( f \) a monodromy so that \( N \setminus K(k) = M_f \). Now all mappings of the form \( fH_k \) are realized in \( N = 4_1(-5) \).
(D-K) Let $M_f$ be the mapping torus of a periodic mapping class $f \in \text{Mod}(\Sigma)$ of order $N$. Then, for any odd integer $r \geq 3$, with $\gcd(r, N) = 1$, we have $TV_r(M_f) \in \mathbb{Z}$, for any choice of root of unity.

**Corollary.** For co-prime integers $p, q$ let $T_{p,q}$ denote the $(p, q)$-torus link. Then, for any odd $r$ co-prime with $p$ and $q$, we have $TV_r(S^3 \setminus T_{p,q}) \in \mathbb{Z}$.

In particular: $TV_r(M_f) \in \mathbb{Z}$, for infinitely many $r$.

If $lTV(M_f) > 0$ at some root of unity, then there can be at most finitely many values $r$ for which $TV_r(M_f) \in \mathbb{Z}$.

**Conjecture.** Suppose that $f \in \text{Mod}(\Sigma)$ contains a PA part. Then, there can be at most finitely many odd integers $r$ such that $TV_r(M_f) \in \mathbb{Z}$.


