Colored Jones polynomials

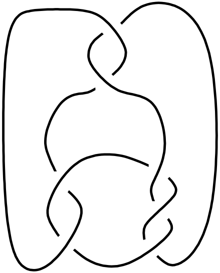
Effie Kalfagianni, Michigan State University

Survey talk for graduate students

AMS meeting, Hartford, CT, April 2019

Talk outline

Knots: Smooth embedding $K: S^1 \to S^3$. Knots K_1, K_2 are equivalent if $f(K_1) = K_2$, f orientation preserving diffeomorphism of S^3.



Outline

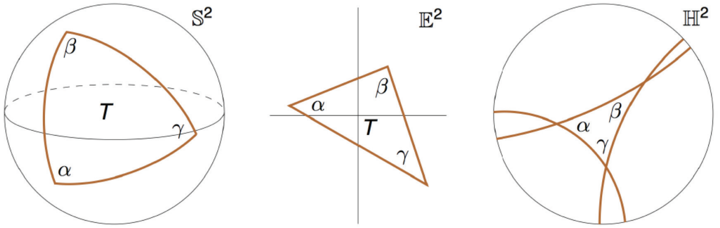
- 3-manifold geometric structures
  - Geometrization of $S^3 \setminus K$
  - Invariants arising from geometry: Hyperbolic volume
  - Incompressible surfaces
- Quantum topology
  - Colored Jones Polynomials
  - Knot diagrammatic approaches
  - CJP and volume (Volume type conjectures)
  - CJP and incompressible surfaces (Slopes conjectures)

Talk: A survey of colored Jones polynomials with emphasis on relations to geometry and topology of knot complements.

2-d Model Geometries:

For this talk, an n-dimensional *model geometry* is a simply connected n-manifold with a "homogeneous" Riemannian metric.
In dimension 2, there are exactly three model geometries, up to scaling:

Spherical	Eucledian	Hyperbolic

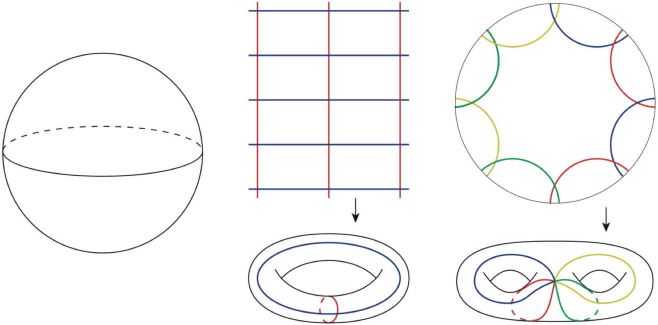


curvature $= +1$	curvature $= 0$	curvature $= -1$
$\text{Area}(T) = (\alpha+\beta+\gamma) - \pi$	$\alpha+\beta+\gamma = \pi$	$\text{Area}(T) = \pi - (\alpha+\beta+\gamma)$

Geometrization (a.k.a. Uniformization) in 2-d:

Every (closed, orientable) surface can be written as $S = X/G$, where X is a model geometry and G is a discrete group of isometries.

$X = \mathbf{S}^2$ $\qquad$ $X = \mathbb{E}^2$ $\qquad$ $X = \mathbb{H}^2$



- Geometry relates to topology: $k \cdot Area(S) = 2\pi\chi(S)$, $k = 1, 0, -1$ (*curvature*).

Geometrization in 3-d:

In dimension 3, there are eight model geometries:

$$X = \mathbf{S}^3 \;\; \mathbb{E}^3 \;\; \mathbb{H}^3, \;\; \mathbf{S^2} \times \mathbb{R}, \;\; \mathbb{H}^2 \times \mathbb{R}, \;\; Sol, \;\; Nil, \;\; \widetilde{SL_2}(\mathbb{R})$$

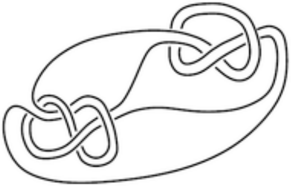
Theorem (Thurston 1980 + Perelman 2003)

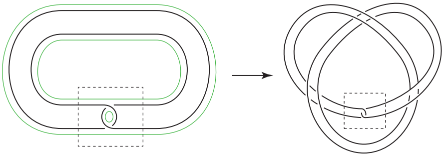
For every (closed, oriented) 3-manifold M, there is a canonical way to cut M along spheres and tori into pieces $M_1, \ldots, M_n$, such that each piece is $M_i = X_i/G_i$, where G_i is a discrete group of isometries of the model geometry X_i .

- The Poincare conjecture is a special case ($\mathbf{S}^3$ is the only compact model).
- Hyperbolic 3-manifolds are a prevalent, rich and very interesting class.
- Because of cutting along tori, manifolds with toroidal boundary will naturally arise. Knot complements fit in this class:
- **Knots complements:** Given K remove an open tube around K to obtain the *Knot complement:* Notation. $M_K = \overline{S^3 \setminus n(K)}$.

Geometrization of knot complements: 80's

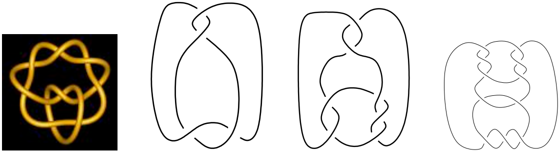
- By Jaco-Shalen-Johannson (1970's)+ W. Thurston (1980's) thre are three distinct classes of knots.
- *Torus knots:* Can be embedded on a standard torus in S^3. Up to symmetries they are classified by co-prime pairs of integers (studied by Burde-Zieschang (1960?)). The geometry of the interior is $\mathbb{H}^2 \times \mathbb{R}$.
- *Satellites:* Knot complement that are glueings of geometric pieces along tori. (Studied earlier by Schubert (1950's)).





Hyperbolic knots and rigidity, con't

- *Hyperbolic:* Interior of M_K admits complete hyperbolic metric of finite volume



- Hyperbolic knots are abundant: E.g. *prime* knots with at most 16 crossings: 20 are satellites, 13 are torus knots, 1,701,903 are hyperbolic.

Theorem (Mostow, Prasad 1973)

Suppose M is compact, oriented, and ∂M is a possibly empty union of tori. If M is hyperbolic (that is: $M \setminus \partial M = \mathbb{H}^3/G$), then G is unique up to conjugation by hyperbolic isometries. In other words, a hyperbolic metric on M is essentially unique.

- By rigidity, every geometric measurement of M (e.g. volume) is a *topological invariant*.

Jones Polynomials–Quantum invariants

1980's: Ideas originated in physics and in representation theory led to vast families invariants of knots and 3-manifolds. (*Quantum invariants*)

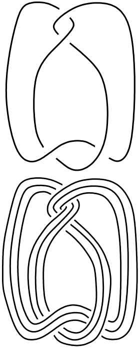
For this talk we discuss:

- The *Colored Jones Polynomials*: Infinite sequence of Laurent polynomials $\{J_{K,n}(t)\}_n$ encoding the *Jones polynomial* of K and these of the links K^s that are the parallels of K.
- Formulae for $J_{K,n}(t)$ come from representation theory of $SU(2)$ (decomposition of tensor products of representations).
  They look like

$$J_{K,1}(t) = 1, \qquad J_{K,2}(t) = J_K(t)\text{- } \textit{Original JP,}$$

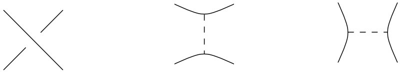
$$J_{K,3}(t) = J_{K^2}(t) - 1, \quad J_{K,4}(t) = J_{K^3}(t) - 2J_K(t), \ \ldots$$

- $J_{K,n}(t)$ can be calculated from any knot diagram via processes such as *Skein Theory*, *State sums*, *R-matrices*, *Fusion rules*....



The skein theory approach

- A or B resolutions, D_A, D_B, of a crossing of $D = D(K)$.



- Kauffman bracket: polynomial $\langle D \rangle \in \mathbb{Z}[t^{\pm 1/4}]$, regular isotopy invariant:
  - $\langle L \coprod \bigcirc \rangle = -(t^{1/2} + t^{-1/2})\langle L \rangle := \delta \langle L \rangle$
  - $\langle L \rangle = t^{-1/4}\langle D_A \rangle + t^{1/4}\langle D_B \rangle$
  - $\langle \bigcirc \rangle = -t^{1/2} - t^{-1/2}$
- Chebyshev polynomials:

$$S_{n+2}(x) = xS_{n+1}(x) - S_n(x), \quad S_1(x) = x, \quad S_0(x) = 1.$$

- D^m diagram obtained from D by taking m parallels copies.
- For $n > 0$, we define (where $w = w(D)$=*writhe*):

$$J_{K,n}(t) := ((-1)^{n-1} t^{(n^2-1)/4})^w (-1)^{n-1} \langle S_{n-1}(D) \rangle$$

- $\langle S_{n-1}(D) \rangle$ is linear extension on combinations of diagrams.

The CJP predicts Volume?

- **Question:** How do the *CJP* relate to geometry/topology of knot complements?
- *Renormalized CJP.*

$$J'_{K,n}(t) := \frac{J_{K,n}(t)}{J_{\bigcirc,n}(t)}.$$

Volume conjecture. [Kashaev+ H. Murakami - J. Murakami] Suppose K is a knot in S^3. Then

$$2\pi \cdot \lim_{n\to\infty} \frac{\log |J'_{K,n}(e^{2\pi i/n})|}{n} = \mathrm{Vol}\,(S^3 \setminus K)$$

- The conjecture is wide open:
- 4_1 (by Ekholm), knots up to 7 crossings (by Ohtsuki)
- *simplicial volume version* torus knots (by Kashaev and Tirkkonen), Whitehead doubles of torus knots of type $(2, b)$ (by Zheng).
- Versions Volume Conjectures for all 3-manifolds (talk by T. Yang, here).

Next: *Stable* coefficients of CJP coarsely predict volume.

Colored Jones polynomial prelims

For a knot K, and $n = 1, 2, \ldots$, we write its *n-colored Jones polynomial*:

$$J_{K,n}(t) := \alpha_n t^{m_n} + \beta_n t^{m_n - 1} + \cdots + \beta'_n t^{k_n+1} + \alpha'_n t^{k_n} \in \mathbb{Z}[t, t^{-1}]$$

- (Garoufalidis-Le, 04): The sequence $\{J_{K,n}(t)\}_n$ has a *recursive relation.*
- **Example:** For K=the trefoil knot

$$J_{K,n} = t^{-6(n^2-1)} \sum_{j=-\frac{n-1}{2}}^{\frac{n-1}{2}} t^{24j^2+12j} \frac{t^{8j+2} - t^{-(8j+2)}}{t^2 - t^{-2}}.$$

- The relation is

$$(t^{8n+12} - 1) J_{K,n+2} + (t^{-4n-6} - t^{-12n-10} - t^{8n+10} + t^{-2}) J_{K,n+1}$$
$$-(t^{-4n+4} - t^{-12n-8}) J_{K,n} = 0.$$

- Each of $\alpha'_n, \beta'_n \ldots$ satisfies a *linear recursive relation* in n, with integer coefficients .

$$\left(\text{ e. g. } \quad \alpha'_{n+1} + (-1)^n \alpha'_n = 0\right).$$

Knots "generic" to the eyes of CJP

Given a knot K with

$$J_{K,n}(t) = \alpha_n t^{m_n} + \beta_n t^{m_n - 1} + \cdots + \beta'_n t^{k_n+1} + \alpha'_n t^{k_n},$$

and any diagram $D(K)$, there exist explicitly given functions $M(n, D)$

$$m_n \leq M(n, D).$$

- **Definition.** Knots with $m_n = M(n, D)$, are called *semi-adequate*. They have *stable coefficients* of $J_{K,n}(t)$.
- (Dasbach-Lin, Armond) If $m_n = M(n, D)$, then

  $$\alpha_K = |\alpha_n| = 1 \quad \text{and} \beta_K := |\beta_n| = |\beta_2|,$$

  for every $n > 1$. Similar statements for α'_n, β'_n.
- **Remark:** Each coefficient of $J_{K,n}(t)$ stabilizes eventually. Stable coefficients form q-series (Armond, Dasbach, Garoufalidis Le). Generalized stability phenomena in CJP (Hajij, Lee, Walsh, Lee-van der Veen, Garoufalidis-Le, Vuong...)
- Stable coefficients control the volume of the link complement.

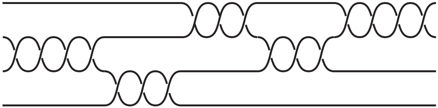
Sample families: alternating and positive braids

Theorem (Menasco, Lackenby, Dasbach-Lin)

If K is a prime, non-torus, non-torus alternating link, then K is hyperbolic, and

$$\frac{v_8}{2}\,(\beta_K + \beta'_K - 1) \;\leq\; Vol(S^3 \setminus K) \;<\; 10v_3\,(\beta_K + \beta'_K - 1)$$

Here, $v_3 \approx 1.0149$ and $v_8 = 3.6638$.



s

Theorem (Futer-K.-Purcell)

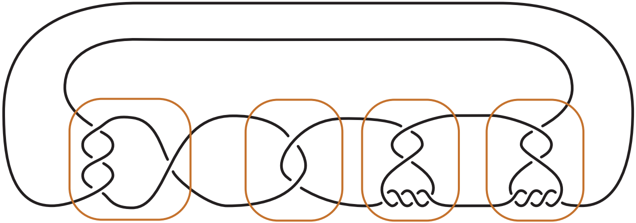
If K is the closure of a positive braid $s = \sigma_{i_1}^{r_1}\sigma_{i_2}^{r_2}\cdots\sigma_{i_t}^{r_t}$, where $r_j \geq 3$ for all j, then K is hyperbolic, and

$$v_8\,(\beta'_K - 1) \;\leq\; Vol(S^3 \setminus K) \;<\; 15v_3\,\beta'_K - 25v_3\,.$$

The gap between the upper and lower bounds is a factor of 4.155...

Sample family: Montesinos links

A Montesinos knot or link is constructed by connecting n rational tangles in a cyclic fashion.



Theorem (FKP + Finlinson)

If K be a hyperbolic Montesinos knot. Then

$$v_8\,(\beta'_K - 2) \;\leq\; Vol(S^3 \smallsetminus K).$$

If K has length at least four we get two-sided volume estimates:

$$v_8\,(\max\{\beta_K, \beta'_K\} - 2) \;\leq\; Vol(S^3 \smallsetminus K) \;<\; 4v_8\,(\beta'_K + \beta_K - 2) + 2v_8.$$

A Coarse Volume Conjecture

Results and experimental evidence prompt (*A coarse Volume conjecture?*):

Question. Does there exist function $B(K)$ of the coefficients of the colored Jones polynomials of a knot K, that is easy to calculate from a "nice" knot diagram such that for hyperbolic knots, $B(K)$ is coarsely related to hyperbolic volume $\mathrm{Vol}\,(S^3 \smallsetminus K)$?
Are there constants $C_1 \geq 1$ and $C_2 \geq 0$ such that

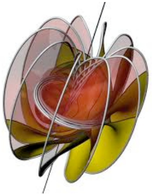
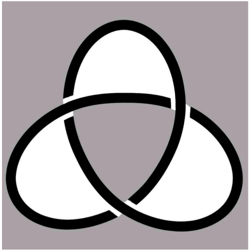
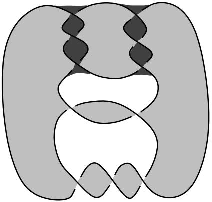
$$C_1^{-1} B(K) - C_2 \ \leq \ \mathrm{Vol}\,(S^3 \smallsetminus K) \ \leq \ C_1 B(K) + C_2,$$

for all hyperbolic K?

- Results and stabilization properties of CJP prompt more guided speculations as to where one might look for $B(K)$.
- For more classes of knots Giambrone and more recently Lee...

Surfaces in knot complements

- There are several properly embedded surfaces in knot complements– some non-orientable.

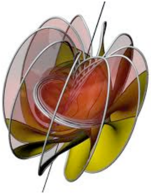
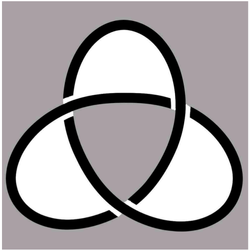
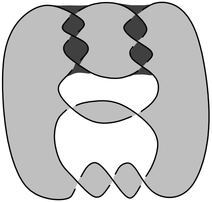
- **Definition.** A surface S, properly embedded in $\mathbf{S}^3 \setminus K$ is called is *essential* if inclusion induces injection

$$\pi_1(S, \partial S) \longrightarrow \pi_1(\mathbf{S}^3 \setminus K, \partial(\mathbf{S}^3 \setminus K)).$$

- **Definition.** A (primitive) class in $H_1(\partial(S^3 \setminus K)) \cong \mathbb{Z} \times \mathbb{Z}$, determined by an element in $s \in \mathbf{Q} \cup \{\infty\}$, is called *a boundary slope of K* if there is an essential suface S such that each component of ∂S represents s.

Surfaces in knot complements

- There are several properly embedded surfaces in knot complements–some non-orientable.

- **Definition.** A surface S, properly embedded in $\mathbf{S}^3 \setminus K$ is called is *essential* if inclusion induces injection

$$\pi_1(S, \partial S) \longrightarrow \pi_1(\mathbf{S}^3 \setminus K, \partial(\mathbf{S}^3 \setminus K)).$$

- **Definition.** A (primitive) class in $H_1(\partial(S^3 \setminus K)) \cong \mathbb{Z} \times \mathbb{Z}$, determined by an element in $s \in \mathbf{Q} \cup \{\infty\}$, is called *a boundary slope of K* if there is an essential suface S such that each component of ∂S represents s.

The topology of the degree of CJP

- $d_+[J_{K,n}]$ =maximum degree of CJP
- The q-holonomicity property of CJP implies:
- Given K there is $N_K > 0$, such that, for $n \geq N_K$,

$$d_+[J_{K,n}] = a_K(n)\, n^2 + b_K(n)n + c_K(n),$$

- where $a_K(n), b_K(n), c_K(n) : \mathbf{N} \to \mathbb{Q}$ are periodic functions.
- Similarly, $d_-[J_{K,n}]$ =maximum degree of CJP:

$$d_-[J_{K,n}] = a^*_K(n)\, n^2 + b^*_K(n)n + c^*_K(n),$$

- where $a^*_K(n), b^*_K(n), c^*_K(n) : \mathbf{N} \to \mathbb{Q}$ are periodic functions.
- We have finitely many cluster points

$$js_K = \{4a_K(n)\}' \quad \text{and} \quad js^*_K = \{4a^*_K(n)\}',$$

- (called Jones Slopes) and finitely many cluster points

$$js_K = \{2b_K(n)\}',\ js^*_K = \{4b^*_K(n)\}'.$$

Slopes Conjectures

- **Definition.** A Jones surface of K is an essential surface $S \subset M_K = S^3 \smallsetminus K$
- ∂S represents a Jones slope $4a(n) = a/b \in js_K$, with $b > 0$, $\gcd(a, b) = 1$, and
$$\frac{\chi(S)}{|\partial S| b} = 2b_K(n).$$
- Similarly, for a a Jones slope $4a^(n) = a^*/b^* \in js_K$, with $b^* > 0$, $\gcd(a^*, b^*) = 1$, and
$$\frac{\chi(S)}{|\partial S| b^*} = -2b_K^*(n).$$
- ***Strong*** **Slope Conjecture.**
- (Garoufalidis) All Jones slopes are boundary slopes.
- (*K+Tran*) *All Jones slopes are realized by Jones surfaces*.
- **Remark.** No knots with more than one Jones slope are known.

Simple Examples

- **Example 1.** For the torus knot $T_{p,q}$, the Jones slopes are $\{0, pq\}$ and the corresponding Jones surfaces are a minimum genus Seifert surface and the cabling annulus, respectively.
- **Example 2.** For the $K = (-2, 3, 7)$-pretzel knot we have

$$\begin{aligned} 4d_+[J_{K,n}] &= 37/2n^2 + 34n + e(n), \\ 4d_-[J_{K,n}] &= 0n^2 + 5n, \end{aligned}$$

  where $e(n)$ is a periodic sequence of period 4.
- The $(-2, 3, 7)$-pretzel knot is a Montesinos knot with boundary slopes

$$\{0, 16, 37/2, 20\}.$$

- For Montesinos knots boundary slopes essential surfaces can be found using the Hatcher-Oertel algorithm.
- For computational purposes there is implementation of the algorithm (Dunfield)

Status

- **Strong slope conjecture known for:**
- Alternating knots (Garoufalidis)
- Adequate knots (Futer-K-Purcell)
- Knots up to nine crossings (Garoufalidis, Tran-K., Howie)
- Montesinos knots (Lee-van der Veen, Garoufalidis-Lee-van der Veen, Leng-Yang-Liu)
- Iterated torus knots
- Graph knots (Motegi-Takata, Baker-Motegi-Takata)
- families of non-Montesinos knots, non-adequate knots (Howie-Do, Lee)
- **SSC is closed under:**
- Connect sums (Motegi-Takata)
- Cabling operations (Tran-K.)
- Whiterhead doubling (Baker-Motegi-Takata)

Implications of SSC

- The degree $d_+[J_{K,n}]$ detects the unknot: For,
- Suppose that $d_+[J_{K,n}] = \mathrm{d}_+[J_{\bigcirc,n}] = 0.5n$. Then $jx_K = \{1\}$, and the Strong Slopes Conjecture holds for K, then we have a Jones surface S for K with boundary slope 0 and with $\chi(S) > 0$. Then S must be a collection of discs which means that a Seifert surface for K is a disc and thus K is the unknot.
- The degrees $d_+[J_{K,n}], d_-[J_{K,n}]$ detect all torus knots!
- Proof of following theorem begins with the fact that an essential surface with $\chi(S) = 0$, implies that there is a cabling annulus!

Theorem (K.–)

Suppose K satisfies the strong slope conjecture and $T_{p,q}$ is the (p,q)-torus knot. If

$$d_+[J_{K,n}] = d_+[J_{T_{p,q},n}] \text{ and } d_-[J_{K,n}] = d_-[J_{T_{p,q},n}],$$

for all n, then, up to orientation change of the knot, K is isotopic to $T_{p,q}$.

Figure-8/alternating

- Howie and Greene gave a characterization of alternating knots in terms of their (spanning) surfaces. Their result, implies
- If K satisfies the SSC and $d_{\pm}[J_{K,n}] = d_{\pm}[J_{4_1,n}]$, then K is isotopic to 4_1.
- **Definition.** A Jones surface S of a knot K is called *characteristic* if the number of sheets $b|\partial S|$ divides the *Jones period* of K.
- For all the knots the SSC is known, the Jones surfaces can be taken to be characteristic!
- The stronger version of SCC, together with Howie's theorem, imply the following (CJP characterization of alternating knot).

Theorem

A knot K that satisfies the SSC, with characteristic surfaces, is alternating if and only if there are $a, b \in \mathbb{Z}$ (depending only on K) such that, for all $n > 0$,

$$a + b = 1 \quad \text{and} \quad d_{+}[J_{K,n}] - d_{-}[J_{K,n}] = an^2 + bn - (b + c),$$