Geometric structures of 3-manifolds and quantum invariants

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3-manifolds: $M$ = compact, orientable, with empty or tori boundary.

Links: Smooth embedding $K : \bigsqcup S^1 \to M$.

Link complements: $M \setminus n(K)$; toroidal boundary

Talk: Relations among three perspectives.

Combinatorial presentations
- knot diagrams, triangulations

3-manifold topology/geometry
- Geometric structures on $M$ and geometric invariants (e.g. hyperbolic volume)

Physics originated invariants
- Quantum invariants of knots/3-manifolds
For this talk, an $n$-dimensional *model geometry* is a simply connected $n$-manifold with a “homogeneous” Riemannian metric. In dimension 2, there are exactly three model geometries, up to scaling:

- **Spherical**: Curvature $= +1$
  
  $\text{Area}(T) = (\alpha + \beta + \gamma) - \pi$

- **Euclidean**: Curvature $= 0$
  
  $\alpha + \beta + \gamma = \pi$

- **Hyperbolic**: Curvature $= -1$
  
  $\text{Area}(T) = \pi - (\alpha + \beta + \gamma)$
Geometrization (a.k.a. Uniformization) in 2-d:

Every (closed, orientable) surface can be written as $S = X/G$, where $X$ is a model geometry and $G$ is a discrete group of isometries.

- $X = S^2$
- $X = \mathbb{E}^2$
- $X = \mathbb{H}^2$

- **Curvature**: $k = 1, 0, -1$
- Geometry vs topology: $k \cdot \text{Area}(S) = 2\pi \chi(S)$,
Geometrization in 3-d:

In dimension 3, there are eight model geometries:

\[ X = S^3, \mathbb{E}^3, \mathbb{H}^3, S^2 \times \mathbb{R}, H^2 \times \mathbb{R}, Sol, Nil, \overline{SL_2(\mathbb{R})} \]

Recall \( M = \) compact, oriented, \( \partial M = \) empty or tori

**Theorem (Thurston 1980 + Perelman 2003)**

*For every 3-manifold \( M \), there is a canonical way to cut \( M \) along spheres and tori into pieces \( M_1, \ldots, M_n \), such that each piece is \( M_i = X_i / G_i \), where \( G_i \) is a discrete group of isometries of the model geometry \( X_i \).*

- **Canonical**: “Unique” collection of spheres and tori.
- Poincare conjecture: \( S^3 \) is the only compact mode.
- **Hyperbolic** 3-manifolds form a rich and very interesting class.
- Cutting along tori, manifolds with toroidal boundary will naturally arise. Knot complements fit in this class.
Given $K$ remove an open tube around $K$ to obtain the **Knot complement**:  
Notation. $M_K = S^3 \setminus n(K)$.

Knot complements can be visualized! (Picture credit: J. Cantarella, UGA)
Theorem (Kneser, Milnor 60’s, Jaco-Shalen, Johanson 1970, Thurston 1980 + Perelman 2003)

$M = \text{oriented, compact, with empty or toroidal boundary.}$

1. **There is a unique collection of 2-spheres that decompose $M$**

$$M = M_1 \# M_2 \# \ldots \# M_p \# (\# S^2 \times S^1)^k,$$

where $M_1, \ldots, M_p$ are compact orientable irreducible 3-manifolds.

2. **For $M = \text{irreducible}$, there is a unique collection of disjointly embedded essential tori $T$ such that all the connected components of the manifold obtained by cutting $M$ along $T$, are either Seifert fibered manifolds or hyperbolic.**

- **Seifert fibered manifolds**: For this talk, think of it as

  $$S^1 \times \text{surface with boundary} + \text{union of solid tori}.$$ 

  Complete topological classification [Seifert, 60’]

- **Hyperbolic**: Interior admits complete, hyperbolic metric of finite volume.
Thee types of knots:

**Satellite Knots:** Complement contains embedded “essential” tori; There is a *canonical* (finite) collection of such tori.

**Torus knots:** Knot embeds on standard torus in $T$ in $S^3$ and is determined by its class in $H_1(T)$. Complement is SFM.

**Hyperbolic knots:** Rest of them.
Rigidity for hyperbolic 3-manifolds:

**Theorem (Mostow, Prasad 1973)**

Suppose $M$ is compact, oriented, and $\partial M$ is a possibly empty union of tori. If $M$ is hyperbolic (that is: $M \setminus \partial M = \mathbb{H}^3 / G$), then $G$ is unique up to conjugation by hyperbolic isometries. In other words, a hyperbolic metric on $M$ is essentially unique.

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$M =$hyperbolic 3-manifold:

- By rigidity, every geometric measurement of $M$ is a **topological invariant**
- Example: **Volume** of hyperbolic manifolds (**important for this talk**).
- In practice $M$ is represented by combinatorial data such as, **a triangulation**, or a **knot diagram** (in case of knot complements in $S^3$).

**Question:** How do we “see” geometry in the combinatorial descriptions of $M$? Can we calculate/estimate geometric invariants from combinatorial ones?
Recall $M$ uniquely decomposes along spheres and tori into disjoint unions of Seifert fibered spaces and hyperbolic pieces $M = S \cup H$,

**Gromov norm of $M$:** (Gromov, Thurston, 80’s)

\[ \nu_{tet} ||M|| = \text{Vol}(H), \text{ where} \]

- $\text{Vol}(H) = \text{sum of the hyperbolic volumes of components of } H$,
- $\nu_{tet} = \text{volume of the regular hyperbolic tetrahedron}$.

$||M||$ is additive under disjoint union and connected sums of manifolds.

If $M$ hyperbolic $\nu_{tet} ||M|| = \text{Vol}(M)$.

If $M$ Seifert fibered then $||M|| = 0$.

**Cutting along tori:** If $M'$ is obtained from $M$ by cutting along an embedded torus $T$ then

\[ ||M|| \leq ||M'||, \]

with equality if $T$ is incompressible.
Quantum invariants: Jones Polynomials

1980’s: Ideas originated in physics and in representation theory led to vast families invariants of knots and 3-manifolds. (Quantum invariants)

- **Jones Polynomials**: Discovered by V. Jones (1980’s); using braid group representations coming from the theory of certain operator algebras (sub factors).
- Can be calculated from any link diagram using, for example, Kaufman states:
- Two choices for each crossing, A or B resolution.

Choice of A or B resolutions for all crossings: state $\sigma$.
- Assign a “weight” to every state.
- JP calculated as a certain “state sum” over all states of any diagram.
For this talk we discuss:

- **The Colored Jones Polynomials**: Infinite sequence of Laurent polynomials \( \{J^K_n(t)\}_n \) encoding the Jones polynomial of \( K \) and those of the links \( K^s \) that are the parallels of \( K \).

- Formulae for \( J^K_n(t) \) come from representation theory of Lie Groups!: representation theory of \( SU(2) \) (decomposition of tensor products of representations). For example, they look like

\[
J^K_1(t) = 1, \quad J^K_2(t) = J_K(t) - \text{Original JP},
\]

\[
J^K_3(t) = J_{K^2}(t) - 1, \quad J^K_4(t) = J_{K^3}(t) - 2J_K(t), \quad \ldots
\]

- \( J^K_n(t) \) can be calculated from any knot diagram via processes such as *Skein Theory, State sums, R-matrices, Fusion rules*....
Question: How do the CJP relate to geometry/topology of knot complements?

Kashaev+ H. Murakami - J. Murakami (2000) proposed

Volume Conjecture. Suppose $K$ is a knot in $S^3$. Then

$$2\pi \cdot \lim_{n \to \infty} \frac{\log |J^n_K(e^{2\pi i/n})|}{n} = v_{tet}||S^3 \setminus n(K)||$$

- Wide Open!
- $4_1$ (by Ekholm), knots up to 7 crossings (by Ohtsuki)
- torus knots (by Kashaev and Tirkkonen); special satellites of torus knots (by Zheng).

Some difficulties:
- For families of links we have $J^n_K(e^{2\pi i/n}) = 0$, for all $n$.
- “State sum” for $J^n_K(e^{2\pi i/n})$ has oscillation/cancelation.
- No good behavior of $J^n_K(e^{2\pi i/n})$ with respect to geometric decompositions.
Coarse relations: Colored Jones polynomial

For a knot $K$, and $n = 1, 2, \ldots$, we write its $n$-colored Jones polynomial:

$$J^n_K(t) := \alpha_n t^{m_n} + \beta_n t^{m_n-1} + \cdots + \beta'_n t^{k_n+1} + \alpha'_n t^{k_n} \in \mathbb{Z}[t, t^{-1}]$$

- (Garoufalidis-Le, 04): Each of $\alpha'_n, \beta'_n \ldots$ satisfies a linear recursive relation in $n$, with integer coefficients.

$$(\text{e.g. } \alpha'_{n+1} + (-1)^n \alpha'_n = 0).$$

- Given a knot $K$ any diagram $D(K)$, there exist explicitly given functions $M(n, D)$ $m_n \leq M(n, D)$. For nice knots where $m_n = M(n, D)$ we have stable coefficients.

- (Dasbach-Lin, Armond) If $m_n = M(n, D)$, then

$$\beta'_K := |\beta'_n| = |\beta'_2|, \quad \text{and} \quad \beta_K := |\beta_n| = |\beta_2|,$$

for every $n > 1$.

- Stable coefficients control the volume of the link complement.
A Coarse Volume Conjecture

**Theorem (Dasbach-Lin, Futer-K.-Purcell, Giambrone, 05-’15’)**

There universal constants $A, B > 0$ such that for any hyperbolic link that is nice we have

$$A (\beta'_K + \beta_K) \leq \text{Vol}(S^3 \setminus K) < B (\beta'_K + \beta_K).$$

**Question.** Does there exist function $B(K)$ of the coefficients of the colored Jones polynomials of a knot $K$, that is easy to calculate from a “nice” knot diagram such that for hyperbolic knots, $B(K)$ is coarsely related to hyperbolic volume $\text{Vol}(S^3 \setminus K)$?

Are there constants $C_1 \geq 1$ and $C_2 \geq 0$ such that

$$C_1^{-1} B(K) - C_2 \leq \text{Vol}(S^3 \setminus K) \leq C_1 B(K) + C_2,$$

for all hyperbolic $K$?

- C. Lee, Proved CVC for classes of links that don’t satisfy the standard “nice” hypothesis (2017)
Turaev-Viro invariants: A Volume Conjecture for all 3-manifolds

- (Turaev-Viro, 1990): For odd integer $r$ and $q = e^{\frac{2\pi i}{r}}$

\[
TV_r(M) := TV_r(M, q),
\]

a real valued invariant of compact oriented 3-manifolds $M$

- $TV_r(M, q)$ are combinatorially defined invariants and can be computed from triangulations of $M$ by a state sum formula. Sums involve quantum 6j-symbols. Terms are highly "oscillating" and there is term cancellation. Combinatorics have roots in representation theory of quantum groups.

- For experts: We work with the $SO(3)$ quantum group.

- (Q. Chen- T. Yang, 2015): compelling experimental evidence supporting

**Volume Conjecture**: For $M$ compact, orientable

\[
\lim_{r \to \infty} \frac{2\pi}{r} \log(TV_r(M, e^{\frac{2\pi i}{r}})) = v_{tet} ||M||,
\]

where $r$ runs over odd integers.
What we know:

The Conjecture is verified for the following.

- **(Detcherry-K.-Yang, 2016)** (First examples) of hyperbolic links in $S^3$: The complement of $4_1$ knot and of the Borromean rings.

- **(Ohtsuki, 2017)** Infinite family of closed hyperbolic 3-manifolds: Manifolds obtained by Dehn filling along the $4_1$ knot complement.

- **(Belletti-Detcherry-K.-Yang, 2018)** Infinite family of cusped hyperbolic 3-manifolds that are universal: They produce all $M$ by Dehn filling!

- **(Kumar, 2019)** Infinite families of hyperbolic links in $S^3$.

- **(Detcherry-K, 2017)** All links zero Gromov norm links in $S^3$ and in connected sums of copies of $S^1 \times S^2$.

- **(Detcherry, Detcherry-K, 2017)** Several families of 3-manifolds with non-zero Gromov, with or without boundary.

- For links in $S^3$ Turaev-Viro invariants relate to colored Jones polynomials (Next)
Links complements in $S^3$:

For link complements $TV_r(S^3 \setminus K, e^{\frac{2\pi i}{r}})$ are obtained from (multi)-colored Jones link polynomial. For simplicity, we state only for knots here.

**Theorem (Detcherry-K., 2017)**

*For $K \subset S^3$ and $r = 2m + 1$ there is a constant $\eta_r$ independent of $K$ so that*

$$TV_r(S^3 \setminus K, e^{\frac{2\pi i}{r}}) = \eta_r^2 \sum_{n=1}^{m} |J^n_K(e^{\frac{4\pi i}{r}})|^2.$$  

- Theorem implies that the invariants $TV_r((S^3 \setminus K)$ are not identically zero for any link in $S^3$!
- The quantity $\log(TV_r((S^3 \setminus K))$ is always well defined.
- **Remark.** The values of CJP in Theorem are different that these in “original” volume conjecture.
- Not known how the two conjectures are related for knots in $S^3$. 

Building blocks of TV invariants relate to volumes

- Color the edges of a triangulation with certain “quantum” data

Colored tetrahedra get “6j-symbol” $Q := Q(a_1, a_2, a_3, a_4, a_5, a_6)$ = function of the $a_i$ and $r$. $TV_r(M)$ is a weighted sum over all tetrahedra of triangulation (State sum).

**BDKY** Asymptotics of $Q$ relate to volumes of geometric polyhedra:

$$\frac{2\pi}{r} \log (Q) \leq v_{oct} + O\left(\frac{\log r}{r}\right).$$

- Proved VC for “octahedral” 3-manifolds, where $TV_r$ have “nice” forms. In general, hard to control term cancellation in state sum.
LTV(M) = \limsup_{r \to \infty} \frac{2\pi}{r} \log(\text{TV}_r(M)), \text{ and } ITV(M) = \liminf_{r \to \infty} \frac{2\pi}{r} \log(\text{TV}_r(M))

**Conjecture:** There exists universal constants \( B, C, E > 0 \) such that for any compact orientable 3-manifold \( M \) with empty or toroidal boundary we have

\[ B \|M\| - E \leq ITV(M) \leq LTV(M) \leq C \|M\|. \]

In particular, \( ITV(M) > 0 \) iff \( \|M\| > 0 \).

- Half is done:

**Theorem (Detcherry-K., 2017)**

*There exists a universal constant \( C > 0 \) such that for any compact orientable 3-manifold \( M \) with empty or toroidal boundary we have*

\[ LTV(M) \leq C\|M\|. \]
Why are TV invariants “better”? 

- TV invariants are defined for all compact, oriented 3-manifolds.
- TV invariants are defined on triangulations of 3-manifolds: For hyperbolic 3-manifolds the (hyperbolic) volume can be estimated/calculated from appropriate triangulations.
- TV invariants are part of a Topological Quantum Field Theory (TQFT) and they can be computed by cutting and gluing 3-manifolds along surfaces. The TQFT behaves particularly well when cutting along spheres and tori. In particular it behaves well with respect to prime and JSJ decompositions.
- For experts: The TQFT is the $SO(3)$- Reshetikhin-Turaev and Witten TQFT as constructed by Blanchet, Habegger, Masbaum and Vogel (1995)
Outline of last theorem:

1. Study the large-r asymptotic behavior of the quantum 6j-symbols, and using the state sum formulae for the invariants $TV_r$, to prove give linear upper bound of $LTV(M)$:

   $$ITV(M) \leq LTV(M) < v_8(\# \text{ of tetrahedra needed to triangulate } M).$$

2. Use a theorem of Thurston to show that there is $C > 0$ such that for any hyperbolic 3-manifold $M$

   $$LTV(M) \leq C\|M\|.$$

3. Use TQFT properties to show that if $M$ is a Seifert fibered manifold, then

   $$LTV(M) = \|M\| = 0.$$

4. Show that if $M$ contains an embedded tori $T$ and $M'$ is obtained from $M$ by cutting along $T$ then

   $$LTV(M) \leq LTV(M').$$

5. $LTV(M)$ is (sub)additive under connected sums.

6. Use parallel behavior of $LTV(M)$ and $\|M\|$ under geometric decomposition of 3-manifolds.
Exponential growth results:

- The Invariants $TV_r(M)$ grow exponentially in $r$, iff
  \[ ITV(M) := \liminf_{r \to \infty} \frac{2\pi}{r} \log(TV_r(M)) > 0. \]

- **AMU Conjecture relation**: The statement
  \[ ITV(M) > 0 \text{ iff } ||M|| > 0, \]
  implies a conjecture of Andersen-Masbaum-Ueno on the geometric content of the *quantum representations* of surface mapping class groups.

- Detcherry-K. showed that for $M, M'$ compact orientable with empty or toroidal boundary, and such that $M''$ is obtained by drilling a link from $M$ we have $ITV(M') > ITV(M)$.

- This led to many constructions of manifolds with $ITV(M) > 0$. Used these constructions to build substantial evidence for AMU conjecture.