# Crosscap numbers and relations to other knot invariants

joint w/ Christine Lee (UT, Austin)

Invariants in Low Dimensional Geometry, Gazi University, Ankara, Turkey, August 10-14, 2015

# Setting and outline of talk

$C(K)$= *crosscap number* (a. k. a. non-orientable genus) of a knot $K$= smallest genus over all non-orientable surfaces spanned by $K$.

**Plan:**

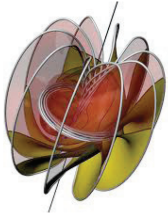
- Review what is known- Compare with the (oriented) *genus*.
- There is an algorithm to compute knot *genus*.
- No algorithm is known to compute crosscap number. Indicate progress/difficulties.
- Discuss calculations for knots up to 12 crossings.

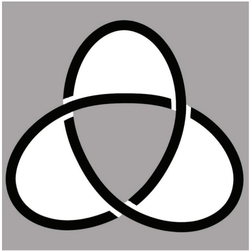
**Restrict to alternating knots:**

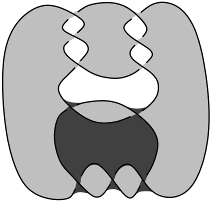
- Classical genus results:
- Genus is calculated from alternating diagrams (*Seifert's algorithm*).
- Genus is calculated from the Alexander polynomial.
- Discuss non-orientable counterparts:
- Crosscap number is calculated from alternating diagrams (*state surfaces*).
- Crosscap number is estimated/calculated from the Jones polynomial.

# Definitions etc

- Knots span surfaces: both orientable and non-orientable.







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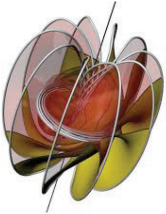
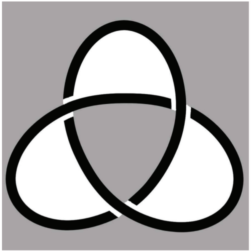
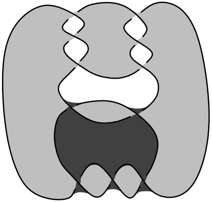

- **Definition.** $S$ non-orientable surface spanned by a $k$-component link $K$. *crosscap number of S*

$$C(S) = 2 - \chi(S) - k.$$

- The *crosscap number of a link K* is the minimum crosscap number over all non-orientable surfaces spanned by $K$.
- Crosscap numbers first studied by B. E. Clark– made several observations (1978).

# Facts, bounds and algorithms:

- Convention: $C(\mathrm{Unknot}) = 0$.
- $g(K)$= *genus* of $K$. Then, $C(K) \leq 2g(K) + 1$.
- $C(K) = 1$ iff $K$ is a $(2, p)$ torus knot or a $(2, p)$ cable.
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- If $K$ alternating, then $C(K) = 1$ iff $K$ is a $(2, p)$ torus knot.
- (H. Murakami- Yasuhara) If $c(K)$=crossing number of $K$, then

$$C(K) \leq \left\lfloor \frac{c(K)}{2} \right\rfloor.$$

  and the bound is sharp.
- Crosscap numbers are known for families: (*e.g. 2-bridge knots, pretzel knots* )– Bessho, Hirasawa, Teragaito, Ichiharra, Mizushima.....

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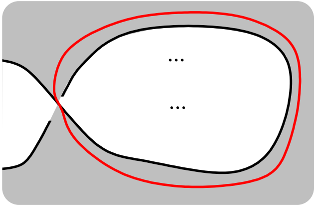
- $C(K)$ not known for a lot of knots up to 12 crossings ($g(K)$ is known).
- There is no known algorithm to calculate $C(K)$ (there is for $g(K)$)
- **Issue:** A surface realizing $C(K)$ need not be $\partial$- incompressible (for $g(K)$ is).

# Facts, bounds and algorithms con't:

- **Pathology:** In fact, all surfaces realizing $C(K)$ may be obtained from oriented ones by adding a "*trivial crosscap*".
- This creates a $\partial$-compression disk in $M_K = S^3 \setminus K$. ( Red line below).




- **Pathology Example:** The knot $K = 7_4$: We have $g(K) = 1$. Murasugi-Yasuhara calculated $C(K) = 3 = 2g(K) + 1$.
- All surfaces for $7_4$, realizing $C(K) = 3$, are obtained from a genus 1 Seifert surface by adding a trivial crosscap.

# Facts, bounds and algorithms: Normal surface theory

- Oriented genus $g(K)$:
- Algorithm and computational complexity (Hass-Lagarias-Pippenger -1999).
- An important point noted by H-L-P is that “normalization” process gives:

**Theorem**

*Let $\mathcal{T}$ be a triangulation of a knot complement $M_K$. Then there is a fundamental, normal, orientable spanning surface of genus $g(K)$.*

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- Basic steps of **Algorithm:** Given $K$,

1. Obtain a "suitable" triangulation $\mathcal{T}$ of $M_K$.
2. Enumerate all fundamental normal surfaces in $\mathcal{T}$.
3. Identify the spanning oriented ones among surfaces in step 2.
4. Identify the smallest genus surface that appears in step 3.

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- B. Burton and Burton-Ozlen (2012) made progress. First they note the following:

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*Let $\mathcal{T}$ be a triangulation of a knot complement $M_K$. Then, either*

- *there is a fundamental, normal, non-orientable spanning surface with $C(S) = C(K)$; or*
- $C(K) \in \{2g(K),\ 2g(K)+1\}$.

- They obtain an **Algorithm:** Given $K$

1. Obtain a single value that is $C(K)$; or
2. Narrow the values for $C(K)$ to two possible ones.

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- Burton-Olsen used integer programing techniques to get upper bounds for $C(K)$ calculated several previously unknown values.

# Low crossing data: up to 12 crossings

Info copied from KnotInfo ( Cha- Livingston).

- $C(7_4) = 3$ (Murakami-Yasuhara)
- 2-bridge cases; $C(K)$ determined by Teragaito and Hirasawa
- Typically KnotInfo gives upper bounds that were obtained by finding non-orientable surfaces *state surfaces*.
- Burton-Ozlen: Used normal surfaces and integer programming to find non-orientable surfaces of small crosscap number. They got **new bounds for 778** of the knots in the table.
- (2012) Adams and Kindred: Method that determines the crosscap number of an alternating knot. They got previously unknown values for:

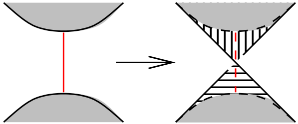
$$\mathbf{8_{10,15,16,17,18}} \ \textbf{and} \ \mathbf{9_{16,22,24,25,28,29,30,32,33,34,36,37,38,39,40,41}}.$$

- (2014) K.- Lee: Bounds in terms of the Jones polynomial. Improved the bounds for **almost half** of the table knots, and precisely determined the number for **283** of the 12-crossing knots.

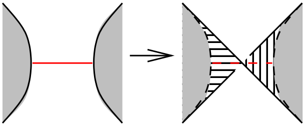
# State surfaces

For a Kauffman state $\sigma$ of a link diagram, form a *state surface* $S_\sigma$:

- Each state circle bounds a disk in $S_\sigma$ (nested disks drawn on top).
- At each edge (for each crossing) attach twisted band.



*A*–resolution



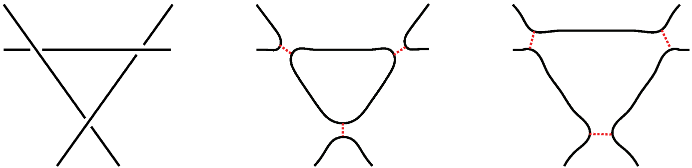
*B*–resolution

- **Special Cases:** Seifert state, checkerboard states of alternating knots.

# Alternating knots

- [Murasugi (1960)]. The Seifert state applied to a reduced alternating diagram $D(K)$ gives a minimum genus surface.
- [Adams-Kindred (2013)]. Gave an algorithm to calculate $C(K)$ of alternating knots, from state surfaces.
- **The Algorithm:** $D = D(K)$ alternating knot diagram. Think of $D$ as a 4=valent graph.
- If $D$ has regions that 1-gons or 2-gons resolve the corresponding crossings so that the region becomes a state circle.
- Suppose $D$ has no 1-gons or 2-gons; then it has triangles.
- Pick a triangle region on $D$. Create two branches as shown below:

# Algorithm con't:



- Repeat until each branch reaches a projection without crossings.
- Choose the resulting surfaces $S$ that have maximal Euler characteristic.

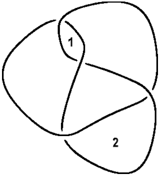
## Theorem (Adams-Kindred, 2013)

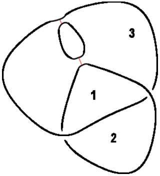
*After applying the algorithm to an alternating diagram of $k$-component link $K$:*

1. *If there is $S$ as above that is non-orientable then $C(K) = 2 - \chi(S) - k$.*
2. *If all surfaces produced by the algorithm are orientable, $S$ is a minimal genus Seifert surface of $K$ and $C(K) = 2g(K) + 1$.*

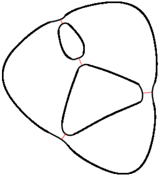
# An example: Fig-8:

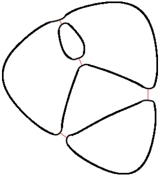
- Bigons labeled 1 and 2 and diagram resulting from applying the first step of the Algorithm. New bigon regions labeled 1, 2, and 3.




- State surfaces from different choices of bigon regions.





- Left one gives a non-orientable surface of maximal Euler characteristic $\chi(S) = -1$. Hence, $C(K) = C(S) = 2$.

# Knot polynomial bounds:

- **Genus:** (Crowell, Murasugi, 1960) For $K$ alternating, $g(K)$ is half the degree span of the *Alexander polynomial of K*.

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- Let
$$J_K(t) = \alpha_K t^n + \beta_K t^{n-1} + \ldots + \beta'_K t^{s+1} + \alpha'_K t^s$$
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denote *the Jones polynomial of $K$.*
- Set $T_K := |\beta_K| + |\beta'_K|$.

### Theorem (K.-Lee, 2014)

*Let $K$ be a non-split, prime, non-torus, alternating link with $k$-components and with crosscap number $C(K)$. We have*

$$\left\lceil \frac{T_K}{3} \right\rceil + 2 - k \ \leq \ C(K) \ \leq \ T_K + 2 - k,$$

*Furthermore, both bounds are sharp.*

# Sharpness:

- **Knots:** For $K$ =alternating, non-torus knot we have

$$\left\lceil \frac{T_K}{3} \right\rceil + 1 \;\leq\; C(K) \;\leq\; \min\left\{ T_K + 1,\; \left\lfloor \frac{s_K}{2} \right\rfloor \right\}$$

where $T_K$ as above and $s_K$ =degree span of $J_K(t)$. Bounds are sharp.

- **Some examples:** Knotinfo $C(K)$ upper bound agrees with above lower bound. $T_K$ value also from Knotinfo. We determine that $C(K) = 3$.

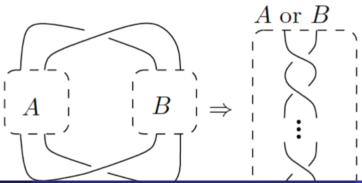
| $K$ | $T_K$ | $K$ | $T_K$ | $K$ | $T_K$ | $K$ | $T_K$ |
|---|---|---|---|---|---|---|---|
| $10_{85}$ | 6 | $10_{93}$ | 6 | $10_{100}$ | 6 | $11a_{74}$ | 5 |
| $11a_{97}$ | 5 | $11a_{223}$ | 5 | $11a_{250}$ | 5 | $11a_{259}$ | 5 |
| $11a_{263}$ | 4 | $11a_{279}$ | 6 | $11a_{293}$ | 6 | $11a_{313}$ | 6 |
| $11a_{323}$ | 6 | $11a_{330}$ | 6 | $11a_{338}$ | 4 | $11a_{346}$ | 6 |
| $12a_{0636}$ | 5 | $12a_{0641}$ | 4 | $12a_{0753}$ | 5 | $12a_{0827}$ | 5 |
| $12a_{0845}$ | 5 | $12a_{0970}$ | 6 | $12a_{0984}$ | 6 | $12a_{1017}$ | 6 |
| $12a_{1031}$ | 5 | $12a_{1095}$ | 6 | $12a_{1107}$ | 6 | $12a_{1114}$ | 6 |
| $12a_{1142}$ | 5 | $12a_{1171}$ | 6 | $12a_{1179}$ | 6 | $12a_{1205}$ | 6 |
| $12a_{1220}$ | 6 | $12a_{1240}$ | 6 | $12a_{1243}$ | 4 | $12a_{1247}$ | 6 |

# Calculating $T_K$ and $s_K$:

- Let $D = D(K)$ reduced alternating knot diagram.
- (Murasugi, Kauffman '80s) We have $s_k = c(D) = c(K)$=number of crossings
- Let $\mathbb{G}_A$ and $\mathbb{G}_B$ the *reduced checkerboard graphs* (a.k.a. reduced *Tait graphs*) of $D$.
- (Dasbach-Lin) We have

$$T_K = 2 - \chi(\mathbb{G}_A) - \chi(\mathbb{G}_B).$$

- If $D$ is *twist reduced*, with twist number $t = t(D)$, then $T_k = t$.
- **Definition.** *twist region* = maximal string of bigons
  *Twist reduced:* $A$ or $B$ must be a string of bigons.



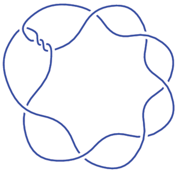
# Twist number and crosscap number

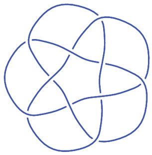
## Theorem (K.- Lee, 2014)

*Let $D(K)$ a twist reduced, prime, alternating diagram with twist number $t \geq 2$ and crossing number $c$. We have sharp bounds:*

$$1 + \left\lceil \frac{t}{3} \right\rceil \leq C(K) \leq \min\left\{ t+1, \left\lfloor \frac{c}{2} \right\rfloor \right\}.$$

- **Sharp upper bound:** $K = 10_3$( left )– $C(K) = 2g(K) + 1 = 3 = t + 1$.
- **Sharp lower bound:** $K = 10_{123}$– Both bounds give 5. We get $C(K) = 5$.





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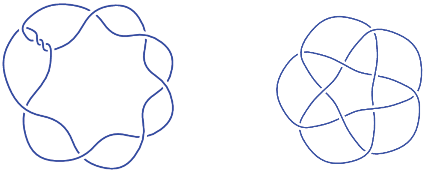


- **Note:** Upper bound of theorem follows easily. Discuss the lower bound.

# Getting the lower bound: Outline

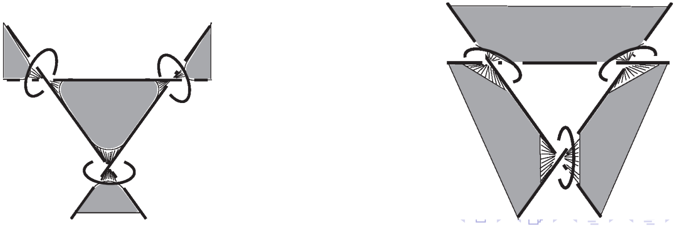
$D = D(K)$ prime,reduced, twist-reduced alternating diagram, with $t > 1$.

- **Step 1.** Show there is a surface $S$ coming from the Adams-Kindred algorithm, and an *augmented link* $L$, obtained from $D$, such that "augmentation components" added to $D$ don't intersect $S$.
- **Step 2.** Use geometry of $L$ (*angled polyhedral structures*) and normal surface theory to obtain a surface $S'$, such that
  1. $S'$ is a normal surface,
  2. $C(K)$ can be calculated from $S'$
- **Step 3.** To obtain the lower bound of $C(K)$ in terms of $t$, combine
  1. a combinatorial notion of area that satisfies Gauss-Bonnet ( Casson),
  2. Estimates of slope lengths on cusps of augmented links (Futer-Purcell based using work of Lackenby).

# Step1: Augmenting:

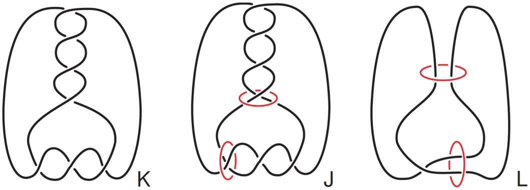
- Starting with $D = D(K)$ a prime,reduced, twist-reduced alternating diagram, we want to augment “around” the Adams-Kindred algorithm.
- Augmenting around bigon regions of $D$ and creating a state surface disjoint from the augmentation component:

- Augmenting around triangle regions and creating a state surface disjoint from the augmentation components:



# "Nice" polyhedral decomposition:

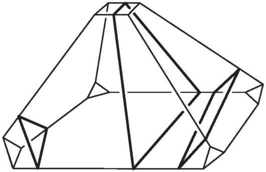
Alternating link $K$, *augmented* and *fully augmented* links I$J$ and a $L$.




- $M_L = S^3 \setminus L$ has a "nice" decomposition (Adams) into two convex ideal polyhedra $P_1$ and $P_2$ in the hyperbolic 3-space. (truncated vertices).
- Dihedral angles of $P_i$ are $\pi/2$. Thus $M_L$ is hyperbolic.
- Edges of $P_i \cap \partial M_L$ called *boundary edges*.
- Faces of $P_i \cap \partial M_L$ called *boundary faces*. They subdivide $\partial M_L$ into rectangles.
- *Interior faces* of $P_i$ admit checker-board coloring: opposite sides of

# Step 2:"Normalizing $C(K)$:"

- **Recall:** For $K$=alternating, have augmented link $L$ and surface $S$ in $M_L$ such that $C(S) = C(K)$.( $S$ need not be $\partial$-incompressible).
- Going through the normalization process: There is a normal surface, $S'$ in $M_L$ so that either $C(K) = 1 - \chi(S')$ or $C(K) = 2 - \chi_(S')$.
- *combinatorial area* $A_c(S')$= Sum of areas of all *normal disks* of $S'$.
- Normal disks look like:



- Combinatorial area of a normal disk $D$ that crosses $m$ interior edges of $P_i$:

$$A_c(D) = \frac{m\pi}{2} + \pi|D \cap \partial E(L)| - 2\pi.$$

# Estimate of $-\chi(S')$:

- We have

$$A_c(S') = -2\pi\chi(S')$$

- There is a notion of combinatorial length also due to Lackenby, such that

$$A_c(S') > \text{total length of } \partial S' \text{ on } \partial M_L.$$

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- **Non-alternating knots:** Futer-Purcell used similar methods to estimate the oriented genus of "highly twisted" knots (a. k. a. knots with diagrams that have at least 7 crossings per twist region).
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- **Question:** Does the Jones polynomial (coarsely) determine the crosscap number of all knots?What about the Khovanov homology?