Geometric structures of 3-manifolds and quantum invariants

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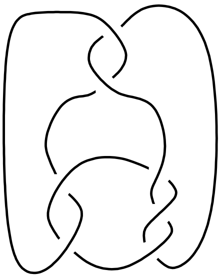
Settings and talk theme

3-manifolds: M=compact, orientable, with empty or tori boundary.

Links: Smooth embedding $K : \coprod S^1 \to M$.

Link complements: $\overline{M \setminus n(K)}$; toroidal boundary



Talk: Relations among three perspectives.

Combinatorial presentations

- knot diagrams, triangulations

3-manifold topology/geometry

- Geometric structures on M and geometric invariants (e.g. hyperbolic volume)

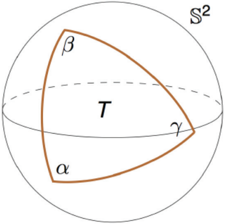
Physics originated invariants

- Quantum invariants of knots/3-manifolds

Warm up: 2-d Model Geometries:

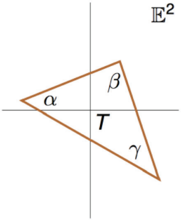
For this talk, an n-dimensional *model geometry* is a simply connected n-manifold with a "homogeneous" Riemannian metric.
In dimension 2, there are exactly three model geometries, up to scaling:

Spherical



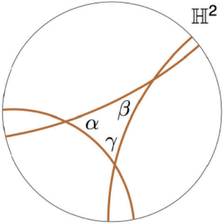
curvature $= +1$
$\text{Area}(T) = (\alpha+\beta+\gamma) - \pi$

Eucledian



curvature $= 0$
$\alpha+\beta+\gamma = \pi$

Hyperbolic

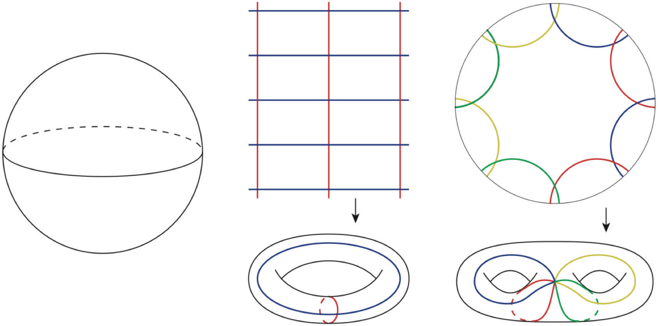


curvature $= -1$
$\text{Area}(T) = \pi - (\alpha+\beta+\gamma)$

Geometrization (a.k.a. Uniformization) in 2-d:

Every (closed, orientable) surface can be written as $S = X/G$, where X is a model geometry and G is a discrete group of isometries.

$X = \mathbf{S}^2$ $\qquad$ $X = \mathbb{E}^2$ $\qquad$ $X = \mathbb{H}^2$



- *Curvature:* $k = 1, 0, -1$
- Geometry vs topology: $k \cdot Area(S) = 2\pi\chi(S)$,

Geometrization in 3-d:

In dimension 3, there are eight model geometries:

$$X = \mathbf{S}^3 \ \ \mathbb{E}^3 \quad \mathbb{H}^3 \ , \ \mathbf{S}^2 \times \mathbb{R}, \ \ \mathbb{H}^2 \times \mathbb{R}, \ \ Sol, \ \ Nil, \ \ \widetilde{SL_2(\mathbb{R})}$$

Recall M= compact, oriented, ∂M=empty or tori

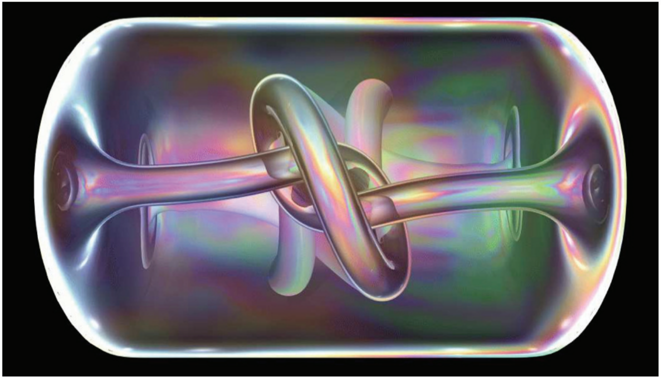
Theorem (Thurston 1980 + Perelman 2003)

For every 3-manifold M, there is a canonical way to cut M along spheres and tori into pieces $M_1, \ldots, M_n$, such that each piece is $M_i = X_i / G_i$, where G_i is a discrete group of isometries of the model geometry X_i .

- *Canonical* : "Unique" collection of spheres and tori.
- Poincare conjecture: $\mathbf{S}^3$ is the only compact mode.
- Hyperbolic 3-manifolds form a rich and very interesting class.
- Cutting along tori, manifolds with toroidal boundary will naturally arise. Knot complements fit in this class.

Knots complements; nice 3-manifolds with boundary:

Given K remove an open tube around K to obtain the *Knot complement:*
Notation. $M_K = S^3 \setminus n(K)$.



Knot complements can be visualized! (Picture credit: J. Cantarella, UGA)

Geometric decomposition picture for this talk:

Theorem (Knesser, Milnor 60's, Jaco-Shalen, Johanson 1970, Thurston 1980 + Perelman 2003)

M=oriented, compact, with empty or toroidal boundary.

1. *There is a unique collection of 2-spheres that decompose M*

$$M = M_1 \# M_2 \# \dots \# M_p \# (\# S^2 \times S^1)^k,$$

   where $M_1, \dots, M_p$ are compact orientable irreducible 3-manifolds.

2. *For M=irreducible, there is a unique collection of disjointly embedded essential tori $\mathcal{T}$ such that all the connected components of the manifold obtained by cutting M along $\mathcal{T}$, are either Seifert fibered manifolds or hyperbolic.*

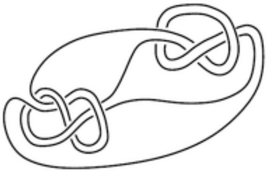
- *Seifert fibered manifolds:* For this talk, think of it as

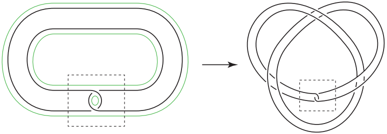
$$S^1 \times \text{surface with boundary} + \text{union of solid tori.}$$

  Complete topological classification [Seifert, 60']
- *Hyperbolic:* Interior admits complete, hyperbolic metric of finite volume.

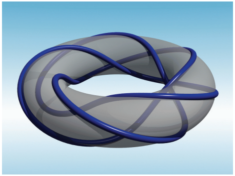
Thee types of knots:

Satellite Knots: Complement contains embedded “essential” tori; There is a *canonical* (finite) collection of such tori.





Torus knots: Knot embeds on standard torus in T in S^3 and is determined by its class in $H_1(T)$. Complement is SFM.



Hyperbolic knots: Rest of them.

Rigidity for hyperbolic 3-manifolds:

Theorem (Mostow, Prasad 1973)

Suppose M is compact, oriented, and ∂M is a possibly empty union of tori. If M is hyperbolic (that is: $M \setminus \partial M = \mathbb{H}^3/G$), then G is unique up to conjugation by hyperbolic isometries. In other words, a hyperbolic metric on M is essentially unique.

M =hyperbolic 3-manifold:

- By rigidity, every geometric measurement of M is a *topological invariant*
- Example: *Volume* of hyperbolic manifolds (important for this talk).
- In practice M is represented by combinatorial data such as, *a triangulation*, or a *knot diagram* (in case of knot complements in S^3).

Question: How do we "see" geometry in the combinatorial descriptions of M? Can we calculate/estimate geometric invariants from combinatorial ones?

Gromov Norm/Volume highlights:

- Recall M uniquely decomposes along spheres and tori into disjoint unions of Seifert fibered spaces and hyperbolic pieces $M = S \cup H$,
- *Gromov norm of M:* (Gromov, Thurston, 80's)

$$v_{\text{tet}}||M|| = \text{Vol}\,(H), \quad \text{where}$$

  - $\text{Vol}\,(H)$ = sum of the hyperbolic volumes of components of H,
  - v_{tet} = volume of the regular hyperbolic tetrahedron.
- $||M||$ is additive under disjoint union and connected sums of manifolds.
- If M hyperbolic $v_{\text{tet}}||M|| = \text{Vol}\,(M)$.
- If M Seifert fibered then $||M|| = 0$
- *Cutting along tori*: If M' is obtained from M by cutting along an embedded torus T then

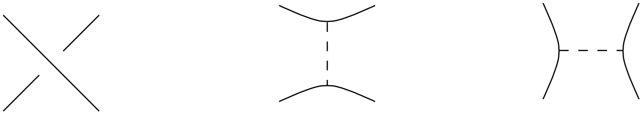
$$||M|| \leqslant ||M'||,$$

  with equality if T is incompressible.

Quantum invariants: Jones Polynomials

1980's: Ideas originated in physics and in representation theory led to vast families invariants of knots and 3-manifolds. (*Quantum invariants*)

- *Jones Polynomials*: Discovered by V. Jones (1980's); using braid group representations coming from the theory of certain operator algebras (sub factors).
- Can be calculated from any link diagram using, for example, Kaufman states:
- Two choices for each crossing, A or B resolution.



- Choice of A or B resolutions for all crossings: *state* σ.
- Assign a "*weight*" to every state.
- JP calculated as a certain "*state sum*" over all states of any diagram.

Quantum invariants: Colored Jones Polynomials

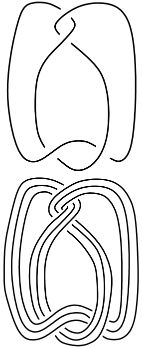
For this talk we discuss:

- The *Colored Jones Polynomials*: Infinite sequence of Laurent polynomials $\{J_K^n(t)\}_n$ encoding the *Jones polynomial* of K and these of the links K^s that are the parallels of K.
- Formulae for $J_K^n(t)$ come from representation theory of Lie Groups!: representation theory of $SU(2)$ (decomposition of tensor products of representations). For example,
  They look like

$$J_K^1(t) = 1, \qquad J_K^2(t) = J_K(t) \text{- } \textit{Original JP,}$$

$$J_K^3(t) = J_{K^2}(t) - 1, \quad J_K^4(t) = J_{K^3}(t) - 2J_K(t), \ \ldots$$

- $J_K^n(t)$ can be calculated from any knot diagram via processes such as *Skein Theory*, *State sums*, *R-matrices*, *Fusion rules*....



The CJP predicts Volume?

Question: How do the *CJP* relate to geometry/topology of knot complements?

Kashaev+ H. Murakami - J. Murakami (2000) proposed

Volume Conjecture. Suppose K is a knot in S^3. Then

$$2\pi \cdot \lim_{n\to\infty} \frac{\log |J_K^n(e^{2\pi i/n})|}{n} = v_{\mathrm{tet}}||S^3 \setminus n(K)||$$

- Wide Open!
- 4_1 (by Ekholm), knots up to 7 crossings (by Ohtsuki)
- torus knots (by Kashaev and Tirkkonen); special satellites of torus knots (by Zheng).

.

Some difficulties:

- For families of links we have $J_K^n(e^{2\pi i/n}) = 0$, for all n.
- "State sum" for $J_K^n(e^{2\pi i/n})$ has oscillation/cancelation.
- No good behavior of $J_K^n(e^{2\pi i/n})$ with respect to geometric decompositions.

Coarse relations: Colored Jones polynomial

For a knot K, and $n = 1, 2, \ldots$, we write its *n-colored Jones polynomial*:

$$J_K^n(t) := \alpha_n t^{m_n} + \beta_n t^{m_n - 1} + \cdots + \beta'_n t^{k_n + 1} + \alpha'_n t^{k_n} \in \mathbb{Z}[t, t^{-1}]$$

- (Garoufalidis-Le, 04): Each of $\alpha'_n, \beta'_n \ldots$ satisfies a *linear recursive relation* in n, with integer coefficients .

  (e. g. $\alpha'_{n+1} + (-1)^n \alpha'_n = 0$).

- Given a knot K any diagram $D(K)$, there exist explicitly given functions $M(n, D)$ $m_n \leq M(n, D)$. For nice knots where $m_n = M(n, D)$ we have *stable coefficients*
- (Dasbach-Lin, Armond) If $m_n = M(n, D)$, then

  $$\beta'_K := |\beta'_n| = |\beta'_2|, \quad \text{and} \quad \beta_K := |\beta_n| = |\beta_2|,$$

  for every $n > 1$.
- Stable coefficients control the volume of the link complement.

A Coarse Volume Conjecture

Theorem (Dasbach-Lin, Futer-K.-Purcell, Giambrone, 05-’15’)

There universal constants $A, B > 0$ such that for any hyperbolic link that is nice we have

$$A\,(\beta'_K + \beta_K) \;\leq\; Vol\,(S^3 \smallsetminus K) \;<\; B\,(\beta'_K + \beta_K).$$

Question. Does there exist function $B(K)$ of the coefficients of the colored Jones polynomials of a knot K, that is easy to calculate from a “nice” knot diagram such that for hyperbolic knots, $B(K)$ is coarsely related to hyperbolic volume $\mathrm{Vol}\,(S^3 \smallsetminus K)$?
Are there constants $C_1 \geq 1$ and $C_2 \geq 0$ such that

$$C_1^{-1} B(K) - C_2 \;\leq\; \mathrm{Vol}\,(S^3 \smallsetminus K) \;\leq\; C_1 B(K) + C_2,$$

for all hyperbolic K?

- C. Lee, Proved CVC for classes of links that don’t satisfy the standard “nice” hypothesis (2017)

Turaev-Viro invariants: A Volume Conjecture for all 3-manifolds

- (Turaev-Viro, 1990): For odd integer r and $q = e^{\frac{2\pi i}{r}}$

$$TV_r(M) := TV_r(M, q),$$

  a real valued invariant of compact oriented 3-manifolds M
- $TV_r(M, q)$ are combinatorially defined invariants and can be computed from triangulations of M by a *state sum* formula. Sums involve *quantum 6j-sympols*. Terms are highly "oscillating" and there is term canellation. Combinatorics have roots in representation theory of quantum groups.
- For experts: We work with the $SO(3)$ quantum group.
- (Q. Chen- T. Yang, 2015): compelling experimental evidence supporting
- **Volume Conjecture :** For M compact, orientable

$$\lim_{r\to\infty} \frac{2\pi}{r} \log(TV_r(M, e^{\frac{2\pi i}{r}})) = v_{\text{tet}}||M||,$$

  where r runs over odd integers.

What we know:

The Conjecture is verified for the following.

- *(Detcherrry-K.-Yang, 2016)* (First examples) of hyperbolic links in S^3: The complement of 4_1 knot and of the Borromean rings.
- *(Ohtsuki, 2017)* Infinite family of closed hyperbolic 3-manifolds: Manifolds obtained by *Dehn filling* along the 4_1 knot complement.
- *(Belletti-Detcherry-K- Yang, 2018)* Infinite family of cusped hyperbolic 3-manifolds that are universal: They produce all M by Dehn filling!
- *(Kumar, 2019)* Infinite families of hyperbolic links in S^3.
- *(Detcherry-K, 2017)* All links zero Gromov norm links in S^3 and in connected sums of copies of $S^1 \times S^2$.
- *(Detcherry, Detcherry-K, 2017)* Several families of 3-manifolds with non-zero Gromov, with or with or without boundary.

- For links in S^3 Turaev-Viro invariants relate to colored Jones polynomials (Next)

Links complements in S^3:

For link complements $TV_r(S^3 \setminus K, e^{\frac{2\pi i}{r}})$ are obtained from (multi)-colored Jones link polynomial. For simplicity, we state only for knots here.

Theorem (Detcherry-K., 2017)

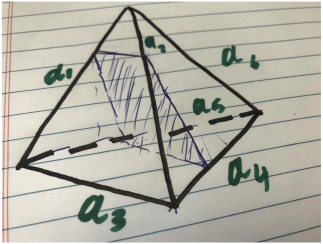
For $K \subset S^3$ and $r = 2m + 1$ there is a constant η_r independent of K so that

$$TV_r(S^3 \setminus K, e^{\frac{2\pi i}{r}}) = \eta_r^2 \sum_{n=1}^{m} |J_K^n(e^{\frac{4\pi i}{r}})|^2.$$

- Theorem implies that the invariants $TV_r((S^3 \setminus K)$ are not identically zero for any link in S^3!
- The quantity $\log(TV_r((S^3 \setminus K))$ is always well defined.
- **Remark.** The values of CJP in Theorem are different that these in "original" volume conjecture.
- Not known how the two conjectures are related for knots in S^3.

Building blocks of TV invariants relate to volumes

- Color the edges of a triangulation with certain “quantum ” data



- Colored tetrahedra get “6j-symbol” $\mathbf{Q} := Q(a_1, a_2, a_3, a_4, a_5, a_6)$= function of the a_i and r. $TV_r(M)$ is a weighted sum over all tetrahedra of triangulation (*State sum*).
- *(BDKY)* Asympotics of **Q** relate to volumes of geometric polyhedra:

$$\frac{2\pi}{r}\log(\mathbf{Q}) \leqslant v_{\text{oct}} + O(\frac{\log r}{r}).$$

- Proved VC for “octahedral” 3-manifolds, where TV_r have “nice” forms. In general, hard to control term cancellation in state sum.

A more Robust statement?:

$$LTV(M) = \limsup_{r\to\infty} \frac{2\pi}{r} \log(TV_r(M)), \quad \text{and} \quad lTV(M) = \liminf_{r\to\infty} \frac{2\pi}{r} \log(TV_r(M))$$

Conjecture: There exists universal constants $B, C, E > 0$ such that for any compact orientable 3-manifold M with empty or toroidal boundary we have

$$B\,||M|| - E \leqslant lTV(M) \leqslant LTV(M) \leqslant C\,||M||.$$

In particular, $lTV(M) > 0$ iff $||M > 0$.

- Half is done:

Theorem (Detcherry-K., 2017)

There exists a universal constant $C > 0$ such that for any compact orientable 3-manifold M with empty or toroidal boundary we have

$$LTV(M) \leqslant C||M||,$$

Why are TV invariants “better”?

- TV invariants are defined for all compact, oriented 3-manifolds.
- TV invariants are defined on triangulations of 3-manifolds: For hyperbolic 3-manifolds the (hyperbolic) volume can be estimated/calculated from appropriate triangulations.
- TV invariants are part of a Topological Quantum Field Theory (TQFT) and they can be computed by cutting and gluing 3-manifolds along surfaces. The TQFT behaves particularly well when cutting along spheres and tori. In particular it behaves well with respect to prime and JSJ decompositions.
- For experts: The TQFT is the $SO(3)$- Reshetikhin-Turaev and Witten TQFT as constructed by Blanchet, Habegger, Masbaum and Vogel (1995)

Outline of last theorem:

1. Study the large-r asymptotic behavior of the quantum $6j$-symbols, and using the state sum formulae for the invariants TV_r, to prove give linear upper bound of $LTV(M)$:
$$lTV(M) \leq LTV(M) < v_8(\# \text{ of tetrahedra needed to triangulate } M).$$
2. Use a theorem of Thurston to show that there is $C > 0$ such that for any hyperbolic 3-manifold M
$$LTV(M) \leq C||M||.$$
3. Use TQFT properties to show that if M is a Seifert fibered manifold, then
$$LTV(M) = ||M|| = 0.$$
4. Show that If M contains an embedded tori T and M' is obtained from M by cutting along T then
$$LTV(M) \leqslant LTV(M').$$
5. $LTV(M)$ is (sub)additive under connected sums.
6. Use parallel behavior of $LTV(M)$ and $||M||$ under geometric decomposition of 3-manifolds.

Exponential growth results:

- The Invariants $TV_r(M)$ grow exponetiallly in r, iff

$$ITV(M) := \liminf_{r\to\infty} \frac{2\pi}{r} \log(TV_r(M)) > 0.$$

- *AMU Conjecture relation*: The statement

$$ITV(M) > 0 \ \text{ iff } \ ||M|| > 0,$$

  implies a conjecture of Andersen-Masbaum-Ueno on the geometric content of the *quantum representations* of surface mapping class groups.
- *Detcherry-K*. showed that for M, M' compact orientable with empty or toroidal boundary, and such that M'' is obtained by drilling a link from M we have $ITV(M') > ITV(M)$.
- This led to many constructions of manifolds with $ITV(M) > 0$. Used these constructions to build substantial evidence for AMU conjecture.