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ON ESSENTIAL COEXISTENCE OF ZERO AND NONZERO LYAPUNOV EXPONENTS

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Abstract. We show that there exists a C^∞ volume preserving diffeomorphism P of a compact smooth Riemannian manifold $\mathcal{M}$ of dimension 4, which is close to the identity map and has nonzero Lyapunov exponents on an open and dense subset $\mathcal{G}$ of not full measure and has zero Lyapunov exponent on the complement of $\mathcal{G}$. Moreover, $P|\mathcal{G}$ has countably many disjoint open ergodic components.

1. **Introduction.** The problem of essential coexistence emerged from the following result in [3, 10, 24, 25]: on any manifold $\mathcal{M}$ and for any sufficiently large r there are open sets of C^r volume preserving diffeomorphisms of $\mathcal{M}$, all of which possess a Cantor set of codimension-1 invariant tori of positive volume; on each such torus the diffeomorphism is C^1 conjugate to a Diophantine translation; all of the Lyapunov exponents are zero on the invariant tori. Those codimension-1 invariant tori cannot be destroyed by small perturbations of the system. The existence of invariant tori was first observed and established in Hamiltonian dynamics, from which the classical KAM theory arises. We can view the above result as a discrete version of the KAM theory.

It is expected that outside the set of invariant tori the system is nonuniformly completely hyperbolic, i.e., the Lyapunov exponents are nonzero almost everywhere and the system has at most countably many ergodic components (see [17]). In other words, the invariant tori are surrounded by the so-called "chaotic sea". It is still an open problem whether generically this picture is true. In this paper, we construct a smooth conservative system with the coexistence of elliptic behaviors and complete hyperbolicity, which is "essential" in the sense that the chaotic sea is dense.

Main Theorem. *Given $\alpha > 0$, there exist a compact smooth Riemannian manifold $\mathcal{M}$ of dimension 4 and a C^∞ diffeomorphism $P : \mathcal{M} \to \mathcal{M}$ preserving the Riemannian volume m such that*

(1) $\|P - Id\|_{C^1} \le \alpha$ and P is homotopic to Id.
(2) P has nonzero Lyapunov exponents almost everywhere on an open dense subset $\mathcal{G} \subset \mathcal{M}$. Moreover, $\mathcal{G}$ consists of countably many open connected components, on which P is ergodic.

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(3) The set $\mathcal{G}^c \subset \mathcal{M}$ has positive volume and is a union of 3-dimensional invariant submanifolds. Also $P|G^c = Id$ and the Lyapunov exponents of P on $\mathcal{G}^c$ are all zero.

We emphasize that the expected picture is still out of reach in our example, since the 3-dimensional invariant submanifolds are not invariant tori but copies of the suspension manifold over an Anosov automorphism of 2-torus (see section 3).

Our main theorem is parallel and somewhat complementary to the result in [12], where a volume preserving diffeomorphism of a 5-dimensional compact manifold is constructed in such a way that this diffeomorphism is close and homotopic to Id, has nonzero Lyapunov exponents and is ergodic (indeed Bernoulli) on an open and dense subset $\mathcal{G}$ of the manifold. The complement $\mathcal{G}^c$ has positive volume, and the diffeomorphism is Id and has zero Lyapunov exponents on $\mathcal{G}^c$. In that construction, $\mathcal{G}^c$ is the direct product of a 3-dimensional compact manifold and a Cantor set of positive volume in 2-torus and thus has codimension 2. On the other hand, the set $\mathcal{G}^c$ in our example is the direct product of a 3-dimensional compact manifold and a Cantor set of positive Lebesgue measure in a circle, and thus has codimension 1. Therefore the map P cannot be ergodic but has countably many ergodic components. The codimension 1 property of the invariant submanifolds makes our example closer to the "real KAM-like" picture than the one in [12].

The coexistence of zero and nonzero Lyapunov exponents is one of the most interesting phenomena in dynamical systems. While in the absence of a general theory, the coexistence phenomenon has been observed in various examples[1], among which the area preserving surface diffeomorphisms have been mostly studied. Przytycki [18] studied a specially chosen one-parameter family of C^∞ area preserving diffeomorphisms of $\mathbb{T}^2$, which demonstrates a route from Anosov diffeomorphisms to nonuniform hyperbolicity and then to the coexistence of elliptic islands and the chaotic sea[2]. In fact, the appearance of chaotic sea in the 2-dimensional case is equivalent to positivity of the metric entropy, which is an outstanding and extremely difficult problem for the standard twist maps (see [20]). Various results in this direction have been achieved for other surface diffeomorphisms, most of which can be viewed as modifications of the standard maps, for instance, see [22, 23, 7, 15, 8, 9]. One would often observe the coexistence of elliptic islands and the chaotic sea in these results. The coexistence phenomenon in the continuous-time dynamical systems have also been found in [5, 6, 2], etc.

We emphasize that establishing the essential coexistence in [12] and this paper requires somewhat different and delicate techniques. Unlike the 2-dimensional situation, we start with a weak version of partial hyperbolicity - pointwise partial hyperbolicity (see Section 2) - on an open dense subset while its complement have all zero exponents. In general, a pointwise partially hyperbolic system just has some but not all nonzero Lyapunov exponents. We must make great efforts to ensure that the Lyapunov exponents are all nonzero in the chaotic sea $\mathcal{G}$ after perturbations without changing the zero Lyapunov exponents in the regular region $\mathcal{G}^c$. The matter is that a typical trajectory that originates in the chaotic sea $\mathcal{G}$ will spend a long time in the vicinity of the regular region $\mathcal{G}^c$ where contraction and expansion rates are very small. To make sure all nonzero Lyapunov exponents, the trajectory must spend even longer periods away from $\mathcal{G}^c$ as a compensation.

[1]See, for example, the expository survey [21].

[2]The elliptic island is an open neighborhood of $0 \in \mathbb{T}^2$, most of which is filled with KAM invariant circles. Therefore, the chaotic sea is not dense.

Due to the construction of the manifold $\mathcal{M}$ and non-ergodicity of the map P in this paper, we need to deal with connected components of the set $\mathcal{G}$ one by one. By modifying the construction in [12], we perturb the identity map and obtain nonzero Lyapunov exponents and ergodicity near any open connected component of $\mathcal{G}$. In this way we can construct consecutive small perturbations P_j, which possesses hyperbolicity and ergodicity on the first j connected components of $\mathcal{G}$ and is identity elsewhere. The main technical issue, which does not appear in [12], is how to effect this inductive argument and guarantee that the sequence P_j converges to the desired map P in our Main Theorem. Indeed this inductive procedure relies on the special structure of the Cantor set in a circle, that is, it can be produced by consecutively removing disjoint open intervals from the circle (see section 3). Therefore we can easily label the connected components of $\mathcal{G}$, each closure of which possesses a neighborhood disjoint from other components. The inductive step from P_{j-1} to P_j can hence be restricted in a neighborhood of the j-th component of $\mathcal{G}$, at which P_{j-1} is identity. Controlling the C^j-norm of $P_j - P_{j-1}$ carefully, one can obtain the desire map P as the limit map of the sequence P_j.

The paper is organized as follows. In section 2 we introduce some background information and basic notations in the theory of partially hyperbolic systems. In section 3 we describe the construction of the 4-dimensional manifold $\mathcal{M}$ and the open and dense subset $\mathcal{G}$, and show that the construction of the diffeomorphism P can be reduced to the construction of a perturbation H at some neighborhood of each connected components of $\mathcal{G}$. In the remaining sections, we apply the approach in [12] to obtain the perturbation map H in the reduction. To be precise, we describe the three steps to obtain H in section 4, create positive central exponents in section 5, and produce the accessibility property in section 6.

2. **Preliminaries.** See [16, 1, 12] for more details.

Let f be a diffeomorphism of a compact smooth Riemannian manifold $\mathcal{M}$ and $\mathcal{S} \subset \mathcal{M}$ an f-invariant open subset. The map f is said to be *pointwise partially hyperbolic* on $\mathcal{S}$ if for every $x \in \mathcal{S}$ the tangent space at x admits an invariant splitting

$$T_x\mathcal{M} = E^s(x) \oplus E^c(x) \oplus E^u(x)$$

and there are continuous positive functions $\lambda(x) < \lambda'(x) \leq 1 \leq \mu'(x) < \mu(x)$, $x \in \mathcal{S}$ such that

$$\begin{aligned} \|dfv\| &\leq \lambda(x)\|v\|, && v \in E^s(x), \\ \lambda'(x)\|v\| \leq \|dfv\| &\leq \mu'(x)\|v\|, && v \in E^c(x), \\ \mu(x)\|v\| \leq \|dfv\|&, && v \in E^u(x). \end{aligned}$$

In particular, if there is an f-invariant compact subset $\Lambda \subset \mathcal{S}$ such that the functions $\lambda(x), \lambda'(x), \mu(x), \mu'(x)$ are constants on Λ, then we say f is *uniformly partially hyperbolic* on Λ.

Given a subset $\mathcal{S}$ we call a partition $\mathcal{P}$ of $\mathcal{S}$ a $(\delta,\, q)$-*foliation with smooth leaves* if there exist continuous functions $\delta = \delta(x) > 0$, $q = q(x) > 0$, and an integer $k > 0$ such that for each $x \in \mathcal{S}$:

(1) there exists a smooth immersed k-dimensional manifold $W(x)$ containing x for which $\mathcal{P}(x) = W(x)$ where $\mathcal{P}(x)$ is the element of the partition $\mathcal{P}$ containing x. The manifold $W(x)$ is called the *global leaf* of the foliation at x; the connected component of the intersection $W(x) \cap B(x, \delta(x))$ that contains x is called the *local leaf* at x and is denoted by $V(x)$;

(2) there exists a continuous map $\phi_x : B(x, q(x)) \to C^1(D, \mathcal{M})$ (where D is the unit ball) such that $V(y)$ is the image of the map $\phi_x(y) : D \to \mathcal{M}$ for each $y \in B(x, q(x))$; the number $q(x)$ is called the *size* of $V(x)$.

We say that a foliation with smooth leaves is *absolutely continuous* if for almost every $x \in \mathcal{S}$ and almost every $y \in B(x, q(x))$ the conditional measure generated on $V(y)$ by volume m (with respect to the partition of $B(x; q(x))$ by local leaves) is absolutely continuous with respect to the leaf volume $m_{V(y)}$ on $V(y)$.

In general a pointwise partially hyperbolic diffeomorphism f on an open set $\mathcal{S}$ might not have strongly stable and unstable local manifolds at every point in $\mathcal{S}$. However, for all pointwise partially hyperbolic diffeomorphisms that we will construct in this paper, their global strongly stable and unstable manifolds form transversal foliations with smooth leaves, denoted by W^s and W^u respectively. In this case, we say f has the *accessibility property* via W^s and W^u if any two points $z, z' \in \mathcal{S}$ are *accessible*, i.e.

(1) there exists a collection of points $z_1, \dots, z_n \in \mathcal{S}$ such that $z = z_1$, $z' = z_n$ and $z_k \in W^i(z_{k-1})$ for $i = s$ or u and $k = 2, \dots, n$;
(2) the points z_{k-1} and z_k can be connected by a smooth curve $\gamma_k \subset W^i(z_{k-1})$ for $i = s$ or u and $k = 2, \dots, n$. Here the leaf-wise path γ_k is called the $(u, s)_f$-*path* or simply (u, s)-*path*.

A uniformly partially hyperbolic diffeomorphism f is called *dynamically coherent* if the subbundles $E^{cu} = E^c \oplus E^u$, E^c, and $E^{cs} = E^c \oplus E^s$ are integrable to continuous foliations with smooth leaves W^{cu}, W^c and W^{cs}, called respectively the *center-unstable*, *center* and *center-stable foliations*. Furthermore, the foliation W^c and W^u are subfoliations of W^{cu}, while W^c and W^s are subfoliations of W^{cs}. The following theorem (see [11, 19]) shows that dynamical coherence is robust.

Theorem 2.1. *Suppose that f is a partially hyperbolic diffeomorphism. If the center foliation W^c is smooth, then f is dynamically coherent. Moreover, any diffeomorphism that is close to f in the C^1 topology is dynamically coherent.*

We denote by

$$\lambda(x, v) = \limsup_{n\to\infty} \frac{1}{n} \log \|df^n v\|$$

the *Lyapunov exponent* of a nonzero vector v at $x \in \mathcal{M}$ and by $\lambda_i(x) = \lambda_i(x, f)$, $i = 1, \dots, \dim\mathcal{M}$, the values of the Lyapunov exponents at x. Note that the functions $\lambda_i(x, f)$ are invariant. We assume that these values are ordered so that

$$\lambda_1(x, f) \geq \cdots \geq \lambda_{\dim\mathcal{M}}(x, f).$$

We also denote by

$$L_k(f) := \int_{\mathcal{M}} \sum_{i=1}^{k} \lambda_i(x, f) dm(x) \tag{2.1}$$

where m is the Riemannian volume. We call this number the *k-th average Lyapunov exponent* of f.

Consider a volume preserving C^2 diffeomorphisms f of a compact smooth manifold $\mathcal{M}$ that is pointwise partially hyperbolic on an open set S. We say that f has *positive central exponents* if there is an invariant set $\mathcal{A} \subset \mathcal{S}$ of positive volume such that for every $x \in \mathcal{A}$ and every $v \in E^c(x)$ the Lyapunov exponent $\lambda(x, v) > 0$. The following theorem (see [12]) plays an important part in the proof of the Main Theorem.

Theorem 2.2. *Assume that the following conditions hold:*

1. *f has strongly stable and unstable (δ, q)-foliations W^s and W^u where $\delta = \delta(x)$ and $q = q(x)$ are continuous functions on $\mathcal{S}$;*
2. *the foliations W^s and W^u are absolutely continuous;*
3. *f has the accessibility property via the foliations W^s and W^u;*
4. *f has positive central exponents.*

Then f has positive central exponents at almost every point $x \in \mathcal{S}$. $f|_{\mathcal{S}}$ is ergodic and indeed, is a Bernoulli diffeomorphism.

3. **Construction of The Map P: Proof of Main Theorem.** In this section, we first describe the construction of the 4-dim manifold $\mathcal{M}$ and the related sets, that is, the open dense subset $\mathcal{G}$ and its connected components $\mathcal{G}_j$, $j = 1, 2, \ldots$. Then we will show that the desired map P of our Main Theorem can be obtained by inductive perturbations on the components $\mathcal{G}_j$, in other words, we need to construct a sequence of diffeomorphisms P_j converging to P. Due to the special structures of the manifold $\mathcal{M}$ and the open dense set $\mathcal{G}$, we can actually reduce our construction of P_j to the construction of the perturbation that changes P_{j-1} to P_j, which is much simpler.

3.1. **The 4-dim Manifold $\mathcal{M}$.** Take an Anosov automorphism A of the 2-torus $X = \mathbb{T}^2$ with the constant expanding rate η_A along the unstable direction. We consider the suspension flow T^τ over A with roof function 1. This flow acts on the suspension manifold

$$\mathcal{N} = X \times [0, 1]/\sim,$$

where "$\sim$" is the identification $(x, 1) \sim (Ax, 0)$.

Set $Y = S^1 = [0, 1]/\{0 \sim 1\}$. For each $n \in \mathbb{N}$, pick finitely many non-overlapping closed intervals $C_{i_1\ldots i_n} \subset Y$ in such a way that $C_{i_1\ldots i_n i_{n+1}} \subset C_{i_1\ldots i_n}$, then we obtain a Cantor set $C \subset Y$ by letting

$$C = \bigcap_{n=1}^{\infty} \bigcup_{(i_1\ldots i_n)} C_{i_1\ldots i_n}.$$

Alternatively, one can obtain this Cantor set by consecutively removing disjoint open intervals from Y. More precisely, denote by $I_1, I_2, \ldots$ those open intervals that are removed from Y, then set $\mathcal{I} = \biguplus_{j=1}^{\infty} I_j$ and $C = Y \backslash \mathcal{I}$. Moreover, let us assume that $\sum\limits_{j=1}^{\infty} |I_j| < 1$ so that the Cantor set C is of positive Lebesgue measure, where $|I_j|$ is the length of the interval I_j.

Finally, we take $\mathcal{M} = \mathcal{N} \times Y$, $\mathcal{G} = \mathcal{N} \times \mathcal{I}$ and $\mathcal{G}_j = \mathcal{N} \times I_j$, $j = 1, 2, \ldots$. Clearly $\{\mathcal{G}_j\}_{j=1}^{\infty}$ are open connected components of $\mathcal{G}$. Also the complement $\mathcal{G}^c = \mathcal{N} \times C$ is of positive Riemannian volume.

3.2. **The Sequence of Diffeomorphisms P_j.** Starting from $P_0 = Id$ on $\mathcal{M}$, we intend to obtain inductively the map P_j as a small homotopic perturbation of P_{j-1} for $j \geq 1$. Furthermore, P_j differs from P_{j-1} only on the j-th connected component $\mathcal{G}_j$ of $\mathcal{G}$. More precisely, we will show

Proposition 3.1. *Given $\alpha > 0$, there exists a sequence of C^∞ volume preserving diffeomorphisms $P_j : \mathcal{M} \to \mathcal{M}$, $j = 0, 1, 2, \ldots$ such that*

1. *$P_0 = Id$; $P_j = P_{j-1}$ outside $\mathcal{G}_j$, in particular, $P_j = Id$ outside $\biguplus_{k=1}^{j} \mathcal{G}_k$; Also, P_j is homotopic to P_{j-1};*

(2) P_j *is ergodic and has nonzero Lyapunov exponents almost everywhere on each* $\mathcal{G}_k$*,* $k = 1, \dots, j$*;*
(3) $\|P_j - P_{j-1}\|_{C^j} \le \alpha/2^j$*.*

Once such a sequence of diffeomorphisms $\{P_j\}_{j=0}^{\infty}$ is constructed, we can take the pointwise limit $P = \lim_{j\to\infty} P_j$. We shall show that P is the desired map of our Main Theorem.

Proof of Main Theorem. By Proposition 3.1(3), for any $r \ge 1$ and $l > j \ge r-1$, we have

$$\|P_l - P_j\|_{C^r} \le \sum_{k=j}^{l-1} \|P_{k+1} - P_k\|_{C^{k+1}} \le \sum_{k=j}^{l-1} \frac{\alpha}{2^{k+1}} \le \frac{\alpha}{2^j}.$$

It follows that $\|P - P_j\|_{C^r} \le \alpha/2^j$, in particular, $\|P - Id\|_{C^1} \le \alpha$. Hence P_j converges to P in the C^r topology, and therefore P is a C^∞ diffeomorphism since r is arbitrary. Clearly P is volume preserving. In addition, by Proposition 3.1(1), $P = P_j$ on $\biguplus_{k=1}^{j} \mathcal{G}_k$ and $P = Id$ outside $\biguplus_{j=1}^{\infty} \mathcal{G}_j = \mathcal{G}$. Since every P_j is homotopic to Id, P is also homotopic to Id. These imply statement (1) and (3) of our Main Theorem.

By Proposition 3.1(2), P_j is ergodic and has nonzero Lyapunov exponents almost everywhere on each $\mathcal{G}_k$, $k = 1, \dots, j$, and so is P. Since j is arbitrary, statement (2) of our Main Theorem follows. □

3.3. **The Reduction.** The proof of Proposition 3.1 boils down to a construction of a suitable homotopic perturbation $H_j : \mathcal{M} \to \mathcal{M}$ that changes P_{j-1} to P_j, i.e. $P_j = H_j \circ P_{j-1}$ and H_j is homotopic to Id, for each $j = 1, 2, \dots$.

Fix j, and pick an open interval $\breve{I}_j \supset \overline{I_j}$ in such a way that $\overline{\breve{I}_j} \cap \overline{I_k} = \emptyset$ for all $1 \le k < j$. Set $\breve{\mathcal{G}}_j = \mathcal{N} \times \breve{I}_j$, then $\breve{\mathcal{G}}_j \supset \overline{\mathcal{G}_j}$ and $\overline{\breve{\mathcal{G}}_j} \cap \overline{\mathcal{G}_k} = \emptyset$ for all $1 \le k < j$. Note that

(1) on the set $\breve{\mathcal{G}}_j$, $P_{j-1} = Id$ and hence $H_j = P_j$.
(2) outside the set $\mathcal{G}_j$, $P_j = P_{j-1}$ and hence $H_j = Id$.

Therefore, by Proposition 3.1, we only need to restrict the construction of H_j on $\breve{\mathcal{G}}_j$ such that H_j is homotopic to Id, ergodic and has nonzero Lyapunov exponents almost everywhere on $\mathcal{G}_j$. Also $H_j = Id$ on $\breve{\mathcal{G}}_j \backslash \mathcal{G}_j$, and $\|H_j|\mathcal{G}_j - Id|\mathcal{G}_j\|_{C^j} \le \alpha/2^j$. More generally, we can show that

Proposition 3.2. *Set* $\mathcal{Z} = \mathcal{N} \times I$ *and* $\breve{\mathcal{Z}} = \mathcal{N} \times \breve{I}$*, where* $I, \breve{I}$ *are two arbitrary open intervals satisfying* $\overline{I} \subset \breve{I}$*. Given* $\delta > 0$*,* $r \in \mathbb{N}$*, there exists a* C^∞ *volume preserving diffeomorphism* $H : \breve{\mathcal{Z}} \to \breve{\mathcal{Z}}$ *such that*

(1) H *is homotopic to* Id*, and* $H = Id$ *on* $\breve{\mathcal{Z}} \backslash \mathcal{Z}$*;*
(2) $H|\mathcal{Z}$ *is ergodic and has nonzero Lyapunov exponents almost everywhere;*
(3) $\|H - Id\|_{C^r} \le \delta$*.*

One can see that Proposition 3.1 immediately follows from Proposition 3.2. After proper scalings, we may assume $I = (-1, 1)$ and $\breve{I} = (-2, 2)$. We are going to prove this proposition in section 4.

4. **Construction of The Map H: Proof of Proposition 3.2.** We describe the construction of the map H splitting into several steps, following [12]. From now on, let us fix δ and r in the assumption of Proposition 3.2.

4.1. **Step 1: The Original Map T.** Pick an open subinterval $\widetilde{I} \subset I = (-1, 1)$, for example, $\widetilde{I} = (-5/8, 5/8)$, and a C^∞ function $\kappa : \breve{I} = (-2, 2) \to \mathbb{R}$ satisfying:

(1) $\kappa(y) > 0$ if $y \in I$ and $\kappa(y) = 0$ if $y \in \breve{I} \backslash I$;
(2) $\|\kappa\|_{C^r} \leq 1$;
(3) $\kappa(y) = \kappa_0$ for $y \in \widetilde{I}$, where κ_0 is a constant.

We define a map $T : \breve{\mathcal{Z}} \to \breve{\mathcal{Z}}$ by

$$((x, t), y) \mapsto (T^{\kappa(y)}(x, t), y),$$

where $(x, t) \in \mathcal{N}$, $y \in \breve{I}$, and T^τ is the suspension flow on $\mathcal{N}$.

Recall that for each $\tau \neq 0$ the map T^τ is uniformly partially hyperbolic with one-dimensional stable $E^s_{T^\tau}$, one-dimensional unstable $E^u_{T^\tau}$ and one-dimensional center $E^c_{T^\tau}$ subbundles, and these subbundles are integrable to smooth stable $W^s_{T^\tau}$, unstable $W^u_{T^\tau}$ and center $W^c_{T^\tau}$ foliations of $\mathcal{N}$. Moreover, we can choose a suitable Riemannian metric on $\mathcal{N}$ such that at every $(x, t) \in \mathcal{N}$, T^τ expands at the rate η_A^τ along the unstable direction and contracts at the rate $\eta_A^{-\tau}$ along the stable direction.

By the above properties of T^τ and the construction of $T : \breve{\mathcal{Z}} \to \breve{\mathcal{Z}}$, we immediately have

Proposition 4.1. *The map T is a C^∞ volume preserving diffeomorphism of $\breve{\mathcal{Z}}$ satisfying:*

(1) given $\delta_T > 0$, one can choose the function κ such that $\|T - Id\|_{C^r} \leq \delta_T$; moreover, T is homotopic to Id;
(2) T preserves the fibers $\mathcal{N} \times \{y\}$;
(3) T is pointwise partially hyperbolic on $\mathcal{Z}$ with one-dimensional stable E^s_T, one-dimensional unstable E^u_T and two-dimensional center E^c_T subspaces; the subspaces E^s_T and E^u_T are integrable to strongly stable and unstable foliations W^s_T and W^u_T with smooth leaves; these foliations are uniformly transversal and their local leaves have uniform size; in addition, these foliations are absolute continuous;
(4) T is uniformly partially hyperbolic on any invariant subset $\mathcal{N} \times J$ where $J \subset I$ is a closed subinterval; moreover, T is dynamically coherent with the center foliation $W^c_T = W^c_{T^\tau} \times \breve{I}$;
(5) $T|(\breve{\mathcal{Z}} \backslash \mathcal{Z}) = Id$ and $dT_z = Id$ for all $z \in \breve{\mathcal{Z}} \backslash \mathcal{Z}$; in particular, the Lyapunov exponents of $T|(\breve{\mathcal{Z}} \backslash \mathcal{Z})$ are all zero;
(6) for every $z = ((x, t), y) \in \mathcal{Z}$, the Lyapunov exponents of T are as follows:

$$\begin{aligned} \lambda_1(z, T) = \lambda^u(z, T) = \kappa(y) \log \eta_A > 0 = \lambda_2(z, T) = \lambda_3(z, T) \\ > \lambda_4(z, T) = \lambda^s(z, T) = -\kappa(y) \log \eta_A. \end{aligned}$$

where $\lambda^u(z, T)$ and $\lambda^s(z, T)$ corresponds to the direction E^u_T and E^s_T respectively and $\lambda_2(z, T)$ and $\lambda_3(z, T)$ corresponds to the direction of the flow T^τ and the I-direction respectively. Consequently,

$$L_1(T) = L_2(T) = L_3(T) > 0 = L_4(T),$$

where $L_k(\cdot)$ is the k-th average Lyapunov exponent given by (2.1).

We say that a diffeomorphism $F : \breve{\mathcal{Z}} \to \breve{\mathcal{Z}}$ is a *gentle perturbation* of T if

(1) F is C^1 close to T;
(2) $F(\mathcal{Z}) = \mathcal{Z}$ and F is pointwise partially hyperbolic in $\mathcal{Z}$;

(3) the one-dimensional strongly stable and unstable subbundles of F are integrable to one-dimensional strongly stable and unstable foliations with smooth leaves on $\mathcal{Z}$; the two-dimensional central subbundle of F is integrable to a central foliation;

(4) $F|(\breve{\mathcal{Z}}\backslash\mathcal{Z}) = Id$.

Let $F : \breve{\mathcal{Z}} \to \breve{\mathcal{Z}}$ be a diffeomorphism that is C^1 close to T. Given any closed interval $J \subset I$, set $\Lambda = \mathcal{N} \times J$. Assume that $F = T$ on $\breve{\mathcal{Z}}\backslash\Lambda$, in particular, Λ is invariant under F, then F is a gentle perturbation of T and in fact, $F|\Lambda$ is uniformly partially hyperbolic.

4.2. **Step 2: The Perturbation Q.** In this step we try to create a gentle perturbation Q of T with nonzero Lyapunov exponents, more precisely, one negative and three positive average Lyapunov exponents. However, Q is not necessarily ergodic.

We take $I_0 = (-0.5, 0.5) \subset \widetilde{I} \subset I$, and $\mathcal{Z}_0 = \mathcal{N} \times I_0$. The following statements describe properties of the map Q, which will be proved in section 5.

Proposition 4.2. *Given $\delta_Q > 0$, there exists a C^∞ volume preserving diffeomorphism Q of $\breve{\mathcal{Z}}$ satisfying:*

(1) $\|Q - T\|_{C^r} \le \delta_Q$ and Q is homotopic to T;

(2) $Q = T$ on the set $\breve{\mathcal{Z}}\backslash\mathcal{Z}_0$; in particular, Q preserves the fibers $\mathcal{N}\times\{y\}$ if $y \in \breve{I}\backslash I_0$, and Q is a gentle perturbation of T.

(3) Q satisfies statements (3)-(5) of Proposition 4.1;

(4) for any $z \in \breve{\mathcal{Z}}$ we have

$$E_Q^{uty}(z) = E_T^{uty}(z), \quad \det(dQ|E_Q^{uty}(z)) = \det(dT|E_T^{uty}(z));$$

(5) $L_1(Q) < L_2(Q) < L_3(Q) = L_3(T)$ and $L_4(Q) = 0$, where $L_k(\cdot)$ is given by (2.1).

4.3. **Step 3: The Final Perturbation H.** We go on to perturb the map Q to a map H that is pointwise partially hyperbolic on $\mathcal{Z}$, and possesses two transversal stable and unstable foliations W_H^s and W_H^u of $\mathcal{Z}$. Furthermore, we will ensure that $H|\mathcal{Z}$ has the accessibility property via these two transversal foliations. We shall also show that H can be constructed in such a way that $\int_{\mathcal{Z}} \lambda_i(z, H)dm(z) > 0$ for $i = 1, 2, 3$, and hence $H|\mathcal{Z}$ has positive central exponents. Then we can get the ergodicity of H by Theorem 2.2.

To effect our construction of H, we choose four sequences of subintervals of I as follows: for $n = 0, 1, 2, \dots$, set

$$I_n = (-1 + \frac{1}{2^{n+1}}, 1 - \frac{1}{2^{n+1}}), \quad \breve{I}_n = (-1 + \frac{3}{2^{n+3}}, 1 - \frac{3}{2^{n+3}}), \tag{4.1}$$
$$\bar{I}_n = (-1 + \frac{7}{2^{n+4}}, 1 - \frac{7}{2^{n+4}}), \quad \widetilde{I}_n = (-1 + \frac{15}{2^{n+5}}, 1 - \frac{15}{2^{n+5}}).$$

Clearly we have $I_n \subset \widetilde{I}_n \subset \bar{I}_n \subset \breve{I}_n \subset I_{n+1}$ and $\cup_{n\ge0} I_n = I$. We set

$$\mathcal{Z}_n = \mathcal{N} \times I_n, \quad \breve{\mathcal{Z}}_n = \mathcal{N} \times \breve{I}_n, \tag{4.2}$$
$$\bar{\mathcal{Z}}_n = \mathcal{N} \times \bar{I}_n, \quad \widetilde{\mathcal{Z}}_n = \mathcal{N} \times \widetilde{I}_n,$$

Apparently $\mathcal{Z}_n \subset \widetilde{\mathcal{Z}}_n \subset \bar{\mathcal{Z}}_n \subset \breve{\mathcal{Z}}_n$, and each of these sequences of sets exhausts $\mathcal{Z}$. We will construct a sequence of diffeomorphisms H_n, $n = 0, 1, 2, \dots$, with the following properties:

Proposition 4.3. *Given $\delta_H > 0$, there exist two sequences of positive numbers δ_n with $\delta_n \le \delta_H/2^n$ and $\delta_n \le d(I_n, \partial I)^2 = 1/2^{2n+2}$, and θ_n and a sequence of C^∞ volume preserving diffeomorphisms $H_n : \breve{\mathcal{Z}} \to \breve{\mathcal{Z}}$, $n = 0, 1, 2, \dots$, such that*

(1) $\|H_0 - Q\|_{C^r} \le \delta_0$, and $\|H_n - H_{n-1}\|_{C^{r+n}} \le \delta_n$ for $n \ge 1$; moreover, H_n is homotopic to Q;

(2) $H_n(\breve{\mathcal{Z}}_n) = \breve{\mathcal{Z}}_n$, $H_n = T$ on $\breve{\mathcal{Z}} \backslash \breve{\mathcal{Z}}_n$, and $H_n = H_{n-1}$ on $\breve{\mathcal{Z}}_{n-2}$; in particular, H_n is a gentle perturbation of T;

(3) H_n satisfies statements (3)-(5) of Proposition 4.1;

(4) for every $z \in \breve{\mathcal{Z}}$,

$$E^{uty}_{H_n}(z) = E^{uty}_Q(z), \quad \det(dH_n | E^{uty}_{H_n}(z)) = \det(dQ | E^{uty}_Q(z));$$

(5) for all $z \in \breve{Z}_j$, $j = 0, \dots, n$ and $i = u, s, c$,

$$\angle(E^i_{H_{n+1}}(z), E^i_{H_n}(z)) \le \theta_j / 2^{n-j}.$$

(6) if the number $\delta_Q > 0$ (in Proposition 4.2) is sufficiently small, then each map H_n is stably accessible in the following sense: let $H^\natural$ be a C^2 volume preserving diffeomorphism of $\breve{\mathcal{Z}}$ that is a gentle perturbation of T; assume for all $z \in \breve{\mathcal{Z}}_n$ and $i = u, s, c$,

$$\angle(E^i_{H^\natural}(z), E^i_{H_n}(z)) \le \theta_n;$$

then any two points $z_1, z_2 \in \widetilde{\mathcal{Z}}_n$ are accessible via a $(u,s)_{H^\natural}$-path in $\mathcal{Z}$; in particular, H_n has the accessibility property on $\widetilde{\mathcal{Z}}_n$.

We will prove the previous proposition in section 6. By statement (1) and (2) of Proposition 4.3, we can take the uniform limit $H = \lim_{n\to\infty} H_n$, which will be the desired map in Proposition 3.2. The proof is essentially given in section 3.5 of [12], so we just outline it here.

Proof of Proposition 3.2. First by Proposition 4.3 (1), we can show that H_n converges to H in the C^k topology for any $k \in \mathbb{N}$, and hence H is a C^∞ diffeomorphism. Clearly H is volume preserving. Also we can have $\|H - Id\|_{C^r} \le \delta$ if we choose sufficiently small numbers $\delta_T, \delta_Q, \delta_H$. Moreover, $H = H_n$ on $\breve{\mathcal{Z}}_{n-1}$, then H is homotopic to Q, to T and hence to Id. Also, since H_n is pointwise partially hyperbolic, H is also pointwise partially hyperbolic with one-dimensional strongly stable E^s_H and unstable E^u_H subbundles. One can show that the Lyapunov exponents $\lambda^s(z) < 0$ in the direction $E^s_H(z)$ and $\lambda^u(z) > 0$ in the direction $E^u_H(z)$ for almost every point $z \in \mathcal{Z}$.

By Proposition 4.3(3),(5) for any $z \in \mathcal{Z}$, the strongly stable local manifolds $V^s_{H_n}(z)$ have uniform size and converges in C^1 topology to a local manifold, which gives the strongly stable local manifold of H at z. Similarly one can get the strongly unstable local manifold of H in this way. Hence the strongly stable E^s_H and unstable E^u_H subbundles are integrable to strongly stable W^s_H and unstable W^u_H foliations with smooth leaves, which are transversal and absolute continuous (see [1]).

To get the accessibility property of H via W^s_H and W^u_H, one use Proposition 4.3 (6) to show that $\angle(E^i_H(z), E^i_{H_n}(z)) \le \theta_n$ for $z \in \breve{\mathcal{Z}}_n$, $i = u, s, c$, and so H has accessibility property on each $\breve{\mathcal{Z}}_n$, hence on $\mathcal{Z}$.

To show that H has positive central Lyapunov exponents, by Proposition 4.3 (4), semicontinuity of $L_k(\cdot)$ (given by (2.1)) and the fact $L_3(Q) - L_2(Q) > 0$, one can show that

$$L_3(H) - L_2(H) = \int \lambda_3(z, H)\, dm(z) > 0.$$

If follows that there is a subset $\mathcal{A} \subset \mathcal{Z}$ of positive volume such that $\lambda_1(z) \geq \lambda_2(z) \geq \lambda_3(z) > 0$ for all $z \in \mathcal{A}$. And since H preserves volume, one must have $\lambda_4(z) < 0$ for all $z \in \mathcal{A}$.

Now by Theorem 2.2, we obtain that H has positive central exponents almost everywhere in $\mathcal{Z}$, and $H|\mathcal{Z}$ is ergodic and indeed, is a Bernoulli diffeomorphism.

Finally by Proposition 4.3 (3) and the fact that $\delta_n \leq d(I_n, \partial I)^2$, we have $H = Id$ on $\breve{\mathcal{Z}} \backslash \mathcal{Z}$ and $dH_z = Id$ for all $z \in \breve{\mathcal{Z}} \backslash \mathcal{Z}$. This completes the proof of Proposition 3.2. □

5. **Construction of The Map Q: Proof of Proposition 4.2.** In this part, using an approach similar to the ones in [13, 12], we obtain the map Q as two consecutive gentle perturbations of T as follows:

$$T \xrightarrow{h_S} S = h_S \circ T \xrightarrow{h_Q} Q = h_Q \circ S,$$

where h_S, h_Q are two C^∞ volume preserving diffeomorphisms of $\breve{\mathcal{Z}}$, which are close to Id. Moreover, h_S and h_Q are concentrated on disjoint small open subsets Ω_S and Ω_Q of $\mathcal{Z}_0$ respectively. If follows that $Q = T$ outside $\Omega_S \cup \Omega_Q$. By this construction, we shall show that $S = h_S \circ T$ has two positive average Lyapunov exponents in the E_T^{ut} subbundle, i.e. $L_1(S) < L_2(S)$), and $Q = h_Q \circ S$ has three positive average Lyapunov exponents, i.e. $L_1(Q) < L_2(Q) < L_3(Q)$.

First note that given $z \in \breve{\mathcal{Z}}$, we can choose a local coordinate system (s, u, t, y) associated to T, i.e.

$$F^s(z) := \frac{\partial}{\partial s} = E_T^s(z), \quad F^u(z) := \frac{\partial}{\partial u} = E_T^u(z), \quad F^t(z) := \frac{\partial}{\partial t} \tag{5.1}$$

are the stable, unstable and central flow direction of T respectively, and

$$F^y(z) := \frac{\partial}{\partial y} \tag{5.2}$$

is tangent to $\breve{I} = (-2, 2)$ as introduced in Subsection 4.1.

Choose periodic points p, p^t, p^y of the Anosov automorphism A of X, which are close to each other and whose orbits are pairwise disjoint. Let $V_A^s(p), V_A^u(p), V_A^s(p^i)$ and $V_A^u(p^i)$, $i = t, y$ be the stable and unstable local manifolds at these periodic points. We may assume that each intersection $V_A^u(p) \cap V_A^s(p^i)$ and $V_A^u(p^i) \cap V_A^s(p)$ consists of exactly one point, denoted by $[p, p^i]$ and $[p^i, p]$ respectively. Let γ^i denote the closed quadrilateral path with the collection of points p, $[p, p^i]$, p^i, $[p^i, p]$ and p, and let

$$\gamma(p) = V_A^u(p) \cup V_A^s(p), \quad \gamma(p^i) = V_A^u(p^i) \cup V_A^s(p^i).$$

Choose sufficiently small number $\nu > 0$, and set for $i = t, y$,

$$\begin{aligned}
\Omega^i(\nu) &= \Big(\bigcup_{t \in [0, \tau(p^i)]} B_{\mathcal{N}}(T^t(p^i, 0), \nu)\Big) \times I \\
\widehat{\Omega}^i(\nu) &= \Big(\bigcup_{(x,t) \in (\gamma(p) \times [0, \tau(p)]) \cup (\gamma(p^i) \times [0, \tau(p^i)])} B_{\mathcal{N}}((x, t), \nu)\Big) \times I \\
\Omega(\nu) &= \Big(\bigcup_{i = t, y} \Omega^i(\nu)\Big) \cup \Big(\bigcup_{i = t, y} \widehat{\Omega}^i(\nu)\Big).
\end{aligned}$$

where $\tau(p)$ and $\tau(p^i)$ are the periods of p and p^i, and $B_{\mathcal{N}}((x, t), r)$ is the ball in $\mathcal{N}$ of radius r centered at $(x, t) \in \mathcal{N}$. Finally, we set

$$\Omega_0 = \Omega_0(\nu) = \Omega(\nu) \cap \mathcal{Z}_0 \tag{5.3}$$

According to sublemma 5.2, given $\epsilon = \delta_Q/2 > 0$, where δ_Q is given in Proposition 4.2, we can choose $\theta_0 > 0$ and an integer $k_0 > 0$ such that

$$\theta = \pi/2k_0 < \theta_0 \tag{5.4}$$

Now choose ν small enough to ensure that

$$20k_0 m(\Omega_0(\nu)) < 1 \tag{5.5}$$

5.1. **Construction of The Map S.** We shall obtain the map S from T via a small perturbation h_S, which is a small rotation in the E_T^{ut} subbundle on a small subset of $\mathcal{Z}_0$.

Recall that $T((x,t),y) = (T^{\kappa_0}((x,t)),y)$ at every $z = ((x,t),y) \in \mathcal{Z}_0$, and the expansion rate in the E_T^u-direction at z is the constant $\eta = \eta_A^{\kappa_0}$. Given $N_0 \geq 20k_0$, pick a point $(x_0,t_0) \in \mathcal{N}$ and $\varepsilon_1 > 0$ such that

$$B_{\mathcal{N}}((x_0,t_0),2\varepsilon_1) \cap \mathrm{Proj}_{\mathcal{N}}\Omega_0 = \emptyset;$$

$$T^{-k\kappa_0}B_{\mathcal{N}}((x_0,t_0),2\varepsilon_1) \cap B_{\mathcal{N}}((x_0,t_0),2\varepsilon_1) = \emptyset, \quad k = 1,\dots,N_0,$$

where $\mathrm{Proj}_{\mathcal{N}}$ is the projection of $\breve{\mathcal{Z}}$ onto $\mathcal{N}$. Now set

$$\Omega_S = B_{\mathcal{N}}((x_0,t_0),\varepsilon_1) \times I_0. \tag{5.6}$$

It is apparent that

$$\Omega_S \cap \Omega_0 = \emptyset, \quad T^{-k}\Omega_S \cap \Omega_S = \emptyset, k = 1,\dots,N_0. \tag{5.7}$$

Choose a C^∞ function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that
(1) $\psi(r) = \psi_0 > 0$ if $0 \leq r \leq 0.9$;
(2) $\psi(r) > 0$ if $0 \leq r < 1$ and $\psi(r) = 0$ if $r \geq 1$;
(3) $\|\psi\|_{C^r} \leq 1$.

Centered at $((x_0,t_0),0) \in \Omega_S$, we switch from the local Cartesian coordinate system (u,t,y,s) to the local cylindrical coordinate system (r,θ,y,s), where $u = r\cos\theta$, $t = r\sin\theta$. Given $\tau > 0$, define a small rotation $h_{S,\tau}$ in the E_T^{ut} subbundle on Ω_S by

$$h_{S,\tau}(r,\theta,y,s) = (r,\theta+\tau\sigma,y,s), \tag{5.8}$$

where

$$\sigma = \sigma(r,s,y) = 0.25\varepsilon_1^2\psi(\frac{r}{\varepsilon_1})\psi(\frac{|s|}{\varepsilon_1})\psi(2|y|).$$

And we can extend $h_{S,\tau}$ to $\breve{\mathcal{Z}}$ by simply letting $h_{S,\tau} = Id$ outside Ω_S. Clearly h_τ is a C^∞ volume preserving diffeomorphism such that
(1) $\|h_{S,\tau} - Id\|_{C^r} \to 0$ as $\tau \to 0$;
(2) $dh_{S,\tau}$ preserves E_T^{ut} subbundle;
(3) $\det(dh_{S,\tau}|E_T^{ut}(z)) = 1$ for any $z \in \breve{\mathcal{Z}}$.

We then define $S_\tau = T \circ h_{S,\tau}$. Also we set $I_0' = 0.9I_0 = (-0.45, 0.45)$.

Lemma 5.1. *Given $0 < \delta_S < \delta_Q/2$, there exists $\tau > 0$ such that the map $S = S_\tau$ is a C^∞ volume preserving diffeomorphism of $\breve{\mathcal{Z}}$ satisfying:*
(1) $\|S - T\|_{C^r} \leq \delta_S$ and S is homotopic to T;
(2) $S = T$ outside Ω_S, in particular, $S = T$ on the sets $\breve{\mathcal{Z}}\backslash\mathcal{Z}_0$ and Ω_0, which indicates that S is a gentle perturbation of T;
(3) S satisfies statements (3)-(5) of Proposition 4.1;
(4) for any $z \in \breve{\mathcal{Z}}$,

$$E_S^{ut}(z) = E_T^{ut}(z), \quad \det(dS|E_S^{ut}(z)) = \det(dT|E_T^{ut}(z));$$

(5) for any $y_1, y_2 \in I_0'$,

$$\mathrm{Proj}_{\mathcal{N}}(S((x,t),y_1)) = \mathrm{Proj}_{\mathcal{N}}(S((x,t),y_2));$$

(6) $L_1(S) < L_1(T)$ *and hence,*

$$L_1(S) < L_2(S) = L_3(S) = L_3(T) > 0 = L_4(S) = L_4(T);$$

(7) there exists a number $\lambda_S > 0$ *and a set* $\Pi_S = \mathrm{Proj}_{\mathcal{N}}\Pi_S \times I_0'$ *such that*

$$m(\Pi_S) \geq 20k_0 m(\Pi_S \cap \Omega_S) > 0,$$

where m *is the Riemannian volume on* $\breve{\mathcal{Z}}$*, and for any* $z \in \Pi_S$ *the map* S *has two positive Lyapunov exponents* $\lambda_1(z,S), \lambda_2(z,S) \geq \lambda_S$ *along the* $E_S^{ut} = E_T^{ut}$ *subbundle.*

Proof. The proof is an adaptation of arguments in [4, 12] to our case. We shall just outline the proof here. Statements (1)-(5) follows easily from the construction of the map $h_{S,\tau}$ when τ is sufficiently small. Moreover, S is dynamical coherent by Theorem 2.1. It remains to show Statements (6) and (7).

For statement (6), since $S = S_\tau = T$ on $\breve{\mathcal{Z}} \backslash \mathcal{Z}_0$, we only need to show that

$$L_1(S_\tau|\mathcal{Z}_0) < L_1(T|\mathcal{Z}_0) \tag{5.9}$$

for sufficiently small τ. It is easy to see that

$$L_1(S_\tau|\mathcal{Z}_0) = \int_{\mathcal{Z}_0} \lambda_1(z,S_\tau)dm(z) = \int_{\mathcal{Z}_0} \log|dS_\tau(z)|E^u_{S_\tau}(z)|dm(z).$$

By the fact that $h_{S,\tau}$ and S_τ preserve the E_T^{ut}-subbundle, we denote by $e_\tau(z)$ the unique number such that the vector $v_\tau(z) = (1, e_\tau(z), 0, 0)^t \in E^u_{S_\tau}(z)$ for all $z \in \mathcal{Z}$ in the (u,t,y,s) local coordinate system. Also for $z \in \Omega_S \subset \mathcal{Z}_0$, we have

$$dT|E_T^{ut}(z) = \begin{pmatrix} \eta & 0 \\ 0 & 1 \end{pmatrix}, \quad dh_\tau|E_T^{ut}(z) = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where

$$\begin{aligned} A = A(\tau,z) &= 1 - \tau r\sigma_r \sin\theta\cos\theta - \frac{\tau^2\sigma^2}{2} - \tau^2 r\sigma\sigma_r\cos^2\theta + O(\tau^3), \\ B = B(\tau,z) &= -\tau\sigma - \tau r\sigma_r\sin^2\theta - \tau^2 r\sigma\sigma_r\sin\theta\cos\theta + O(\tau^3), \\ C = C(\tau,z) &= \tau\sigma + \tau r\sigma_r\cos^2\theta - \tau^2 r\sigma\sigma_r\sin\theta\cos\theta + O(\tau^3), \\ D = D(\tau,z) &= 1 + \tau r\sigma_r\sin\theta\cos\theta - \frac{\tau^2\sigma^2}{2} - \tau^2 r\sigma\sigma_r\sin^2\theta + O(\tau^3), \end{aligned}$$

and hence

$$dS_\tau|E_T^{ut}(z) = \begin{pmatrix} \eta A & \eta B \\ C & D \end{pmatrix}.$$

Repeating the arguments of Lemma B.7 in [4], one can show that

$$L_\tau = L_1(S_\tau|\mathcal{Z}_0) = \int_{\mathcal{Z}_0} \log\eta dm(z) - \int_{\mathcal{Z}_0} \log[D(\tau,z) - \eta B(\tau,z)e_\tau(S_\tau(z))]dm(z).$$

Note that $D(0,z) = 1$ and $B(0,z) = 0$, we get

$$L_0 = \int_{\mathcal{Z}_0} \log\eta dm(z) = L_1(T|\mathcal{Z}_0).$$

Applying the same arguments in Lemma B.8 in [4], we can show

$$\frac{dL_\tau}{d\tau}|_{\tau=0} = 0, \quad \frac{d^2L_\tau}{d\tau^2}|_{\tau=0} < 0,$$

which immediately implies that (5.9) for sufficiently small $\tau > 0$.

In fact, one get that

$$\frac{dL_\tau}{d\tau}|_{\tau=0} = -\int_{\mathcal{Z}_0} D_\tau|_{\tau=0}\ dm(z) = 0$$

and

$$\frac{d^2L_\tau}{d\tau^2}|_{\tau=0} = \int_{\mathcal{Z}_0} [(D_\tau)^2 - D_{\tau\tau} + 2\eta B_\tau \frac{\partial}{\partial \tau}(e_\tau(S_\tau(z)))]_{\tau=0}\ dm(z).$$

Similar to Lemma B.9 in [4], this integral can be written as

$$\begin{aligned} &\int_{\mathcal{Z}_0} [(D_\tau(0,z))^2 - D_{\tau\tau}(0,z) + 2\eta B_\tau(0,z)C_\tau(0,z)]\ dm(z) \\ + \quad &\int_{\mathcal{Z}_0} \sum_{i=1}^{\infty} \frac{1}{\eta^i} 2B_\tau(0,z)C_\tau(0,T^{-i}(z))\ dm(z). \end{aligned}$$

The first term is bounded above by

$$-(1-\varepsilon_1)\int_{\mathcal{Z}_0} \sigma^2 dm(z) - \frac{1}{8}\int_{\mathcal{Z}_0} r^2\sigma_r^2 dm(z).$$

For the second term, note that

$$\int_{\mathcal{Z}_0} 2B_\tau(0,z)C_\tau(0,T^{-i}(z))dm(z) \le 4\int_{\mathcal{Z}_0} (\sigma^2 + r^2\sigma_r^2)\ dm(z),$$

and $B_\tau(0,z)C_\tau(0,T^{-i}(z)) = 0$ for all $z \in \mathcal{Z}_0 \backslash \Omega_S$ and any i. Also note that $B_\tau(0,z)C_\tau(0,T^{-i}(z)) = 0$ at every $z \in \Omega_S$ for $i = 1,\dots,N_0-1$ since $T^{-i}\Omega_S \cap \Omega_S = \emptyset$. This allows we take $N_0 > 0$ big enough such that the second term is bounded by

$$\frac{1}{10}\int_{\Omega_S} (\sigma^2 + r^2\sigma_r^2)\ dm(z).$$

Hence

$$\frac{d^2L_\tau}{d\tau^2}|_{\tau=0} \le -(\frac{9}{10} - \varepsilon_1)\int_{\mathcal{Z}_0} \sigma^2 dm(z) - \frac{1}{40}\int_{\Omega_S} r^2\sigma_r^2 dm(z) < 0.$$

It follows that $L_1(S_\tau) < L_1(T)$ for sufficiently small $\tau > 0$. Moreover, since $\det(dS_\tau|E_T^i) = \det(dT|E_T^i)$ for $i = ut, uty$ and S_τ is volume preserving, we have $L_i(S_\tau) = L_i(T)$ for $i = 2, 3$, and $L_4(S_\tau) = 0$, and hence

$$L_1(S) < L_2(S) = L_3(S) = L_3(T) > 0 = L_4(S) = L_4(T).$$

To prove statement (7) we notice that for any S_τ preserves the fiber $\mathcal{N} \times \{y\}$ for any $y \in I$, and by statement (5), we know that for any $y_1, y_2 \in I_0'$,

$$\mathrm{Proj}_{\mathcal{N}}(S((x,t),y_1)) = \mathrm{Proj}_{\mathcal{N}}(S((x,t),y_2)),$$

in other words, the action of S_τ on the fiber $\mathcal{N} \times \{y\}$ are the same for each $y \in I_0'$. Then by the same arguments as above, one could have $L_1(S_\tau|\mathcal{N}\times\{y\}) < L_2(S_\tau|\mathcal{N}\times\{y\})$ for sufficiently small τ, which does not depend on $y \in I_0'$. Moreover, for sufficiently small $\lambda_S > 0$, the sets $\Pi_S(y) = \{z \in \mathcal{N}\times\{y\} : \lambda_2(z, S_\tau) \ge \lambda_S\}$ are of the same form as $\Pi_S(y) = \mathcal{A} \times \{y\}$ for some $\mathcal{A} \subset \mathcal{N}$ of positive Lebesgue measure. Then we set $\Pi_S = \mathcal{A} \times I_0'$. Clearly Π_S is an S_τ-invariant subset. Moreover, by the fact that Π_S is also T-invariant and (5.7), we have

$$m(\Pi_S) \ge \sum_{k=1}^{N_0} m(\Pi_S \cap T^{-k}\Omega_S) = N_0 m(Pi_S \cap \Omega_S) \ge 20k_0 m(\Pi_S \cap \Omega_S).$$

This completes the proof of Statement (7). □

5.2. **Construction of The Map Q.** Following [12], we go on to perturb S to the map Q via a diffeomorphism h_Q. This time we construct h_Q as a composition of rotations in F^{ty}-subbundle on several pairwise disjoint cylinders in Π_S (in Lemma 5.1), which gives a total rotation $\pi/2$. In this way we gain positive central exponents in both central directions F^t and F^y for the map Q. The technique we shall use here is the Rokhlin Halmos tower construction for S on the measurable set Π_S, which allows us to do rotations on finitely many small cylinders.

Let $\lambda = \lambda_S$ and $\Pi = \Pi_S$ be as in Lemma 5.1. Given $K \in \mathbb{N}$, set

$$\begin{aligned}\Lambda' = \Lambda'(K) \quad &= \quad \{z \in \Pi : \left|\frac{1}{k}\log\|dS^k(z,v)\| - \lambda\right| \le 0.1\lambda, \\ &\qquad \text{for all } v \in E^{ut}_S(z), \|v\| = 1 \text{ and all } |k| \ge 0.5K\} \end{aligned} \tag{5.10}$$

and also

$$\Lambda = \Lambda(K) = \bigcap_{i=0}^{k_0-1} S^{-i}\Lambda'(K), \tag{5.11}$$

where k_0 is given by (5.4). Since $m(\Lambda'(K)) \to m(\Pi)$ as $K \to \infty$, we also have $m(\Lambda(K)) \to m(\Pi)$ as $K \to \infty$. Remember that δ_T and δ_Q are given in Proposition 4.1 and 4.2 respectively, and one can choose K so large that

$$K\lambda \ge \max\{5k_0\lambda, 10\log 2, -10k_0\log(1-\delta_T-\delta_Q)\}, \tag{5.12}$$

$$\lambda m(\Pi) + 40\log(1-\delta_T-\delta_Q)m(\Pi\backslash\Lambda) > 0, \tag{5.13}$$

$$20m(\Pi\backslash\Lambda) \le m(\Pi). \tag{5.14}$$

Set

$$\Lambda^* = \Lambda \backslash \cup_{i=0}^{k_0-1} S^{-i}(\Omega_0 \cup \Omega_S) \tag{5.15}$$

where Ω_0 and Ω_S are given by (5.3) and (5.7) respectively. By Lemma 5.1 (7), and also choosing ν in (5.3) small enough, we have

$$m(\Omega_S \cap \Pi) \le m(\Pi)/20k_0, \quad m(\Omega_0 \cap \Pi) \le m(\Pi)/20k_0 \tag{5.16}$$

Combining above relations, we find that $m(\Lambda^*) \ge 0.8m(\Pi)$.

Now let us use Rokhlin-Halmos towers (see [14]) to approximate the measurable set Π. More precisely, we can choose a measurable subset $\Gamma' \subset \Pi$ such that $S^i\Gamma'$ are pairwise disjoint for $-K \le i \le 6K + k_0 - 1$ and

$$m\left(\biguplus_{i=-K}^{6K+k_0-1} S^i\Gamma'\right) \ge 0.9m(\Pi) \tag{5.17}$$

Take Γ_0 as the set of first entries to Λ^* of the trajectories $\{S^i(z)\}_{i=0}^{5K-1}$ with $z \in \Gamma'$, i.e.

$$\Gamma_0 = \{S^j(z) : z \in \Gamma', 0 \le j \le 5K-1, S^j(z) \in \Lambda^*, S^i(z) \in \Lambda^* \text{ for } i < j\}$$

By Lemma 5.1 (5), one can find that $\Gamma_0 = \mathrm{Proj}_{\mathcal{N}}\Gamma_0 \times I_0'$. Furthermore, we can approximate Γ_0 by finitely many disjoint cylinders Σ_{0j} of the form

$$\Sigma = B_{F^u}(u_j, r_j') \times B_{F^s}(s_j, r_j'') \times B_{F^{ty}}((t_j, y_j), r_j)$$

where $z_j = (u_j, s_j, t_j, y_j) \in \breve{\mathcal{Z}}$, $r'_j \geq r_j$, $r''_j \geq r_j \eta^{\kappa_0}$, and $j = 1, \ldots, J$. Let

$$\Gamma_i = S^i \Gamma_0, \Gamma = \biguplus_{i=-K}^{K+k_0-1} \Gamma_i, \tag{5.18}$$

where Γ_i are pairwise disjoint for $-K \leq i \leq K + k_0 - 1$. Then we can also approximate Γ_i by $\Sigma_{ij} = S^i \Sigma_{0j}$, $j = 1, \ldots, J$, which are also cylinders. Moreover, we may choose the sets Σ_{0j} in such a way that

$$\Sigma_{ij} \cap \Sigma_{kl} = \emptyset, \ (i,j) \neq (k,l), -K \leq i, k \leq K + k_0 - 1, 1 \leq j, l \leq J,$$

and $\Sigma_{ij} \cap (\Omega_0 \cup \Omega_S) = \emptyset$ for $0 \leq i \leq k_0 - 1$, $1 \leq j \leq J$. Set $\Delta_i = \biguplus_{j=1}^{J} \Sigma_{ij}$ for $i = 0, \ldots, k_0 - 1$, we can also assume that

$$m(\Gamma_i \triangle \Delta_i) \leq 0.05 \max\{m(\Gamma_i), m(\Delta_i)\} \tag{5.19}$$

We need the following sublemma (see sublemma 4.5 in [12]) to construct the perturbation h_Q.

Sublemma 5.2. *For any $\epsilon > 0$, there is $\theta_0 > 0$ such that for any $0 \leq \theta \leq \theta_0$, any cylinder $\Sigma \subset \mathbb{R}^4$ of the form*

$$\Sigma = B_1(z_1, s_1) \times B_2(z_2, s_2) \times B_{34}((z_3, z_4), s_3) \text{ with } s_1, s_2 \geq s_3,$$

there exists a subcylinder $\Sigma' \subset \Sigma$ of the form

$$\Sigma' = B_1(z_1, s'_1) \times B_2(z_2, s'_2) \times B_{34}((z_3, z_4), s'_3)$$

and a C^∞ map $\rho : \mathbb{R}^4 \to \mathbb{R}^4$ satisfying:

1. *ρ is a rotation by the angle θ in $z_3 z_4$-plane on Σ', i.e.*
$$\rho(z_1, z_2, z_3, z_4) = (z_1, z_2, z_3 \cos\theta - z_4 \sin\theta, z_3 \sin\theta + z_4 \cos\theta);$$
2. *$\rho = Id$ outside Σ;*
3. *$m(\Sigma')/m(\Sigma) \geq 0.75$;*
4. *$s'_i/s_i \geq 0.9$ for $i = 1, 2, 3$;*
5. *$\|\rho - Id\|_{C^r} \leq \epsilon$, where r is in Proposition 3.2.*

Applying this sublemma on each cylinder Σ_{ij}, $i = 0, \ldots, k_0 - 1$, $j = 1, \ldots, J$, we obtain a map ρ_{ij} and a subcylinder $\Sigma'_{ij} \subset \Sigma_{ij}$ such that $\|\rho_{ij} - Id\| \leq \delta_Q/2$ and $m(\Sigma'_{ij})/m(\Sigma_{ij}) \geq 0.75$. Moreover, by (5.4), one can make $\rho_{ij}|\Sigma'_{ij}$ the rotation by the angle $\theta = \pi/2k_0$ along the F^{ty}-subspace and $\rho_{ij} = Id$ outside Σ_{ij}. With good choices of Σ'_{ij} one may assume that $S(\Sigma'_{ij}) = \Sigma'_{i+1,j}$ for $i = 0, \ldots, k_0 - 1$. Let

$$\Omega_Q = \biguplus_{i=0}^{k_0-1} \Delta_i, \quad \Delta'_i = \biguplus_{j=1}^{J} \Sigma'_{ij} \tag{5.20}$$

then $m(\Delta'_i)/m(\Delta_i) \geq 0.75$. Define $h_Q = \rho_{ij}$ on each Σ_{ij}, and $h_Q = Id$ otherwise. Clearly h_Q is a C^∞ volume preserving diffeomorphism and dh_Q preserves E_T^{uty} subbundle with $\det(dh_Q|E_T^{uty}(z)) = 1$ for any $z \in \breve{\mathcal{Z}}$. Finally define $Q = h_Q \circ S$, and we ready to show that Q is the desired map in Proposition 4.2.

Proof of Proposition 4.2. We give a sketch of the proof here, see section 4.2 in [12] for more details. By the construction of Q, statements (1)-(4) hold immediately. To prove statement (5), set $\Delta_0^* = \Delta'_0 \cap \Lambda$, and

$$U_1 = Q^{-K} \Delta_0^*, \qquad U_2 = \Delta_0 \backslash \Delta_0^*,$$
$$U_3 = Q^{k_0}((\Delta_0 \cap \Lambda) \backslash \Delta_0^*), \quad U_4 = Q^{k_0}(\Delta_0 \backslash \Lambda)$$

and consider the first return map $\overline{Q} = Q^\beta$ on the set $U = U_1 \cup U_2 \cup U_3 \cup U_4$, where $\beta = \beta(z)$ is the first return time of $z \in U$ to U under Q. Note that $E_Q^{uty}(z) = E_S^{uty}(z) = E_T^{uty}(z)$ for any $z \in U$.

We intend to show that

$$\int_U \left(\log \| \wedge^3 (d\overline{Q}|E_T^{uta}(z))\| - \log \| \wedge^2 (d\overline{Q}|E_T^{uta}(z))\|\right) dm(z) > 0 \tag{5.21}$$

where $\wedge^k(d\overline{Q}|E_T^{uta}(z))$ is the k-th exterior power of $d\overline{Q}|E_T^{uta}(z)$. Indeed if this is the case, we take $\Pi' = \cup_{i=-\infty}^{\infty} Q^i(U)$, then for $k = 2, 3$,

$$\begin{aligned} \int_U \log \| \wedge^k (d\overline{Q}|E_T^{uta}(z))\| dm(z) &= \int_{\Pi'} \log \| \wedge^k (dQ|E_T^{uta}(z))\| dm(z) \\ &= \int_{\Pi'} \sum_{i=1}^k \lambda_i(z, Q) dm(z) = L_k(Q|\Pi'), \end{aligned}$$

and hence $L_3(Q|\Pi') > L_2(Q|\Pi')$. Since $Q = S$ outside Π', we obtain that $L_3(Q) > L_2(Q)$, which indicates statement (5) of Proposition 4.2.

To show (5.21), one can split the left-hand side into four integrals on U_i, $i = 1, 2, 3, 4$, and estimate lower bounds for each. See [12] for more details. □

6. **Construction of The Maps H_n: Proof of Proposition 4.3.** Recall that the map Q in Proposition 4.2 is pointwise partially hyperbolic with one-dimensional stable, one-dimensional unstable and two-dimensional central subbundles. Q is a gentle perturbation of T and has positive average central Lyapunov exponents on $\mathcal{Z}$, however, $Q|\mathcal{Z}$ does not have the accessibility property, which is needed for the ergodicity of $Q|\mathcal{Z}$ in view of Theorem 2.2.

To this end we construct H_n as a small gentle perturbation of Q for each $n \geq 0$, such that H_n has the accessibility property on an invariant open set $\breve{\mathcal{Z}}_n$, and is stably accessible on an open set $\mathcal{Z}_n$ (see (4.2)). Then the limit diffeomorphism H of the sequence H_n will be accessible on $\mathcal{Z}$ since $\cup_{n\geq 0}\mathcal{Z}_n = \mathcal{Z}$.

6.1. **Construction of The Maps H_n.** First we can decompose the sets $\mathcal{Z}_n$, $\breve{\mathcal{Z}}_n$, $\bar{\mathcal{Z}}_n$ and $\widetilde{\mathcal{Z}}_n$ as follows: let $J_0 = [-0.5, 0.5]$, $\breve{J}_0 = (-5/8, 5/8)$, $\bar{J}_0 = (-9/16, 9/16)$ and $\widetilde{J}_0 = (-17/32, 17/32)$. For $l \geq 1$, set

$$J_l = [1 - \frac{1}{2^l}, 1 - \frac{1}{2^{l+1}}], \quad \breve{J}_l = (1 - \frac{3}{2^{l+1}}, 1 - \frac{3}{2^{l+3}}),$$
$$\bar{J}_l = (1 - \frac{9}{2^{l+3}}, 1 - \frac{7}{2^{l+4}}), \quad \widetilde{J}_l = (1 - \frac{20}{2^{l+4}}, 1 - \frac{15}{2^{l+5}})$$

and $J_l = -J_{-l}$, $\breve{J}_l = -\breve{J}_{-l}$, $\bar{J}_l = -\bar{J}_{-l}$, $\widetilde{J}_l = -\widetilde{J}_{-l}$ for $l \leq -1$. Clearly, we have for all $l \in \mathbb{Z}$,

$$J_l \subset \widetilde{J}_l \subset \bar{J}_l \subset \breve{J}_l.$$

Also note that $\breve{J}_l \cap \breve{J}_{l+2} = \emptyset$. Moreover,

$$I_n = \bigcup_{|l|\leq n} J_l, \quad \breve{I}_n = \bigcup_{|l|\leq n} \breve{J}_l, \quad \bar{I}_n = \bigcup_{|l|\leq n} \bar{J}_l, \quad \widetilde{I}_n = \bigcup_{|l|\leq n} \widetilde{J}_l, \tag{6.1}$$

where $I_n, \breve{I}_n, \bar{I}_n, \widetilde{I}_n$ are given by (4.1). Hence we can write

$$\mathcal{Z}_n = \bigcup_{|l|\le n} \mathcal{N} \times J_l, \quad \breve{\mathcal{Z}}_n = \bigcup_{|l|\le n} \mathcal{N} \times \breve{J}_l,$$
$$\bar{\mathcal{Z}}_n = \bigcup_{|l|\le n} \mathcal{N} \times \bar{J}_l, \quad \widetilde{\mathcal{Z}}_n = \bigcup_{|l|\le n} \mathcal{N} \times \widetilde{J}_l.$$

Also we take $K = (-1/8, 1+1/8)$, $\breve{K} = (-1/4, 1+1/4)$ and $\bar{K} = (-1/16, 1+1/16)$ for the time variable t.

Pick two triples of pairwise disjoint periodic points $\{p_j, p_j^t, p_j^y\}$, $j = 0, 1$, of the Anosov automorphism A of X. We can assume that the orbits of p_0 and p_1 under A are disjoint, and choose $\epsilon_0 > 0$ such that $B_X(A^i p_0, \epsilon_0) \cap B_X(A^i p_1, \epsilon_0) = \emptyset$ for $i = -1, 0, 1$. We may also assume that $p_j^t, p_j^y \in B_X(p_j, \epsilon_0/3)$. Without loss of generality, one can take $\{p_0, p_0^t, p_0^y\} = \{p, p^t, p^y\}$ as in the construction of Ω_0 at the beginning of section 5. For ϵ_0 sufficiently small, we have that $V^u(p_j) \cap V^s(p_j^i)$, $V^s(p_j) \cap V^u(p_j^i)$ consist of exactly one point, denoted by $[p_j, p_j^i]$, $[p_j^i, p_j]$ respectively, where $i = t, y$, $j = 0, 1$.

For any integer $l \ne 0$, let $\eta_-(l) = \min\{\eta_A^{\kappa(y)} : y \in J_l\}$, set

$$\breve{\nu}_u^i(l) = d(p_j^i, [p_j^i, p_j]), \quad \breve{\nu}_s^i(l) = d(p_j^i, [p_j, p_j^i]);$$
$$\nu_u^i(l) = \breve{\nu}_u^i(l)/\eta_-(l), \qquad \nu_s^i(l) = \breve{\nu}_s^i(l)/\eta_-(l),$$

where $j \equiv l \mod 2$, and $i = t, y$. Define rectangles in X:

$$\breve{\Pi}_l^i = B_{F^u}(p_j^i, \breve{\nu}_u^i(l)) \times B_{F^s}(p_j^i, \breve{\nu}_s^i(l)), \quad \Pi_l^i = B_{F^u}(p_j^i, \nu_u^i(l)) \times B_{F^s}(p_j^i, \nu_s^i(l)).$$

In the case $l = 0$, choose the smallest numbers l_u^i and l_s^i such that

$$A^{-l_u^i}[p^i, p] \in B_X(p^i, \nu/2), \quad A^{l_s^i}[p, p^i] \in B_X(p^i, \nu/2),$$

where ν is given by (5.5), and

$$\breve{\nu}_u^i(0) = d(p^i, A^{-l_u^i}[p^i, p]), \quad \breve{\nu}_s^i(0) = d(p^i, A^{l_s^i}[p, p^i]);$$
$$\nu_u^i(0) = \breve{\nu}_u^i(0)/\eta_A^{\kappa_0}, \qquad \nu_s^i(0) = \breve{\nu}_s^i(0)/\eta_A^{\kappa_0},$$

and define rectangles centered at p^i in X:

$$\breve{\Pi}_0^i = B_{F^u}(p^i, \breve{\nu}_u^i(0)) \times B_{F^s}(p^i, \breve{\nu}_s^i(0)), \ \Pi_0^i = B_{F^u}(p^i, \nu_u^i(0)) \times B_{F^s}(p^i, \nu_s^i(0)).$$

Finally, we let

$$\breve{\epsilon}_t(l) = \max\{\kappa(y)/2 : \ y \in J_l\}, \quad \epsilon_t(l) = 5\breve{\epsilon}_t(l)/6.$$

Fix an integer l and write $\nu_\lambda^i = \nu_\lambda^i(l)$, $\breve{\nu}_\lambda^i = \breve{\nu}_\lambda^i(l)$, $i = t, y$, $\lambda = u, s$, and $\epsilon_t = \epsilon_t(l)$, $\breve{\epsilon}_t = \breve{\epsilon}_t(l)$. Choose C^∞ functions on $\mathbb{R}$ as follows:

(1) ϕ^i, ψ^i satisfying:
- $\cdot$ ϕ^i =const. on $(-\nu_u^i, \nu_u^i)$, ψ^i =const. on $(-\nu_s^i, \nu_s^i)$.
- $\cdot$ $\phi^i = 0$ outside $(-\breve{\nu}_u^i, \breve{\nu}_u^i)$, $\psi^i = 0$ outside $(-\breve{\nu}_s^i, \breve{\nu}_s^i)$.
- $\cdot$ $\int_0^{\pm\breve{\nu}_u^i} \phi^i(r)dr = 0$, and $\psi^i > 0$ on $(-\breve{\nu}_s^i, \breve{\nu}_s^i)$.
- $\cdot$ $\|\phi^i\|_{C^{r+|l|}}, \|\psi^i\|_{C^{r+|l|}} \le 1$.

(2) ξ_t, ξ_y satisfying:
- $\cdot$ ξ_t =const. on K, ξ_y =const. on J_l.
- $\cdot$ $\xi_t = 0$ outside $\breve{K}$, $\xi_y = 0$ outside $\breve{J}_l$.
- $\cdot$ $\xi_t > 0$ on $\breve{K}$, $\xi_y > 0$ on $\breve{J}_l$.

- $\|\xi_t\|_{C^{r+|l|}}, \|\xi_y\|_{C^{r+|l|}} \le 1$.

(3) ζ_t, ζ_y satisfying:
- ζ_t =const. on $(-\epsilon_t, \epsilon_t)$, ζ_y =const. on J_l.
- $\zeta_t = 0$ outside $(-\breve{\epsilon}_t, \breve{\epsilon}_t)$, $\zeta_y = 0$ outside $\breve{J}_l$.
- $\zeta_t > 0$ on $(-\breve{\epsilon}_t, \breve{\epsilon}_t)$, $\zeta_y > 0$ on $\breve{J}_l$.
- $\|\zeta_t\|_{C^{r+|l|}}, \|\zeta_y\|_{C^{r+|l|}} \le 1$.

Now Consider the box $\breve{\Omega}_l^y = \breve{\Pi}_l^y \times (\frac{1}{2} - \breve{\epsilon}_t, \frac{1}{2} + \breve{\epsilon}_t) \times \breve{J}_l$ centered at $z_l^y = (p_j^y, \frac{1}{2}, y_l)$, where $j \equiv l \mod 2$ and y_l is the middle point of J_l. Introduce the local coordinate system (u, t, y, s) originated at z_l^y, then

$$\breve{\Omega}_l^y = \{(u, t, y, s) : \ |u| \le \breve{\nu}_u^y, \ |s| \le \breve{\nu}_s^y, \ |t| < \breve{\epsilon}_t, \ y \in \breve{J}_l\}. \tag{6.2}$$

For each $\beta > 0$, define a vector field $X^y = X_{l,\beta}^y$ supported on $\breve{\Omega}_l^y$ by

$$X^y = \beta\psi^y(s)\zeta_t(t)\Big(-\xi_y'(y) \int_0^u \phi^y(r)dr, \ 0, \ \xi_y(y)\phi^y(u), \ \ 0\Big), \tag{6.3}$$

and clearly X^y is constant on the subset $\Omega_l^y = \Pi_l^y \times (\frac{1}{2} - \epsilon_t, \frac{1}{2} + \epsilon_t) \times J_l$. We define $h^y = h_{l,\beta}^y$ on $\breve{\Omega}_l^y$ to be the time-1 map of the flow generated by X^y, and set $h^y = Id$ outside $\breve{\Omega}_l^y$. Since X^y is divergence free, dh^y preserves E_T^{uy}-subbundle and $\det(dh^y|E_T^{uy}(z)) = 1$.

Similarly, take $\breve{\Omega}_l^t = \breve{\Pi}_l^t \times \breve{K} \times \breve{J}_l$ centered at $z_l^t = (p_j^t, \frac{1}{2}, y_l)$, and the local coordinate system (u, t, y, s) is centered at z_l^t, then

$$\breve{\Omega}_l^t = \{(u, t, y, s) : \ |u| \le \breve{\nu}_u^t, |s| \le \breve{\nu}_s^t, |t| < 3/4, y \in \breve{J}_l\}. \tag{6.4}$$

Define $X^t = X_{l,\beta}^t$ supported on Ω_l^t by

$$X^t = \beta\psi^t(s)\zeta_y(y)\Big(-\xi_t'(t) \int_0^u \phi^t(r)dr, \ \ \xi_t(t)\phi^t(u), \ 0, \ 0\Big), \tag{6.5}$$

and clearly X^t is constant on the subset $\Omega_l^t = \Pi_l^t \times K \times J_l$. We define $h^t = h_{l,\beta}^t$ on $\breve{\Omega}_l^t$ to be the time-1 map of the flow generated by X^t, and set $h^t = Id$ outside $\breve{\Omega}_l^t$. Since X^t is divergence free, dh^t preserves E_T^{ut}-subbundle and $\det(dh^t|E_T^{ut}(z)) = 1$.

Take the map Q in Proposition 4.2. Given a sequence of positive numbers $\{\beta_n\}_{n\ge 0}$, start with

$$H_0 = h_{0,\beta_0}^t \circ h_{0,\beta_0}^y \circ Q,$$

and inductively let

$$H_n = h_{-n,\beta_n}^t \circ h_{-n,\beta_n}^y \circ h_{n,\beta_n}^t \circ h_{n,\beta_n}^y \circ H_{n-1}, \tag{6.6}$$

for $n \ge 1$. We are going to show that H_n are the desired maps in Proposition 4.3 with suitable choices of the sequence $\{\beta_n\}_{n\ge 0}$.

6.2. **Proof of Proposition 4.3.** We outline the proof of Proposition 4.3, following [4, 12].

First note that statements (2) and (4) and the fact that H_n is homotopic to Q follow directly from the above construction. Moreover, the original map T is uniformly partially hyperbolic on each $\breve{\mathcal{Z}}_n$ and is dynamical coherent in view of Theorem 2.1. We can choose $\{\beta_n\}_{n\ge 0}$ carefully in (6.6), and a positive sequence $\{\delta_n'\}_{n\ge 0}$ with $\delta_n' \le \delta_{n-1}'/2$ such that

$$\|H_0 - Q\|_{C^r} \le \delta_0', \ \ \|H_n - H_{n-1}\|_{C^{r+n}} \le \delta_n' \tag{6.7}$$

then the statement (3) holds. In particular, H_n is a gentle perturbation of T and dynamical coherent. It remains to show H_n satisfies statement (5) and (6) with good choices of δ_n and θ_n.

Denote $W^c_{H_n}(z)$ the center manifold of H_n at the point $z \in \breve{\mathcal{Z}}$. For any $l \in \mathbb{Z}$, set $z_0(l) = (q_j, \frac{1}{2}, y_l) \in \mathcal{N} \times \breve{J}_l$, where $j \equiv l \mod 2$, and y_l is the middle point of $\breve{J}_l$. Let $n = |l|$, and we denote by $W^c_{H_n}(z_0(l), K, J_l)$ the connected component of $W^c_{H_n}(z_0(l)) \cap (X \times K \times J_l)$ that contains $z_0(l)$. We shall also use similar notations $W^c_{H_n}(z_0(l), \breve{K}, \breve{J}_l)$, etc.

Now let us introduce two important maps Θ and Ψ.

Denote γ^i_j the quadrilateral $(u,s)_A$-path of X with the collection of points p_j, $[p_j, p^i_j]$, p^i_j, $[p^i_j, p_j]$ and p_j, for $i = t, y$ and $j = 0, 1$. Given $l \in \mathbb{Z}$, set $n = |l|$ and $j \equiv l \mod 2$, one can lift the quadrilateral γ^i_j to a quadrilateral $(u,s)_{H_n}$-path $\widehat{\gamma}^i_j$ of $\mathcal{Z}$ with the initial point z_1 by letting

$$\begin{aligned} z_2 &= V^u_{H_n}(z_1) \cap V^{sc}_{H_n}(p^i_j, 1/2, y_l), \\ z_3 &= V^s_{H_n}(z_2) \cap V^{uc}_{H_n}(p^i_j, 1/2, y_l), \\ z_4 &= V^u_{H_n}(z_3) \cap V^{sc}_{H_n}(z_1), \\ z_5 &= V^s_{H_n}(z_4) \cap V^{uc}_{H_n}(z_1). \end{aligned} \tag{6.8}$$

and $\widehat{\gamma}^i_j = \{z_1, \ldots, z_5\}$. This path defines a map $\Theta^i = \Theta^i_{l,H_n}$ given by $\Theta^i(z_1) = z_5$. It is easy to see that $z_5 \in V^c_{H_n}(z_1)$, and hence Θ^i maps $W^c_{H_n}(z_0(l), \breve{K}, \breve{J}_l)$ into itself. Reparameterizing the curve $V^u_{H_n}(z_1)$ from z_1 to z_2 by $\sigma : [0,1] \to V^u_{H_n}(z_1)$ so that $\sigma(0) = z_1$ and $\sigma(1) = z_2$, we obtain a parameterized family of quadrilaterals $\widehat{\gamma}^i_j = \{z_1(\tau), \ldots, z_5(\tau)\}$, $\tau \in [0,1]$, where $z_1(\tau) = z_1$, $z_2(\tau) = \sigma(\tau)$, and $z_i(\tau)$, $i = 3,4,5$ are obtained in the way similar to (6.8). Then we obtain $\Theta^i_\tau = \Theta^i_{\tau,l,H_n}$ given by $\Theta^i_\tau(z_1) = z_5(\tau)$. Clearly $\Theta^i_0 = Id$, $\Theta^i_1 = \Theta^i$, and Θ^i_τ depends continuously on τ and maps $W^c_{H_n}(z_0(l), \breve{K}, \breve{Z}_l)$ into $W^c_{H_n}(z_0(l))$.

On the other hand, given $z = ((x,t),y)$, there is a $(u,s)_T$-path connecting z to $z' = ((p_j,t),y)$ whose length does not exceed $2d(x,p_j)$. This generates a map $\Psi_T = \Psi_{T,j}$ from $\mathcal{Z}$ to $\{p_j\} \times \breve{K} \times I$ given by $\Psi_T(z) = z'$.

Moreover, if $H^\natural$ is a gentle perturbation of H_n, and hence a gentle perturbation of T, by uniform partially hyperbolicity and dynamical coherence of H_n and T, one can define $\Theta^i_{l,H^\natural}$, $\Theta^i_{\tau,l,H^\natural}$ and $\Psi_{H^\natural,j}$ in a similar way, $i = t, y$, $\tau \in [0,1]$ and $j \equiv l \mod 2$. As long as $H^\natural = T$ outside some $\mathcal{Z}_k$ and $\angle(E^\omega_{H^\natural}(z), E^\omega_T(z))$ is sufficiently small for all $z \in \mathcal{Z}_k$, $\omega = u,s,c$, the maps $\Theta^i_{l,H^\natural}$, $\Theta^i_{\tau,l,H^\natural}$ and $\Psi_{H^\natural,j}$ depends uniformly continuously on $H^\natural$, see [12] for more details.

Given a set $\Gamma \subset \breve{\mathcal{Z}}$ and a gentle perturbation $H^\natural$ of T, set

$$\begin{aligned} \mathcal{A}_{H^\natural}(\Gamma) = \{z \in \breve{\mathcal{Z}} \quad : &\quad \text{there exists } y \in \Gamma \text{ such that} \\ &\quad y \text{ is accessible to } z \text{ via a } (u,s)_{H^\natural} - \text{path}\}. \end{aligned} \tag{6.9}$$

Let $n = |l|$, and set $\epsilon_n = \min\{1/2^{n+4}, \breve{\epsilon}_t(l)\}$. Also, let $i = t, y$, $\omega = u,s,c$, $\tau \in [0,1]$.

We now show how to choose δ_n and θ_n. For $n = l = 0$, choose $\theta^*_0 > 0$ such that for any gentle perturbation $H^\natural$ of T with $\angle(E^\omega_{H^\natural}(z), E^\omega_T(z)) \le 2\theta^*_0$ for $z \in \breve{\mathcal{Z}}_0 = \mathcal{N} \times \breve{I}_0$, the maps $\Psi_{H^\natural} = \Psi_{H^\natural,0}$ and $\Theta^i_{\tau,0,H^\natural}$, are well defined. We assume that δ_Q in Proposition 4.2 is so small that $\angle(E^\omega_Q(z), E^\omega_T(z)) \le \theta^*_0$ and $d(\Theta^i_{\tau,0,Q}(z), z) \le \epsilon_0/8$ for $z \in \mathcal{Z}_0 = \mathcal{N} \times I_0$. Now choose θ'_0 such that $0 < \theta'_0 \le \theta^*_0/2$, and if $H^\natural$ is a gentle perturbation of T with $\angle(E^\omega_{H^\natural}(z), E^\omega_Q(z)) \le 2\theta'_0$ for $z \in \breve{\mathcal{Z}}_0$, then

$$d(\Psi_{H^\natural}(z), \Psi_Q(z)) \le 1/2^7, \ z \in \breve{\mathcal{Z}}_0 = \mathcal{N} \times \breve{J}_0. \tag{6.10}$$

Also choose δ'_0 in (6.7) so small that if $\|H_0 - Q\|_{C^r} \le \delta'_0$, then for all $z \in \breve{\mathcal{Z}}_0$, we have $\angle(E^\omega_{H_0}(z), E^\omega_Q(z)) \le \theta'_0$.

Set $\delta_0 = \min\{\delta'_0, \delta''_0\}$ and $\theta_0 = \min\{\theta'_0, \theta''_0\}$ where δ''_0 and θ''_0 are given by Lemma 6.1. For any gentle perturbation $H^\natural$ of H_0 with $\angle(E^\omega_{H^\natural}(z), E^\omega_{H_0}(z)) \le \theta'_0$ for all $z \in \mathcal{Z}_0$, we have

$$\angle(E^\omega_{H^\natural}(z), E^\omega_Q(z)) \le 2\theta'_0 \le \theta^*_0, \tag{6.11}$$

$$\angle(E^\omega_{H^\natural}(z), E^\omega_T(z)) \le 2\theta^*_0. \tag{6.12}$$

By Lemma 6.1, (6.11) implies that $d(\Theta^i_{\tau,0,H_0}(z), z) \le \epsilon_0/4$ for all $z \in W^c_{H_0}(z_0(0), \breve{K}, \breve{J}_0)$, moreover,

$$\mathcal{A}_{H^\natural}(z_0(0)) \supset W^c_{H^\natural}(z_0(0), \bar{K}, \bar{J}_0),$$

Since the distance between $\partial\bar{J}_0$ and $\partial\widetilde{J}_0$ is $1/2^5$, (6.12) implies that

$$\Psi_{P^\natural}(\mathcal{N} \times \widetilde{J}_0) \subset W^c_{H^\natural}(z_0(0), \bar{K}, \bar{J}_0),$$

and by the fact that z and $\Psi_{H^\natural}(z)$ are $(u,s)_{H^\natural}$-accessible and hence

$$\mathcal{A}_{H^\natural}(z_0(0)) \supset \mathcal{N} \times \widetilde{J}_0 = \widetilde{\mathcal{Z}}_0.$$

Proceed inductively in a similar way for $n \ge 1$, we can find δ'_n and θ'_n such that (6.7) and statements (5) and (6) of Proposition 4.3 hold. More precisely, if $\|H_n - H_{n-1}\| \le \delta'_n$ then $\angle(E^\omega_{H_n}(z), E^\omega_{H_{n-1}}(z)) \le \theta'_n$ for $z \in \breve{\mathcal{Z}}_n$. Take $\delta_n = \min\{\delta'_n, \delta''_n\}$ and $\theta_n = \min\{\theta'_n, \theta''_n\}$ where δ''_n and θ''_n are given by Lemma 6.1, and also one can make $\theta'_n < \theta_{n-1}/2$. Moreover, if $H^\natural$ is a gentle perturbation of H_n such that $\angle(E^\omega_{H^\natural}(z), E^\omega_{H_n}(z)) \le \theta_n$ for $z \in \breve{\mathcal{Z}}_n$, then $\angle(E^\omega_{H^\natural}(z), E^\omega_{H_{n-1}}(z)) \le 2\theta'_n \le \theta_{n-1}$, and hence by statement (6), $H^\natural$ has accessibility property on $\widetilde{\mathcal{Z}}_{n-1}$. By the same argument above for $n = 0$, one can get

$$\mathcal{A}_{H^\natural}(z_0(n)) \supset \mathcal{N} \times \widetilde{J}_l, \quad l = \pm n, \tag{6.13}$$

in other words, $H^\natural$ has the accessibility property on $\mathcal{N} \times \widetilde{J}_l$, $l = \pm n$. Note that

$$\widetilde{\mathcal{Z}}_n = \widetilde{\mathcal{Z}}_{n-1} \cup (\mathcal{N} \times \widetilde{J}_n) \cup (\mathcal{N} \times \widetilde{J}_{-n}),$$

and $\widetilde{\mathcal{Z}}_{n-1} \cup (\mathcal{N} \times \widetilde{J}_l) \ne \emptyset$ for $l = \pm n$. Since $\widetilde{\mathcal{Z}}_n$ is connected, we obtain the accessibility of $H^\natural$ on $\widetilde{\mathcal{Z}}_n$. In particular, take $H^\natural = H_n$, and we obtain that H_n has accessibility property on $\widetilde{\mathcal{Z}}_n$.

To complete the proof of Proposition 4.3, it remains to show the following lemma. (Since the construction on $\mathcal{N} \times \breve{J}_{-l}$ and $\mathcal{N} \times \breve{J}_l$ are symmetric for $l \ge 1$, we just need to consider the case when $n = l \ge 0$ for this lemma.)

Lemma 6.1. *Let $H_{-1} = Q$, $z_0(-1) = z_0(0)$, $\epsilon_{-1} = \epsilon_0/2$, and also $i = t, y$, $\omega = u, s, c$, $\tau \in [0,1]$. Suppose for some $n \ge 0$, $d(\Theta^i_{\tau,n,H_{n-1}}(z), z) \le \epsilon_{n-1}/4$ for all $z \in W^c_{H_{n-1}}(z_0(n), \breve{K}, \breve{J}_n)$. Then there exist δ''_n, $\theta''_n > 0$ such that if H_n satisfies $\|H_n - H_{n-1}\|_{C^{r+n}} \le \delta''_n$, then we have*

$$d(\Theta^i_{\tau,n+1,H_n}(z), z) \le \epsilon_n/4, \ \text{for all } z \in W^c_{H_n}(z_0(n+1), \breve{K}, \breve{J}_{n+1}). \tag{6.14}$$

Moreover, for any gentle perturbation $H^\natural$ of H_n with

$$\angle(E^\omega_{H^\natural}(z), E^\omega_{H_n}(z)) \le \theta''_n, \ \text{for all } z \in \mathcal{N} \times \breve{J}_n,$$

we have

$$\mathcal{A}_{H^\natural}(z_0(n) \supset W^c_{H^\natural}(z_0(n), \bar{K}, \bar{J}_n) \tag{6.15}$$

In particular, (6.15) holds with $H^\natural = H_{n+1}$.

This is essentially the same as Lemma 5.2 in [12]. The proof are parallel, hence omitted here.

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